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SOME PROPERTIES OF COMMON HERMITIAN SOLUTIONS OF MATRIX EQUATIONS $A_1XA_1^* = B_1$ AND $A_2XA_2^* = B_2$

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Abstract. In this paper we provide necessary and sufficient conditions for the pair of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$ to have a common hermitian solution in the form $\frac{X_1+X_2}{2}$, where X_1 and X_2 are hermitian solutions of the equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$, respectively. In the last section, we derive necessary and sufficient conditions for common hermitian solution X of this pair of matrix equations to have the forms $\begin{pmatrix} 0 & X_2 \\ X_2^* & 0 \end{pmatrix}$ and $\begin{pmatrix} X_1 & 0 \\ 0 & X_3 \end{pmatrix}$, where X_1, X_2 and X_3 denote some submatrices in X.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{C}_H^n$ stand for the sets of all $m\times n$ complex matrices and all $n \times n$ complex Hermitian matrices, respectively. We denote by, A^*, $r(A)$ and $\mathfrak{R}(A)$ the conjugate transpose, the rank, and the range of A, respectively. The Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by A^+, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four matrix equations

(1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^* = AX$, (4) $(XA)^* = XA$.

Moreover, the two matrices E_A and F_A stand for the two orthogonal projectors $E_A = I_m - AA^+$, $F_A = I_n - A^+A$ induced by a matrix A. Their ranks are given by $r(E_A) = m - r(A)$, $r(F_A) = n - r(A)$.

The majority of properties of the generalized inverse, especially the Moore-Penrose generalized inverse have been treated in the book of A. Ben-Israël and T. N. E. Greville [1], and also in the book of Z. Nashed [11].

The inertia of a Hermitian matrix is defined to be a triplet $In(A) = \{i_+(A), i_-(A), i_0(A)\}$ composed of the numbers of the positive $(i_+(A))$, negative $(i_-(A))$ and zero eigenvalues $(i_0(A))$ of the matrix counted with multiplicities, respectively.

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Recently, research on linear matrix equations has received more attention and has had lots of nice results. For example, Mitra [9, 10] has provided conditions for the existence of a solution and a representation of the general common solution to the pair of matrix equations

$$A_1XB_1 = C_1 \quad \text{and} \quad A_2XB_2 = C_2, \tag{1}$$

where A_i, B_i, C_i are given for $i = 1, 2$ and X is unknown.

Also A. Navarra, P.L. Odell, D.M. Young. [12] gave new necessary and sufficient conditions for the existence of a common solution to (1) and derived a new representation of the general common solution to these two equations. In [6] Liu established the condition for the matrix equations (1) to have a common least-squares solution. In [7] X. Fu Liu, Hu Yang gave an expression of the general common least squares solution to the pair of matrix equations (1). In [3] S. Guerarra and S. Guedjiba determined necessary and sufficient conditions for the pair of matrix equations (1) to have a common least-rank solution, and the expression of this solution is also given.

Consider the pair of matrix equations

$$A_1XA_1^* = B_1 \quad \text{and} \quad A_2XA_2^* = B_2. \tag{2}$$

Various problems related to (2) and applications have been investigated in the literature. In [13] Y. Tian determined the conditions for the existence of a common hermitian solution to the pair of matrix equations (2) and gave a representation of the general common hermitian solution. Furthermore, he established explicit formulas for calculating

$$\max_{X\in S} r(A_1 - B_1XB_1), \qquad \max_{X\in S} i_{\pm}(A_1 - B_1XB_1),$$

$$\min_{X\in S} r(A_1 - B_1XB_1), \qquad \min_{X\in S} i_{\pm}(A_1 - B_1XB_1),$$

where $S = \{X \in \mathbb{C}_H^n \,|\, [B_2XB_2^*, B_3XB_3^*] = [A_2, A_3]\}$.

In [14] Y. Tian gave necessary and sufficient conditions for $S = T$, where

$$S = \{X \in \mathbb{R}_H^n \,|\, AXA^* = B\},$$

$$T = \left\{ \frac{X_1 + X_2}{2} \,\middle|\, T_1AX_1A^*T_1^* = T_1BT_1^*, T_2AX_2A^*T_2^* = T_2BT_2^* \right\}.$$

In [4], S. Guerarra and S. Guedjiba established a set of explicit formulas for calculating the maximal and minimal ranks and inertias of $P - X$ with respect to X, where $P \in \mathbb{C}_H^n$ is given, X is a common hermitian least-rank solution of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$.

In [19], the authors derived the maximal and minimal ranks of the submatrices in a least squares solution to the equaition $AXB = C$. From these formulas, they derived necessary and sufficient conditions for the submatrices to be zero and other special forms.

Motivated by these works we use the matrix rank method to derive necessary and sufficient conditions for (2) to have a common hermitian solution in the form $\frac{X_1 + X_2}{2}$, where X_1 and X_2 are hermitian solutions of the equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$, respectively, and we give necessary and sufficient conditions for the

submatrices in a common hermitian solution to (2) to have a special form.
We first give some lemmas.

LEMMA 1.1 ([8]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ and $C \in \mathbb{C}^{q\times n}$. Then, the following rank expansion formulas hold*

$$r\begin{pmatrix}A, & B\end{pmatrix} = r(A)+r(E_AB) = r(B)+r(E_BA), \tag{3}$$

$$r\begin{pmatrix}A\\ C\end{pmatrix} = r(A)+r(CF_A) = r(C)+r(AF_C), \tag{4}$$

$$r\begin{pmatrix}A & B\\ C & 0\end{pmatrix} = r(B)+r(C)+r(E_BAF_C). \tag{5}$$

LEMMA 1.2 ([16]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $C \in \mathbb{D}^{l\times k}$ be given. Then the rank of the Schur complement $S_A = D-CA^{+}B$ satisfies the equality*

$$r(D-CA^{+}B) = r\begin{pmatrix}A^*AA^* & A^*B\\ CA^* & D\end{pmatrix}-r(A). \tag{6}$$

LEMMA 1.3 ([15]). *Let $A \in \mathbb{C}_H^{m}$, $B \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}_H^{n}$, and let*

$$M_1 = \begin{pmatrix}A & B\\ B^* & 0\end{pmatrix},\ M_2 = \begin{pmatrix}A & B\\ B^* & D\end{pmatrix}. \tag{7}$$

Then, the following expansion formulas hold

$$i_{\pm}(M_1) = r(B)+i_{\pm}(E_BAE_B), \qquad i_{\pm}(M_2) = i_{\pm}(A)+i_{\pm}\begin{pmatrix}0 & E_AB\\ B^*E_A & D-B^*A^{+}B\end{pmatrix}, \tag{8}$$

$$r(M_1) = 2r(B)+r(E_BAE_B), \qquad r(M_2) = r(A)+r\begin{pmatrix}0 & E_AB\\ B^*E_A & D-B^*A^{+}B\end{pmatrix}. \tag{9}$$

Under the condition $A \succcurlyeq 0$,

$$i_{+}(M_1) = r\begin{pmatrix}A & B\end{pmatrix}, \quad i_{-}(M_1) = r(B), \quad r(M_1) = r\begin{pmatrix}A & B\end{pmatrix}+r(B).$$

Under the condition $\mathfrak{R}(B) \subseteq \mathfrak{R}(A)$,

$$i_{\pm}(M_2) = i_{\pm}(A)+i_{\pm}(D-B^*A^{+}B), \quad r(M_2) = r(A)+r(D-B^*A^{+}B).$$

Some general rank and inertia expansion formulas derived from (3)–(5), (7)–(9) *are given below*

$$r\begin{pmatrix}A & B\\ E_PC & 0\end{pmatrix} = r\begin{pmatrix}A & B & 0\\ C & 0 & P\end{pmatrix}-r(P), \quad r\begin{pmatrix}A & BF_Q\\ C & 0\end{pmatrix} = r\begin{pmatrix}A & B\\ C & 0\\ 0 & Q\end{pmatrix}-r(Q),$$

$$r\begin{pmatrix}A & BF_Q\\ E_PC & 0\end{pmatrix} = r\begin{pmatrix}A & B & 0\\ C & 0 & P\\ 0 & Q & 0\end{pmatrix}-r(P)-r(Q), \tag{10}$$

$$i_{\pm}\begin{pmatrix}A & BF_P\\ F_PB^* & 0\end{pmatrix} = i_{\pm}\begin{pmatrix}A & B & 0\\ B^* & 0 & P^*\\ 0 & P & 0\end{pmatrix}-r(P), \tag{11}$$

$$i_{\pm}\begin{pmatrix} E_QAE_Q & E_QB \\ B^*E_Q & D \end{pmatrix} = i_{\pm}\begin{pmatrix} A & B & Q \\ B^* & D & 0 \\ Q^* & 0 & 0 \end{pmatrix} - r(Q).$$

LEMMA 1.4 ([14]). *Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}_H^n$, $Q \in \mathbb{C}^{m\times n}$, and assume that $P \in \mathbb{C}^{m\times m}$ is nonsingular. Then,*

$$i_{\pm}(PAP^*) = i_{\pm}(A), \qquad i_{\pm}(\lambda A) = \begin{cases} i_{\pm}(A) & \text{if } \lambda > 0 \\ i_{\mp}(A) & \text{if } \lambda < 0 \end{cases},$$

$$i_{\pm}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = i_{\pm}(A)+i_{\pm}(B), \qquad i_{+}\begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix} = i_{-}\begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix} = r(Q). \tag{12}$$

LEMMA 1.5. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}_H^m$ be given. Then, the following hold.*
(a) [2, 5], The matrix equation $AXA^* = B$ has a solution $X \in \mathbb{C}_H^n$ iff $\Re(B) \subseteq \Re(A)$, or equivalently, $AA^+B = B$.
(b) [15] Under $AA^+B = B$ the general hermitian solution of $AXA^* = B$ can be written in the following two forms $X = A^+B(A^+)^*+U-A^+AUA^+A$, $X = A^+B(A^+)^*+F_AV+V^*F_A$, where $U \in \mathbb{C}_H^n$ and $V \in \mathbb{C}^{n\times n}$ are arbitrary.

LEMMA 1.6 ([13]). *Let $B_i \in \mathbb{C}_H^{m_i}$, $A_i \in \mathbb{C}^{m_i\times n}$ be given for $i = 1, 2$ and suppose that each of the two matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$, has a solution, i.e., $\Re(B_i) \subseteq \Re(A_i)$ for $i = 1, 2$. Then, the following hold.*
(a) The pair of matrix equations has a common hermitian solution if and only if

$$r\begin{pmatrix} B_1 & 0 & A_1 \\ 0 & -B_2 & A_2 \\ A_1^* & A_2^* & 0 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \tag{13}$$

(b) Under (13), the general common hermitian solution of the pair of equation can be written in the following parametric form $X = X_0+VF_A+F_AV^*+F_{A_1}UF_{A_2}+F_{A_2}U^*F_{A_1}$, where X_0 is a special hermitian common solution to the pair of equations, $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, and $U, V \in \mathbb{C}^{n\times n}$ are arbitrary.

2. Properties of common solutions of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$

LEMMA 2.1 ([17]). *Let $\phi(X_1, X_2) = A-B_1X_1C_1-(B_1X_1C_1)^*-B_2X_2C_2-(B_2X_2C_2)^*$, where $A \in \mathbb{C}_H^n$, $B_i \in \mathbb{C}^{m\times p_i}$ and $C_i \in \mathbb{C}^{q_i\times m}$ are given, and $X_i \in \mathbb{C}^{p_i\times q_i}$ are variable matrices for $i = 1, 2$, and assume that $\Re(B_2) \subseteq \Re(B_1)$, $\Re(C_1^*) \subseteq \Re(B_1)$, $\Re(C_2^*) \subseteq \Re(B_1)$. Also let*

$$N = \begin{pmatrix} A & B_2 & C_1^* & C_2^* \\ C_1 & 0 & 0 & 0 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} A & B_2 & C_1^* & C_2^* \\ B_2^* & 0 & 0 & 0 \\ C_1 & 0 & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} A & B_2 & C_1^* & C_2^* \\ C_1 & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} A & B_1 \\ C_1 & 0 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} A & B_2 & C_1^* \\ B_2^* & 0 & 0 \\ C_1 & 0 & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} A & C_1^* & C_2^* \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{pmatrix}.$$

Then, the global maximal and minimal ranks and inertias of $\phi(X_1, X_2)$ are given by

$$\max_{X_1 \in \mathbb{C}^{p_1 \times q_1}, X_2 \in \mathbb{C}^{p_2 \times q_2}} r[\phi(X_1, X_2)] = \min\{r[A, B_1], r(N), r(M_1), r(M_2)\},$$

$$\min_{X_1 \in \mathbb{C}^{p_1 \times q_1}, X_2 \in \mathbb{C}^{p_2 \times q_2}} r[\phi(X_1, X_2)] = 2r[A, B_1] - 2r(M) + 2r(N) + \max\{s_1, s_2, s_3, s_4\},$$

$$\max_{X_1 \in \mathbb{C}^{p_1 \times q_1}, X_2 \in \mathbb{C}^{p_2 \times q_2}} i_\pm[\phi(X_1, X_2)] = \min\{i_\pm(M_1), i_\pm(M_2)\},$$

$$\begin{aligned} \min_{X_1 \in \mathbb{C}^{p_1 \times q_1}, X_2 \in \mathbb{C}^{p_2 \times q_2}} i_\pm[\phi(X_1, X_2)] = {} & r[A, B_1] - r(M) + r(N) \\ & + \max\{i_\pm(M_1) - r(N_1), i_\pm(M_2) - r(N_2)\}, \end{aligned}$$

where $s_1 = r(M_1) - 2r(N_1)$, $\quad s_2 = r(M_2) - 2r(N_2)$,
$s_3 = i_+(M_1) + i_-(M_2) - r(N_1) - r(N_2)$, $\quad s_4 = i_-(M_1) + i_+(M_2) - r(N_1) - r(N_2)$.

Let $A_i \in \mathbb{C}^{m_i \times p}$, $B_i \in \mathbb{C}_H^{m_i}$, for $i = 1, 2$. Define

$$S_1 = \left\{ Y = \frac{X_1 + X_2}{2} \in \mathbb{C}_H^n \mid A_1 X_1 A_1^* = B_1, A_2 X_2 A_2^* = B_2 \right\}, \tag{14}$$

$$S_2 = \left\{ X \in \mathbb{C}_H^n \mid A_1 X A_1^* = B_1, A_2 X A_2^* = B_2 \right\}. \tag{15}$$

In this case, we give necessary and sufficient conditions for $S_1 \cap S_2 \neq \emptyset$.

Theorem 2.2. *Assume that the pair of matrices in* (15) *has a common hermitian solution, and let S_1 and S_2 be defined as in* (14) *and* (15). *Then, $S_1 \cap S_2 \neq \emptyset$ iff*

$$r \begin{pmatrix} 0 & -A_1^* & A_2^* \\ -A_1 & -\frac{1}{2}B_1 & 0 \\ A_2 & 0 & \frac{1}{2}B_2 \end{pmatrix} = 2r \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \tag{16}$$

Proof. Note that $S_1 \cap S_2 \neq \emptyset$ is equivalent to $\min_{Y \in S_1, X \in S_2} r(Y - X) = 0$. From Lemma 1.5, the general expression of the matrices of the two equations in (14) can be written as $Y = \frac{A_1^\dagger B_1 (A_1^\dagger)^*}{2} + F_{A_1} U_1 + U_1^* F_{A_1} + \frac{A_2^\dagger B_2 (A_2^\dagger)^*}{2} + F_{A_2} U_2 + U_2^* F_{A_2}$, where U_1 and U_2 are arbitrary. Then

$$\begin{aligned} Y - X = {} & \frac{A_1^\dagger B_1 (A_1^\dagger)^*}{2} + F_{A_1} U_1 + U_1^* F_{A_1} + \frac{A_2^\dagger B_2 (A_2^\dagger)^*}{2} + F_{A_2} U_2 + U_2^* F_{A_2} - X_0 \\ & - V F_A - F_A V^* - F_{A_1} U F_{A_2} - F_{A_2} U^* F_{A_1} \\ = {} & \frac{A_1^\dagger B_1 (A_1^\dagger)^*}{2} + \frac{A_2^\dagger B_2 (A_2^\dagger)^*}{2} - X_0 - \left(V,\ -U_1^*,\ -U_2^*\right) \begin{pmatrix} F_A \\ F_{A_1} \\ F_{A_2} \end{pmatrix} \\ & - \left(F_A,\ F_{A_1},\ F_{A_2}\right) \begin{pmatrix} V^* \\ -U_1 \\ -U_2 \end{pmatrix} - F_{A_1} U F_{A_2} - F_{A_2} U^* F_{A_1}. \end{aligned}$$

Let

$$L=\begin{pmatrix} \frac{A_1^+B_1(A_1^+)^*}{2}+\frac{A_2^+B_2(A_2^+)^*}{2}-X_0 & F_{A_1} & F_A & F_{A_1} & F_{A_2} & F_{A_2}\\ F_A & 0 & 0 & 0 & 0 & 0\\ F_{A_1} & 0 & 0 & 0 & 0 & 0\\ F_{A_2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L_1=\begin{pmatrix} \frac{A_1^+B_1(A_1^+)^*}{2}+\frac{A_2^+B_2(A_2^+)^*}{2}-X_0 & F_{A_1} & F_A & F_{A_1} & F_{A_2} & F_{A_2}\\ F_{A_1} & 0 & 0 & 0 & 0 & 0\\ F_A & 0 & 0 & 0 & 0 & 0\\ F_{A_1} & 0 & 0 & 0 & 0 & 0\\ F_{A_2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L_2=\begin{pmatrix} \frac{A_1^+B_1(A_1^+)^*}{2}+\frac{A_2^+B_2(A_2^+)^*}{2}-X_0 & F_{A_1} & F_A & F_{A_1} & F_{A_2} & F_{A_2}\\ F_A & 0 & 0 & 0 & 0 & 0\\ F_{A_1} & 0 & 0 & 0 & 0 & 0\\ F_{A_2} & 0 & 0 & 0 & 0 & 0\\ F_{A_2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G=\begin{pmatrix} \frac{A_1^+B_1(A_1^+)^*}{2}+\frac{A_2^+B_2(A_2^+)^*}{2}-X_0 & I_p\\ F_A & 0\\ F_{A_1} & 0\\ F_{A_1} & 0 \end{pmatrix},$$

$$G_1=\begin{pmatrix} \frac{A_1^+B_1(A_1^+)^*}{2}+\frac{A_2^+B_2(A_2^+)^*}{2}-X_0 & F_{A_1} & F_A & F_{A_1} & F_{A_2}\\ F_{A_1} & 0 & 0 & 0 & 0\\ F_A & 0 & 0 & 0 & 0\\ F_{A_1} & 0 & 0 & 0 & 0\\ F_{A_2} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_2=\begin{pmatrix} \frac{A_1^+B_1(A_1^+)^*}{2}+\frac{A_2^+B_2(A_2^+)^*}{2}-X_0 & F_A & F_{A_1} & F_{A_2} & F_{A_2}\\ F_A & 0 & 0 & 0 & 0\\ F_{A_1} & 0 & 0 & 0 & 0\\ F_{A_2} & 0 & 0 & 0 & 0\\ F_{A_2} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, from Lemma 2.1, we get

$$\begin{aligned}\min_{Y\in S_1, X\in S_2} r(Y-X) =& 2r\left[\frac{A_1^+B_1(A_1^+)^*}{2}+\frac{A_2^+B_2(A_2^+)^*}{2}-X_0, I_p\right]\\ &-2r(G)+2r(L)+\max\{t_1,t_2,t_3,t_4\},\end{aligned} \tag{17}$$

where $t_1=r(G_1)-2r(L_1)$, $t_2=r(G_2)-2r(L_2)$,

$t_3=i_+(G_1)+i_-(G_2)-r(L_1)-r(L_2)$, $t_4=i_-(G_1)+i_+(G_2)-r(L_1)-r(L_2)$.

We will simplify $r(G)$, $r(L)$, $i_\pm(L_i)$ and $i_\pm(G_i)$ for $i=1,2$.

Applying (4), (6), (11) and (12), and simplifying by $[A_1A_1^{+}B_1, A_2A_2^{+}B_2] = [B_1, B_2]$, elementary matrix operations and congruence matrix operations, and the fact that $\Re(F_A) \subseteq \Re(F_{A_1})$ and $\Re(F_A) \subseteq \Re(F_{A_2})$, we obtain

$$r(G) = r\begin{pmatrix} \frac{A_1^{+}B_1(A_1^{+})^*}{2} + \frac{A_2^{+}B_2(A_2^{+})^*}{2} - X_0 & I_p \\ F_A & 0 \\ F_{A_1} & 0 \\ F_{A_2} & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} \frac{A_1^{+}B_1(A_1^{+})^*}{2} + \frac{A_2^{+}B_2(A_2^{+})^*}{2} - X_0 & I_p \\ F_{A_1} & 0 \\ F_{A_2} & 0 \end{pmatrix} = 2p + r(A_2F_{A_1}) - r(A_2),$$

$$i_{\pm}(G_1) = i_{\pm}\begin{pmatrix} \frac{A_1^{+}B_1(A_1^{+})^*}{2} + \frac{A_2^{+}B_2(A_2^{+})^*}{2} - X_0 & F_{A_1} & F_A & F_{A_1} & F_{A_2} \\ F_{A_1} & 0 & 0 & 0 & 0 \\ F_A & 0 & 0 & 0 & 0 \\ F_{A_1} & 0 & 0 & 0 & 0 \\ F_{A_2} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= i_{\pm}\begin{pmatrix} \frac{A_1^{+}B_1(A_1^{+})^*}{2} + \frac{A_2^{+}B_2(A_2^{+})^*}{2} - X_0 & F_{A_1} & F_{A_2} \\ F_{A_1} & 0 & 0 \\ F_{A_2} & 0 & 0 \end{pmatrix}$$

$$= i_{\pm}\begin{pmatrix} \frac{A_1^{+}B_1(A_1^{+})^*}{2} + \frac{A_2^{+}B_2(A_2^{+})^*}{2} - X_0 & I_p & I_p & 0 & 0 \\ I_p & 0 & 0 & A_1^* & 0 \\ I_p & 0 & 0 & 0 & A_2^* \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \end{pmatrix} - r(A_1) - r(A_2)$$

$$= i_{\pm}\begin{pmatrix} 0 & I_p & I_p & -\frac{1}{4}A_1^{+}B_1 + \frac{1}{2}X_0A_1^* & -\frac{1}{4}A_2^{+}B_2 \\ I_p & 0 & 0 & A_1^* & 0 \\ I_p & 0 & 0 & 0 & A_2^* \\ -\frac{1}{4}B_1(A_1^{+})^* + \frac{1}{2}A_1X_0 & A_1 & 0 & 0 & 0 \\ -\frac{1}{4}B_2(A_2^{+})^* & 0 & A_2 & 0 & 0 \end{pmatrix} - r(A_1) - r(A_2)$$

$$= p + i_{\pm}\begin{pmatrix} 0 & -A_1^* & A_2^* \\ -A_1 & -\frac{1}{2}B_1 & 0 \\ A_2 & 0 & \frac{1}{2}B_2 \end{pmatrix} - r(A_1) - r(A_2)$$

By similar steps we obtain

$$i_{\pm}(G_2) = p + i_{\pm}\begin{pmatrix} 0 & -A_1^* & A_2^* \\ -A_1 & -\frac{1}{2}B_1 & 0 \\ A_2 & 0 & \frac{1}{2}B_2 \end{pmatrix} - r(A_1) - r(A_2),$$

$$r(L) = 2p + r\begin{pmatrix} 0 & -A_1^* & A_2^* \\ -A_1 & -\frac{1}{2}B_1 & 0 \\ A_2 & 0 & \frac{1}{2}B_2 \end{pmatrix} - 2r(A_1) - 2r(A_2).$$

It is clear that $r(L) = r(L_1) = r(L_2)$ and $t_1 = t_2 = t_3 = t_4 = -r(L)$. Substituting these relations into (17) yields (16). $\square$

3. Ranks of submatrices in a common hermitian solution of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$

Let

$$S = \{X \in \mathbb{C}_H^n \mid A_1XA_1^* = B_1, A_2XA_2^* = B_2\}. \tag{18}$$

The common hermitian solutions of the pair of matrix equations in S are given by $X = X_0 + VF_A + F_AV^* + F_{A_1}UF_{A_2} + F_{A_2}U^*F_{A_1}$, where X_0 is a special hermitian common solution to the pair of matrix equations in S, $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, and $U, V \in \mathbb{C}^{n\times n}$ are arbitrary.

One of the fundamental concepts in matrix theory is the partition of matrix. Many properties of a matrix can be derived from the submatrices in its partition. The common hermitian solution $X \in S$ is partitioned into 2×2 block of the form $X = \begin{pmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{pmatrix}$, where $X_1 \in \mathbb{C}^{n_1\times n_1}$, $X_2 \in \mathbb{C}^{n_1\times n_2}$ and $X_3 \in \mathbb{C}^{n_2\times n_2}$ with $n_1+n_2 = n$. Since X_1, X_2 and X_3 are submatrices in a common hermitian solution X to the pair of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$, they can be rewritten as

$$X_1 = \begin{pmatrix} I_{n_1}, & 0 \end{pmatrix} X \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} := R_1XR_1^*, \quad X_2 = \begin{pmatrix} I_{n_1}, & 0 \end{pmatrix} X \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} := R_1XR_2^*,$$

$$X_3 = \begin{pmatrix} 0, & I_{n_2} \end{pmatrix} X \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} := R_2XR_2^*.$$

We adopt the following notations for the collections of submatrices X_1, X_2 and X_3.

$$S_i = \left\{ X_i \;\middle|\; X = \begin{pmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{pmatrix} \in S \right\}, \quad i = 1,2,3.$$

The following are some known results for ranks and inertias of matrices, which will be used in this section.

Lemma 3.1 ([17]). *Let $A_i \in \mathbb{C}^{m_i\times n}$ and $B_i \in \mathbb{C}_H^n$ be given for $i = 1,2$ and assume that the pair of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$, have a common solution $X \in \mathbb{C}_H^n$. Also, let S be defined by (18) and define*

$$P_1 = \begin{pmatrix} A & B & 0 & 0 \\ B^* & 0 & A_1^* & A_2^* \end{pmatrix}, \qquad P_2 = \begin{pmatrix} A & 0 & B \\ 0 & -B_1 & A_1 \\ B^* & A_1^* & 0 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} A & 0 & B \\ 0 & -B_2 & A_2 \\ B^* & A_2^* & 0 \end{pmatrix},$$

$$Q_1=\begin{pmatrix} A & 0 & 0 & B & B\\ 0 & -B_1 & 0 & A_1 & 0\\ 0 & 0 & -B_2 & 0 & A_2\\ B^* & A_1^* & A_2^* & 0 & 0\end{pmatrix},\quad Q_2=\begin{pmatrix} A & 0 & B & B\\ 0 & -B_1 & A_1 & 0\\ B^* & A_1^* & 0 & 0\\ 0 & 0 & 0 & A_2\end{pmatrix},\quad Q_3=\begin{pmatrix} A & 0 & B & B\\ 0 & -B_2 & A_2 & 0\\ B^* & A_2^* & 0 & 0\\ 0 & 0 & 0 & A_1\end{pmatrix}.$$

Then, the following hold.

(a) The global maximal rank of $A - BXB^*$ subject to S, defined by (18), *is*

$$\max_{X\in S} r(A-BXB^*)=$$
$$\min\left\{r\begin{pmatrix}A, & B\end{pmatrix}, r(Q_1)-r\begin{pmatrix}A_1\\ A_2\end{pmatrix}-r(A_1)-r(A_2), r(P_2)-2r(A_1), r(P_3)-2r(A_2)\right\}.$$

(b) The global minimal rank of $A - BXB^*$ subject to S, defined by (18), *is*

$$\min_{X\in S} r(A-BXB^*)=$$
$$2r\begin{pmatrix}A, & B\end{pmatrix}-2r(P_1)+2r(Q_1)+\max\left\{r(P_2)-2r(Q_2), r(P_3)-2r(Q_3), u_1, u_2\right\},$$

where
$$u_1=i_+(P_2)+i_-(P_3)-r(Q_2)-r(Q_3),$$
$$u_2=i_-(P_2)+i_+(P_3)-r(Q_2)-r(Q_3).$$

(c) The global maximal inertia of $A - BXB^*$ subject to S, defined by 18, is

$$\max_{X\in S} i_\pm(A-BXB^*)=\min\left\{i_\pm(P_2)-r(A_1), i_\pm(P_3)-r(A_2)\right\}.$$

(d) The global minimal inertia of $A - BXB^*$ subject to S, defined by (18), *is*

$$\min_{X\in S} i_\pm(A-BXB^*)=$$
$$r\begin{pmatrix}A, & B\end{pmatrix}-r(P_1)+r(Q_1)+\max\left\{i_\pm(P_2)-r(Q_2), i_\pm(P_3)-r(Q_3)\right\}.$$

LEMMA 3.2 ([18]). *Let $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{m\times k}$, $C\in\mathbb{C}^{l\times n}$, $B_1\in\mathbb{C}^{m\times p}$, $C_1\in\mathbb{C}^{q\times n}$ be given, $Y\in\mathbb{C}^{k\times n}$, $Z\in\mathbb{C}^{m\times l}$, $U\in\mathbb{C}^{p\times q}$ be variant matrices. Then*

$$\min_{Y,Z,U} r(A-BY-ZC-B_1UC_1)=$$
$$r\begin{pmatrix}A & B & B_1\\ C & 0 & 0\end{pmatrix}+r\begin{pmatrix}A & B\\ C & 0\\ C_1 & 0\end{pmatrix}-r\begin{pmatrix}A & B & B_1\\ C & 0 & 0\\ C_1 & 0 & 0\end{pmatrix}-r(B)-r(C),$$

$$\max_{Y,Z,U} r(A-BY-ZC-B_1UC_1)=\min\left\{m, n, r\begin{pmatrix}A & B & B_1\\ C & 0 & 0\end{pmatrix}, r\begin{pmatrix}A & B\\ C & 0\\ C_1 & 0\end{pmatrix}\right\}.$$

THEOREM 3.3. *Let $A_i\in\mathbb{C}^{m_i\times n}$ and $B_i\in\mathbb{C}_H^{m_i}$ be given for $i=1,2$ and assume that the pair of matrix equations $A_1XA_1^*=B_1$ and $A_2XA_2^*=B_2$, have a common solution $X\in\mathbb{C}_H^n$. Also, let*

$$T_1=\begin{pmatrix}-B_1 & 0 & A_{12} & -A_{11} & 0\\ 0 & -B_2 & 0 & A_{21} & A_{22}\\ A_{12}^* & A_{22}^* & 0 & 0 & 0\end{pmatrix},\qquad T_2=\begin{pmatrix}-B_1 & 0 & A_{11} & 0 & -A_{12}\\ 0 & -B_2 & 0 & A_{21} & A_{22}\\ A_{11}^* & A_{21}^* & 0 & 0 & 0\end{pmatrix},$$

$$M_1 = \begin{pmatrix} -B_1 & A_{12} & -A_{11} & 0 \\ A_{12}^* & 0 & 0 & 0 \\ 0 & 0 & A_{21} & A_{22} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} -B_2 & A_{22} & -A_{21} & 0 \\ A_{22}^* & 0 & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} -B_1 & A_{11} & 0 & -A_{12} \\ A_{11}^* & 0 & 0 & 0 \\ 0 & 0 & A_{21} & A_{22} \end{pmatrix}, \qquad M_4 = \begin{pmatrix} -B_2 & A_{21} & 0 & -A_{22} \\ A_{21}^* & 0 & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} \end{pmatrix},$$

$$L_1 = \begin{pmatrix} -B_1 & A_{12} \\ A_{12}^* & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -B_2 & A_{22} \\ A_{22}^* & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} -B_1 & A_{11} \\ A_{11}^* & 0 \end{pmatrix}, \quad L_4 = \begin{pmatrix} -B_2 & A_{21} \\ A_{21}^* & 0 \end{pmatrix}.$$

Then, the following hold.

(a)

$$\min_{X_1 \in S_1} r(X_1) = 2r(T_1) - 2r\left(A_{12}^*,\ A_{22}^*\right) + \max\left\{t_1, t_2, t_3, t_4\right\}, \tag{19}$$

$$\max_{X_1 \in S_1} r(X_1) = \min\Bigg\{n_1, 2n_1 + r(T) - r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} - r(A_1) - r(A_2),$$
$$2n_1 + r(L_1) - 2r(A_1), 2n_1 + r(L_2) - 2r(A_2)\Bigg\}, \tag{20}$$

where $\quad t_1 = i_+(L_1) + i_-(L_2) - r(M_1) - r(M_2), \qquad t_3 = r(L_1) - 2r(M_1),$

$\quad t_2 = i_-(L_1) + i_+(L_2) - r(M_1) - r(M_2), \qquad t_4 = r(L_2) - 2r(M_2).$

(b)

$$\min_{X_2 \in S_2} r(X_2) = r\begin{pmatrix} 0 & 0 & 0 & A_{11}^* & A_{21}^* \\ A_{12} & -A_{11} & 0 & B_1 & 0 \\ 0 & A_{21} & A_{22} & 0 & B_2 \end{pmatrix} + r\begin{pmatrix} 0 & A_{11}^* & 0 \\ 0 & 0 & A_{21}^* \\ 0 & -A_{12}^* & A_{22}^* \\ A_{12} & B_1 & 0 \\ A_{22}^* & 0 & B_2 \end{pmatrix}$$
$$- r\begin{pmatrix} 0 & 0 & 0 & A_{11}^* & 0 \\ 0 & 0 & 0 & 0 & A_{21}^* \\ 0 & 0 & 0 & -A_{12}^* & A_{22}^* \\ A_{12} & -A_{11} & 0 & B_1 & 0 \\ 0 & A_{21} & A_{22} & 0 & 0 \end{pmatrix} - r\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} - r\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}, \tag{21}$$

$$\max_{X_2 \in S_2} r(X_2) = \min\Bigg\{n_1, n_2, r\begin{pmatrix} 0 & 0 & 0 & A_{11}^* & A_{21}^* \\ A_{12} & -A_{11} & 0 & B_1 & 0 \\ 0 & A_{21} & A_{22} & 0 & B_2 \end{pmatrix} + n - r(A_1) - r(A_2) - r(A),$$
$$r\begin{pmatrix} 0 & A_{11}^* & 0 \\ 0 & 0 & A_{21}^* \\ 0 & -A_{12}^* & A_{22}^* \\ A_{12} & B_1 & 0 \\ A_{22} & 0 & B_2 \end{pmatrix} + n - r(A_1) - r(A_2) - r(A)\Bigg\}. \tag{22}$$

(c)

$$\min_{X_3 \in S_3} r(X_3) = 2r(T_2) - 2r\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} + \max\left\{s_1, s_2, s_3, s_4\right\},$$

$$\max_{X_3 \in S_3} r(X_3) = \min\Bigg\{n_2, 2n_2 + r(T_2) - r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} - r(A_1) - r(A_2),$$

$$2n_2+r(L_3)-2r(A_1), 2n_2+r(L_4)-2r(A_2)\Big\},$$

where $s_1 = i_+(L_3)+i_-(L_4)-r(M_3)-r(M_4)$, $s_3 = r(L_3)-2r(M_3)$,
$s_2 = i_-(L_3)+i_+(L_4)-r(M_3)-r(M_4)$, $s_4 = r(L_4)-2r(M_4)$.

Proof. Let

$$P_1 = \begin{pmatrix} 0 & R_1 & 0 & 0 \\ R_1^* & 0 & A_1^* & A_2^* \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & R_1 \\ 0 & -B_1 & A_1 \\ R_1^* & A_1^* & 0 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 0 & 0 & R_1 \\ 0 & -B_2 & A_2 \\ R_1^* & A_2^* & 0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & R_1 & R_1 \\ 0 & -B_1 & 0 & A_1 & 0 \\ 0 & 0 & -B_2 & 0 & A_2 \\ R_1^* & A_1^* & A_2^* & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & R_1 & R_1 \\ 0 & -B_1 & A_1 & 0 \\ R_1^* & A_1^* & 0 & 0 \\ 0 & 0 & 0 & A_2 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & R_1 & R_1 \\ 0 & -B_2 & A_2 & 0 \\ R_1^* & A_2^* & 0 & 0 \\ 0 & 0 & 0 & A_1 \end{pmatrix}.$$

Applying Lemma 3.1, we obtain

$$\begin{aligned} \min_{X_1 \in S_1} r(X_1) &= \min_{X \in S} r(R_1 X R_1^*) \\ &= 2r\begin{pmatrix} 0, & R_1 \end{pmatrix} - 2r(P_1) + 2r(Q_1) + \max\{t_1, t_2, t_3, t_4\}, \end{aligned} \tag{23}$$

where $t_1 = i_+(P_2)+i_-(P_3)-r(Q_2)-r(Q_3)$, $t_3 = r(P_2)-2r(Q_2)$,
$t_2 = i_-(P_2)+i_+(P_3)-r(Q_2)-r(Q_3)$, $t_4 = r(P_3)-2r(Q_3)$.

$$\begin{aligned} &\max_{X_1 \in S_1} r(X_1) = \max_{X \in S} r(R_1 X R_1^*) \\ &= \min\left\{ r\begin{pmatrix} 0, & R_1 \end{pmatrix}, r(Q_1) - r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} - r(A_1) - r(A_2), r(P_2) - 2r(A_1), r(P_3) - 2r(A_2) \right\}. \end{aligned} \tag{24}$$

Rewrite A_1 and A_2 as

$$A_1 = \begin{pmatrix} A_{11}, & A_{12} \end{pmatrix}, \ A_2 = \begin{pmatrix} A_{21}, & A_{22} \end{pmatrix}, \tag{25}$$

where $A_{11} \in \mathbb{C}^{m_1 \times n_1}$, $A_{12} \in \mathbb{C}^{m_1 \times n_2}$, $A_{21} \in \mathbb{C}^{m_2 \times n_1}$, $A_{22} \in \mathbb{C}^{m_2 \times n_2}$.

Simplifying the block matrices in (23) and (24) by elementary matrix operations and elementary congruence matrix operations, we obtain

$$r(P_1) = 2n_1 + r\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}, \qquad r(Q_1) = 2n_1 + r\begin{pmatrix} -B_1 & 0 & A_{12} & -A_{11} & 0 \\ 0 & -B_2 & 0 & A_{21} & A_{22} \\ A_{12}^* & A_{22}^* & 0 & 0 & 0 \end{pmatrix},$$

$$r(Q_2) = 2n_1 + r\begin{pmatrix} -B_1 & A_{12} & -A_{11} & 0 \\ A_{12}^* & 0 & 0 & 0 \\ 0 & 0 & A_{21} & A_{22} \end{pmatrix}, \quad r(Q_3) = 2n_1 + r\begin{pmatrix} -B_2 & A_{22} & -A_{21} & 0 \\ A_{22}^* & 0 & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} \end{pmatrix},$$

$$i_\pm(P_2) = n_1 + i_\pm\begin{pmatrix} -B_1 & A_{12} \\ A_{12}^* & 0 \end{pmatrix}, \qquad i_\pm(P_3) = n_1 + i_\pm\begin{pmatrix} -B_2 & A_{22} \\ A_{22}^* & 0 \end{pmatrix},$$

$$r(P_2) = 2n_1 + r\begin{pmatrix} -B_1 & A_{12} \\ A_{12}^* & 0 \end{pmatrix}, \qquad r(P_3) = 2n_1 + r\begin{pmatrix} -B_2 & A_{22} \\ A_{22}^* & 0 \end{pmatrix}.$$

Substituting the above results into (23) and (24) yields (19) and (20).

Next, we apply Lemma 3.2 to

$$X_2 = R_1XR_2^* = R_1X_0R_2^*+V_1F_AR_2^*+R_1F_AV_2^*+\begin{pmatrix}R_1F_{A_1}, & R_1F_{A_2}\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & U^*\end{pmatrix}\begin{pmatrix}F_{A_2}R_2^*\\ F_{A_1}R_2^*\end{pmatrix},$$

where $V_1 \in \mathbb{C}^{n_1\times n}$ and $V_2 \in \mathbb{C}^{n_2\times n}$, and we get

$$\begin{aligned}
&\min_{X_2\in S_2} r(X_2)\\
&=\min_{U,V_1,V_2} r\left(R_1X_0R_2^*+V_1F_AR_2^*+R_1F_AV_2^*+\begin{pmatrix}R_1F_{A_1}, & R_1F_{A_2}\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & U^*\end{pmatrix}\begin{pmatrix}F_{A_2}R_2^*\\ F_{A_1}R_2^*\end{pmatrix}\right)\\
&=r\begin{pmatrix}R_1X_0R_1^* & R_1F_A & R_1F_{A_1} & R_1F_{A_2}\\ F_AR_2^* & 0 & 0 & 0\end{pmatrix}+r\begin{pmatrix}R_1X_0R_1^* & R_1F_A\\ F_AR_2^* & 0\\ F_{A_2}R_2^* & 0\\ F_{A_1}R_2^* & 0\end{pmatrix}\\
&-r\begin{pmatrix}R_1X_0R_1^* & R_1F_A & R_1F_{A_1} & R_1F_{A_2}\\ F_AR_2^* & 0 & 0 & 0\\ F_{A_2}R_2^* & 0 & 0 & 0\\ F_{A_1}R_2^* & 0 & 0 & 0\end{pmatrix}-r(R_1F_A)-r(F_AR_2^*),
\end{aligned}\tag{26}$$

$$\begin{aligned}
&\max_{X_2\in S_2} r(X_2)\\
&=\max_{U,V_1,V_2} r\left(R_1X_0R_2^*+V_1F_AR_2^*+R_1F_AV_2^*+\begin{pmatrix}R_1F_{A_1}, & R_1F_{A_2}\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & U^*\end{pmatrix}\begin{pmatrix}F_{A_2}R_2^*\\ F_{A_1}R_2^*\end{pmatrix}\right)\\
&=\min\left\{n_1, n_2, r\begin{pmatrix}R_1X_0R_1^* & R_1F_A & R_1F_{A_1} & R_1F_{A_2}\\ F_AR_2^* & 0 & 0 & 0\end{pmatrix}, r\begin{pmatrix}R_1X_0R_1^* & R_1F_A\\ F_AR_2^* & 0\\ F_{A_2}R_2^* & 0\\ F_{A_1}R_2^* & 0\end{pmatrix}\right\}.
\end{aligned}\tag{27}$$

Applying (10) to the block matrices in (26) and (27), and simplifying by using $[A_1A_1^+B_1, A_2A_2^+B_2] = [B_1, B_2]$, elementary matrix operations, and the fact that $\mathfrak{R}(R_1F_A) \subseteq \mathfrak{R}(R_1F_{A_1})$ and $\mathfrak{R}(R_1F_A) \subseteq \mathfrak{R}(R_1F_{A_2})$, we obtain

$$\begin{aligned}
&r\begin{pmatrix}R_1X_0R_1^* & R_1F_A & R_1F_{A_1} & R_1F_{A_2}\\ F_AR_2^* & 0 & 0 & 0\end{pmatrix} = r\begin{pmatrix}R_1X_0R_1^* & R_1F_{A_1} & R_1F_{A_2}\\ F_AR_2^* & 0 & 0\end{pmatrix}\\
&=r\begin{pmatrix}0 & 0 & 0 & A_{11}^* & A_{21}^*\\ A_{12} & -A_{11} & 0 & A_{11}R_1X_0A_1^* & 0\\ 0 & A_{21} & A_{22} & 0 & A_{21}R_1X_0A_2^*\end{pmatrix}+n-r(A_1)-r(A_2)-r(A)\\
&=r\begin{pmatrix}\begin{pmatrix}0, & 0\end{pmatrix} & 0 & \begin{pmatrix}0, & 0\end{pmatrix} & A_{11}^* & A_{21}^*\\ \begin{pmatrix}0, & A_{12}\end{pmatrix} & -A_{11} & \begin{pmatrix}0, & 0\end{pmatrix} & \begin{pmatrix}A_{11}, & 0\end{pmatrix}X_0A_1^* & 0\\ \begin{pmatrix}0, & 0\end{pmatrix} & A_{21} & \begin{pmatrix}0, & A_{22}\end{pmatrix} & 0 & \begin{pmatrix}A_{21}, & 0\end{pmatrix}X_0A_2^*\end{pmatrix}+n-r(A_1)-r(A_2)-r(A)
\end{aligned}$$

$$=r\begin{pmatrix} 0 & 0 & 0 & A_{11}^* & A_{21}^* \\ A_{12} & -A_{11} & 0 & B_1 & 0 \\ 0 & A_{21} & A_{22} & 0 & B_2 \end{pmatrix} +n-r(A_1)-r(A_2)-r(A),$$

$$r\begin{pmatrix} R_1X_0R_2^* & R_1F_A \\ F_AR_2^* & 0 \\ F_{A_2}R_2^* & 0 \\ F_{A_1}R_2^* & 0 \end{pmatrix} = r\begin{pmatrix} R_1X_0R_2^* & R_1F_A \\ F_{A_1}R_2^* & 0 \\ F_{A_2}R_2^* & 0 \end{pmatrix}$$

$$=r\begin{pmatrix} 0 & A_{11}^* & 0 \\ 0 & 0 & A_{21}^* \\ 0 & -A_{12}^* & A_{22}^* \\ A_{12} & 0 & A_1X_0R_2^*A_{22}^* \\ A_{22} & 0 & A_2X_0R_2^*A_{22}^* \end{pmatrix} +n_1+n_2-r(A)-r(A_1)-r(A_2)$$

$$=r\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} A_{11}^* \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & 0 & A_{21}^* \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -A_{12}^* \end{pmatrix} & \begin{pmatrix} 0 \\ A_{22}^* \end{pmatrix} \\ A_{12} & A_1X_0\begin{pmatrix} 0 \\ A_{12}^* \end{pmatrix} & 0 \\ A_{22} & 0 & A_2X_0\begin{pmatrix} A_{21}^* \\ 0 \end{pmatrix} \end{pmatrix} +n-r(A)-r(A_1)-r(A_2)$$

$$=r\begin{pmatrix} 0 & A_{11}^* & 0 \\ 0 & 0 & A_{21}^* \\ 0 & -A_{12}^* & A_{22}^* \\ A_{12} & B_1 & 0 \\ A_{22} & 0 & B_2 \end{pmatrix} +n-r(A)-r(A_1)-r(A_2),$$

$$r\begin{pmatrix} R_1X_0R_1^* & R_1F_A & R_1F_{A_1} & R_1F_{A_2} \\ F_AR_2^* & 0 & 0 & 0 \\ F_{A_2}R_2^* & 0 & 0 & 0 \\ F_{A_1}R_2^* & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} R_1X_0R_1^* & R_1F_{A_1} & R_1F_{A_2} \\ F_{A_1}R_2^* & 0 & 0 \\ F_{A_2}R_2^* & 0 & 0 \end{pmatrix}$$

$$=r\begin{pmatrix} 0 & 0 & 0 & A_{11}^* & 0 \\ 0 & 0 & 0 & 0 & A_{21}^* \\ 0 & 0 & 0 & -A_{12}^* & A_{22}^* \\ A_{12} & -A_{11} & 0 & A_{11}R_1X_0A_1^* & 0 \\ 0 & A_{21} & A_{22} & 0 & 0 \end{pmatrix} +n_1+n_2-2r(A_1)-2r(A_2)$$

$$=r\begin{pmatrix} 0 & 0 & 0 & A_{11}^* & 0 \\ 0 & 0 & 0 & 0 & A_{21}^* \\ 0 & 0 & 0 & -A_{12}^* & A_{22}^* \\ A_{12} & -A_{11} & 0 & B_1 & 0 \\ 0 & A_{21} & A_{22} & 0 & 0 \end{pmatrix} +n-2r(A_1)-2r(A_2).$$

By Lemma 1.2, we obtain $r(R_1F_A) = n_1+r\begin{pmatrix}A_{12}\\A_{22}\end{pmatrix}-r(A)$, $r(F_AR_2^*) = n_2+r\begin{pmatrix}A_{11}\\A_{21}\end{pmatrix}-r(A)$. Substituting these results into (26) and (27) yields (21) and (22).

The proof of (c) is similar to (a). □

Corollary 3.4. *Let $A_i \in \mathbb{C}^{m_i\times n}$ and $B_i \in \mathbb{C}_H^{m_i}$ be given for $i = 1, 2$ and assume that the pair of matrix equations $A_1XA_1^* = B_1$ and $_2XA_2^* = B_2$, has a common solution $X \in \mathbb{C}_H^n$. Then*

(a) Equation (2) has a common hermitian solution in the form $X = \begin{pmatrix}0 & X_2\\X_2^* & 0\end{pmatrix}$ *iff*

$\max\{t_1, t_2, t_3, t_4\} = 2r\begin{pmatrix}A_{12}^*, & A_{22}^*\end{pmatrix}-2r(T_1)$, *and* $\max\{s_1, s_2, s_3, s_4\} = 2r\begin{pmatrix}A_{11}\\A_{21}\end{pmatrix}-2r(T_2)$.

(b) All the common hermitian solutions of (2) have the form $X = \begin{pmatrix}0 & X_2\\X_2^* & 0\end{pmatrix}$ *iff*

$$\min\left\{2n_1+r(T_1)-r\begin{pmatrix}A_1\\A_2\end{pmatrix}-r(A_1)-r(A_2), 2n_1+r(L_1)-2r(A_1), 2n_1+r(L_2)-2r(A_2)\right\} = 0, \textit{ and}$$

$$\min\left\{2n_2+r(T_2)-r\begin{pmatrix}A_1\\A_2\end{pmatrix}-r(A_1)-r(A_2), 2n_2+r(L_3)-2r(A_1), 2n_2+r(L_4)-2r(A_2)\right\} = 0.$$

(c) Equation (2) has a common hermitian solution in the form $X = \begin{pmatrix}X_1 & 0\\0 & X_3\end{pmatrix}$ *iff*

$$r\begin{pmatrix}0 & 0 & 0 & A_{11}^* & 0\\0 & 0 & 0 & 0 & A_{21}^*\\0 & 0 & 0 & -A_{12}^* & A_{22}^*\\A_{12} & -A_{11} & 0 & B_1 & 0\\0 & A_{21} & A_{22} & 0 & 0\end{pmatrix} = r\begin{pmatrix}0 & 0 & 0 & A_{11}^* & A_{21}^*\\A_{12} & -A_{11} & 0 & B_1 & 0\\0 & A_{21} & A_{22} & 0 & B_2\end{pmatrix}$$
$$+r\begin{pmatrix}0 & A_{11}^* & 0\\0 & 0 & A_{21}^*\\0 & -A_{12}^* & A_{22}^*\\A_{12} & B_1 & 0\\A_{22}^* & 0 & B_2\end{pmatrix}-r\begin{pmatrix}A_{11}\\A_{21}\end{pmatrix}-r\begin{pmatrix}A_{12}\\A_{22}\end{pmatrix}.$$

(d) All the common hermitian solutions of (2) have the form $X = \begin{pmatrix}X_1 & 0\\0 & X_3\end{pmatrix}$ *iff*

$$\min\left\{r\begin{pmatrix}0 & 0 & 0 & A_{11}^* & A_{21}^*\\A_{12} & -A_{11} & 0 & B_1 & 0\\0 & A_{21} & A_{22} & 0 & B_2\end{pmatrix}+n-r(A_1)-r(A_2)-r(A),\right.$$
$$\left.r\begin{pmatrix}0 & A_{11}^* & 0\\0 & 0 & A_{21}^*\\0 & -A_{12}^* & A_{22}^*\\A_{12} & B_1 & 0\\A_{22} & 0 & B_2\end{pmatrix}+n-r(A_1)-r(A_2)-r(A)\right\} = 0.$$

THEOREM 3.5. *Let $A_i \in \mathbb{C}^{m_i \times n}$ and $B_i \in \mathbb{C}_H^{m_i}$ be given for $i = 1, 2$ and assume that the pair of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$, has a common solution $X \in \mathbb{C}_H^n$. Then*
(a) The submatrix X_1 is unique iff $\dim\left\{\mathfrak{R}(A_1^*) \cap \mathfrak{R}(A_2^*)\right\} - \dim\left\{\mathfrak{R}(A_{12}^*) \cap \mathfrak{R}(A_{22}^*)\right\} = n_1$.
(b) The submatrix X_3 is unique iff $\dim\left\{\mathfrak{R}(A_1^*) \cap \mathfrak{R}(A_2^*)\right\} - \dim\left\{\mathfrak{R}(A_{11}^*) \cap \mathfrak{R}(A_{21}^*)\right\} = n_2$.
(c) The submatrix X_2 is unique iff $\dim\left\{\mathfrak{R}(A_1^*) \cap \mathfrak{R}(A_2^*)\right\} - \dim\left\{\mathfrak{R}(A_{12}^*) \cap \mathfrak{R}(A_{22}^*)\right\} = n_1$, *and* $\dim\left\{\mathfrak{R}(A_1^*) \cap \mathfrak{R}(A_2^*)\right\} - \dim\left\{\mathfrak{R}(A_{11}^*) \cap \mathfrak{R}(A_{21}^*)\right\} = n_2$.

Proof. We only prove (a). Note that

$$X_1 = R_1X_0R_1^* + V_1F_AR_1^* + R_1F_AV_1^* + R_1F_{A_1}UF_{A_2}R_1^* + R_1F_{A_2}UF_{A_1}R_1^*.$$

Then X_1 is unique iff $R_1F_A = R_1F_{A_1} = R_1F_{A_2} = 0$.

$$R_1F_A = R_1F_{A_1} = R_1F_{A_2} = 0 \Leftrightarrow r(R_1F_A) = r(R_1F_{A_1}) = r(R_1F_{A_2}) = 0$$

$$\Leftrightarrow \begin{cases} n_1 + r\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} - r(A) = 0 \\ n_1 + r(A_{12}) - r(A_1) = 0 \\ n_1 + r(A_{22}) - r(A_2) = 0 \end{cases} \Leftrightarrow \begin{cases} r(A) = r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + n_1 \\ r(A_1) = r(A_{12}) + n_1 \\ r(A_2) = r(A_{22}) + n_1 \end{cases}$$

$$\Leftrightarrow \dim\left\{\mathfrak{R}(A_1^*) \cap \mathfrak{R}(A_2^*)\right\} - \dim\left\{\mathfrak{R}(A_{12}^*) \cap \mathfrak{R}(A_{22}^*)\right\} = n_1,$$

where we use the partitioning of A_1 and A_2 as in (25). □

REFERENCES

[1] A. Ben-Israel, T. N. E. Greville, *Generalized Inverses, Theory and Applications*, Springer-Verlag NewYork, Inc, 2003.

[2] J. Grob, *A note on the general Hermitian solution to $AXA^* = B$*, Bull, Malays. Math. Soc., **21(2)** (1998), 57–62.

[3] S. Guerarra, S. Guedjiba, *Common least-rank solution of matrix equations $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$ with applications*, Facta Universitatis (Niš). Ser. Math. Inform., **29** (2014), 313–323.

[4] S. Guerarra, S. Guedjiba, *Common harmitian least-rank solution of matrix equation $A_1X_1A_1^*$ and $A_2X_2A_2^*$*, Facta universitatis (Niš). Ser. Math. Inform., **5** (2015), 539–554.

[5] C. G. Khatri, S. K. Mitra, *Hermitian and nonnegative definite solutions of linear matrix equations*, SIAM J. Appl. Math., **31** (1976), 579–585.

[6] Y. H. Liu, *Ranks of least squares solutions of the matrix equation $AXB = C$*, Comput. Math. Appl., **55** (2008), 1270–1278.

[7] X. Fu Liu, Hu Yang, *An expression of the general common least squares solution to a pair of matrix equations with applications*, Comp. Math. Appl., **61** (2011), 3071–3078.

[8] G. Marsaglia, G. P. H Styan, *Equalities and inequalities for ranks of matrices*, Linear Multilinear A., **2** (1974), 269–292.

[9] S. K. Mitra, *Common solutions to a pair of linear matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$*, Proc. Cambridge Philos. Soc., **74** (1973), 213–216.

[10] S. K. Mitra, *A pair of simultaneous linear matrix equations and a matrix programming problem*, Linear Algebra Appl., **131** (1990), 97–123.

[11] M. Z. Nashed, *Generalized Inverses, Theory and Applications*, Academic Press, NY, 1976.

[12] A. Navarra, P. L. Odell, D. M. Young, *A representation of the general common solution to the matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ with applications*, Comput. Math. Appl., **41** (2001), 929–935.

[13] Y. Tian, *Maximization and minimization of the rank and inertias of the Hermitian matrix expression $A - BX - (BX)^*$ with applications*, Linear Algebra Appl., **434** (2011), 2109–2139.

[14] Y. Tian, *Equalities and inequalities for Hermitian solutions and Hermitian definite solutions of the two matrix equations $AX = B$ and $AXA^* = B$*, Aequationes Math, in press.

[15] Y. Tian, *Equalities and inequalities for inertias of Hermitian matrices with applications*, Linear Algebra Appl., **433** (2010), 263–296.

[16] Y. Tian, *Rank Equalities Related to Generalized Inverses of Matrices and Their Applications*, Master Thesis, Montreal, Quebec, Canada 2000.

[17] Y. Tian, *Some optimization problems on ranks and inertias of matrix-valued functions subject to linear matrix equation restrictions*, Banach J. Math. Anal., **8(1)** (2014), 148–178.

[18] Y. Tian, *Ranks and independence of solutions of the matrix equation $AXB + CYD = M$*, Acta Math. Univ. Comenianae., **1** (2006), 75–84.

[19] F. Zhang, Y. Li, W. Guo, J. Zhao, *Least squares solution with special structure to the linear matrix equation $AXB = C$*, Appl. Math. Comput., **217** (2011), 10 049–10 057.

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