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ON THE SPECTRA OF THE OPERATOR $B(\widetilde{r}, \widetilde{s})$ MAPPING IN $(w_\infty(\lambda))_a$ AND $(w_0(\lambda))_a$ WHERE λ IS A NONDECREASING EXPONENTIALLY BOUNDED SEQUENCE

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Abstract. Given any sequence $a = (a_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_a for the set of all sequences $x = (x_n)_{n\geq 1}$ such that $x/a = (x_n/a_n)_{n\geq 1} \in E$. We denote by $W_a(\lambda) = (w_\infty(\lambda))_a$ and $W_a^0(\lambda) = (w_0(\lambda))_a$ the sets of all sequences x such that $\sup_n \left(\lambda_n^{-1} \sum_{k=1}^n |x_k| / a_k\right) < \infty$ and $\lim_{n\to\infty} \left(\lambda_n^{-1} \sum_{k=1}^n |x_k| / a_k\right) = 0$, where λ is a nondecreasing exponentially bounded sequence. In this paper we recall some properties of the Banach algebras $(W_a(\lambda), W_a(\lambda))$, and $\left(W_a^0(\lambda), W_a^0(\lambda)\right)$, where a is a positive sequence. We then consider the operator Δ_ρ, defined by $[\Delta_\rho x]_n = x_n - \rho_{n-1}x_{n-1}$ for all $n \geq 1$ with the convention $x_0, \rho_0 = 0$, and we give necessary and sufficient conditions for the operator $\Delta_\rho : E \to E$ to be bijective, for $E = w_0(\lambda)$, or $w_\infty(\lambda)$. Then we consider the generalized operator of the first difference $B(\widetilde{r}, \widetilde{s})$, where $\widetilde{r}, \widetilde{s}$ are two convergent sequences, and defined by $[B(\widetilde{r}, \widetilde{s})x]_n = r_n x_n + s_{n-1}x_{n-1}$ for all $n \geq 1$ with the convention $x_0, s_0 = 0$. Then we deal with the operator $B(\widetilde{r}, \widetilde{s})$ mapping in either of the sets $W_a(\lambda)$, or $W_a^0(\lambda)$. We then apply the previous results to explicitly calculate the spectrum of $B(\widetilde{r}, \widetilde{s})$ over each of the spaces E_a, where $E = w_0(\lambda)$, or $w_\infty(\lambda)$. Finally we give a characterization of the identity $(W_a(\lambda))_{B(r,s)} = W_b(\lambda)$.

1. Preliminary results

Let $A = (\boldsymbol{a}_{nk})_{n,k\geq 1}$ be an infinite complex matrix and consider the complex sequence $x = (x_n)_{n\geq 1}$. We write $Ax = (A_n(x))_{n\geq 1}$ with $A_n(x) = \sum_{k=1}^\infty \boldsymbol{a}_{nk}x_k$ whenever the series are convergent for all $n \geq 1$. Throughout this paper we use the convention that any term with a subscript less than 1 is equal to naught. Let ω denote the set of all complex sequences. We write φ, c_0, c and ℓ_∞ for the sets of all finite, null, convergent and bounded sequences respectively. For any given subsets E and F of ω, we say that the operator represented by the infinite matrix $A = (\boldsymbol{a}_{nk})_{n,k\geq 1}$ *maps* E *into* F and

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denote this by $A \in (E, F)$, see [8], if the series $A_n(x) = \sum_{k=1}^{\infty} \boldsymbol{a}_{nk} x_k$ are convergent for all $n \geq 1$ and for all $x \in E$, and $Ax \in F$ for all $x \in E$. If F is a subset of ω, we denote the so-called matrix domain of A in F by $F_A = \{x \in \omega\ : y = Ax \in F\}$. For any nonzero sequence a we write E_a for the set of all sequences $x = (x_n)_{n \geq 1}$ such that $x/a = (x_n/a_n)_{n\geq 1} \in E$. Let $E \subset \omega$ be a *Banach space*, with the norm $\|\cdot\|_E$. By $B(E)$ we denote the set of all bounded linear operators, mapping E into itself, with the operator norm $\|L\|^*_{\mathcal{B}(E)} = \sup_{x \neq 0} (\|Lx\|_E / \|x\|_E)$ for all $L \in B(E)$. It is well known that $B(E)$ is a Banach algebra with the operator norm $\|\cdot\|^*_{\mathcal{B}(E)}$.

A Banach space $E \subset \omega$ is a BK space if the projections $P_n : x \mapsto x_n$ from E into $\mathbb{C}$ are continuous for all n. We denote by $e^{(k)}$ the sequence defined by $e^{(k)} = (0, \ldots, 0, 1, 0, \ldots)$, where 1 is in the k-th position and we write $e = (1, 1, \ldots)$. A BK space $E \supset \varphi$ is said to have AK if $x = \lim_{p\to\infty} \sum_{k=1}^{p} x_k e^{(k)}$ for every sequence $x = (x_n)_{n\geq 1} \in E$. It is well known that if E has AK then $\mathcal{B}(E) = (E, E)$. If E is a BK space with the norm $\|\cdot\|_E$, then $(E, E) \subset \mathcal{B}(E)$. Indeed by [19, Theorem 4.2.8 p. 57], since E is a BK space, the matrix map $A \in (E, E)$ is continuous and there is $M > 0$ such that $\|Ax\|_E \leq M \|x\|_E$ for all $x \in E$. By U and U^+ we denote the sets of all nonzero sequences and all positive sequences, respectively. For $a \in U^+$ we write $s_a = (\ell_\infty)_a$, $s^0_a = (c_0)_a$, and $s^{(c)}_a = c_a$. Each of the sets s_a, s^0_a, and $s^{(c)}_a$ is a BK space with the norm $\|x\|_{s_a} = \sup_n (|x_n| / a_n)$. Recall that for a, $b \in U^+$, we have $s_a = s_b$ if and only if there are k_1 and $k_2 > 0$ such that $k_1 \leq a_n/b_n \leq k_2$ for all n. We will use the next argument. Since $s_a \supset s_b$ implies $k_1 \leq a_n/b_n$ for all n, we deduce that if $a/b \in c$ and $s_a \supset s_b$, then $\lim_{n\to\infty} (a_n/b_n) > 0$.

This paper is organized as follows. In Section 2 we consider the operator Δ_ρ, defined by $[\Delta_\rho x]_n = x_n - \rho_{n-1} x_{n-1}$ for all $n \geq 1$, and characterize the map $\Delta_\rho : E \to E$, for $E = w_\infty(\lambda)$, or $w_0(\lambda)$. In Section 3 we apply these results to deal with the operator represented by a double band matrix $B(\widetilde{r}, \widetilde{s})$ on E_a, where E is either of the spaces $w_\infty(\lambda)$, or $w_0(\lambda)$. In Section 4 we explicitly calculate the spectrum of $B(\widetilde{r}, \widetilde{s})$ over the spaces E_a, where E is either of the spaces $w_\infty(\lambda)$, or $w_0(\lambda)$. Finally we characterize the identity $(W_a(\lambda))_{B(r,s)} = W_b(\lambda)$.

2. On the band matrix Δ_ρ considered as an operator in each of the spaces $w_\infty(\lambda)$, or $w_0(\lambda)$

In this section we give necessary and sufficient conditions on the sequence $\rho = (\rho_n)_{n\geq 1}$ for Δ_ρ to be bijective from E to itself, where E is either of the spaces $w_\infty(\lambda)$, or $w_0(\lambda)$.

For any given sequence $\rho = (\rho_n)_{n\geq 1} \in \omega$ we consider the operator Δ_ρ defined by $[\Delta_\rho x]_n = x_n - \rho_{n-1} x_{n-1}$ for all $n \geq 1$. This operator is represented by the infinite matrix

$$\Delta_\rho = \begin{pmatrix} 1 & & & & \\ -\rho_1 & 1 & & & 0 \\ & \cdot & \cdot & & \\ & & -\rho_{n-1} & 1 & \\ 0 & & & \cdot & \cdot \\ & & & & \cdot \quad \cdot \end{pmatrix}.$$

Recall that a matrix $T = (t_{nk})_{nk \geq 1}$ is a *triangle* if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ $(n = 1, 2, \ldots)$.

2.1 The Banach algebras $(w_\infty(\lambda), w_\infty(\lambda))$ and $(w_0(\lambda), w_0(\lambda))$

Let $\lambda = (\lambda_n)_{n \geq 1} \in U^+$. We define by $C(\lambda)$ the triangle whose the nonzero entries are defined by $[C(\lambda)]_{nk} = 1/\lambda_n$ for $k \leq n$ and for all n. It is well known that its inverse is the triangular band matrix $\Delta(\lambda)$ whose nonzero entries are given by $[\Delta(\lambda)]_{n,n-1} = -\lambda_{n-1}$ and $[\Delta(\lambda)]_{nn} = \lambda_n$, (see for instance [15]). For $\lambda = e$ we write $C(\lambda) = \Sigma$ and $\Delta(\lambda) = \Delta$, and Δ is called the *operator of the first difference.* For $\lambda = (\lambda_n)_{n \geq 1} \in U^+$ we consider the sets of *strongly bounded and summable sequences*, respectively, that is,

$$w_\infty(\lambda) = \left\{ x \in \omega : \|x\|_{w_\infty(\lambda)} = \sup_n \left(\frac{1}{\lambda_n} \sum_{k=1}^n |x_k| \right) < \infty \right\},$$

and
$$w_0(\lambda) = \left\{ x \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k=1}^n |x_k| = 0 \right\}.$$

Notice that $x \in w_\infty(\lambda)$ means $C(\lambda)|x| \in \ell_\infty$, where $|x| = (|x_k|)_{k \geq 1}$, and $x \in w_0(\lambda)$ means $C(\lambda)|x| \in c_0$. These sets were studied by Malkowsky, with the concept of *exponentially bounded sequences*, see [16]. Recall that Maddox [7], defined and studied the sets $w_\infty(\lambda) = w_\infty$ and $w_0(\lambda) = w_0$ where $\lambda_n = n$ for all n.

Recall that a non-decreasing sequence $\lambda = (\lambda_n)_{n \geq 1} \in U^+$ of positive reals tending to infinity is called *exponentially bounded* if there is an integer $m \geq 2$ such that for all non-negative integers ν there is at least one term $\lambda_n \in I_m^{(\nu)} = \left[m^\nu, m^{\nu+1} - 1\right]$. It was shown (cf. [16, Lemma 1]) that a non-decreasing sequence $\lambda = (\lambda_n)_{n \geq 1}$ is *exponentially bounded* if and only if there are reals $s \leq t$ such that for some subsequence $(\lambda_{n_i})_{i \geq 1}$ we have

$$0 < s \leq \frac{\lambda_{n_i}}{\lambda_{n_{i+1}}} \leq t < 1 \text{ for all } i = 1, 2, \ldots;$$

such a sequence is called an *associated subsequence.* It can easily be shown that any sequence $\lambda = \left(n^\xi\right)_{n \geq 1}$ with $\xi > 0$ is exponentially bounded. Indeed, for $n_i = 2^i$ we have $\lim_{n \to \infty} \left(2^i/2^{i+1}\right)^\xi = 1/2^\xi \in]0, 1[$. In [17] it was shown that if $\lambda = (\lambda_n)_{n \geq 1} \in U^+$ is *exponentially bounded* then the class $(w_\infty(\lambda), w_\infty(\lambda))$ is a *Banach algebra* with the norm $\|A\|_{(w_\infty(\lambda), w_\infty(\lambda))} = \sup_{x \neq 0} \left(\|Ax\|_{w_\infty(\lambda)} / \|x\|_{w_\infty(\lambda)} \right)$. When λ is an exponentially bounded sequence we obtain similar results on the Banach algebra

$(w_0(\lambda), w_0(\lambda))$ with the norm $\|A\|_{(w_\infty(\lambda), w_\infty(\lambda))}$. In the following we will write $a_n^\bullet = a_{n-1}/a_n$ for all n, with $a_0 = 1$, and $a^\bullet = (a_n^\bullet)_{n\geq 1}$ for $a \in U^+$. We also use the sets $\Gamma = \left\{a \in U^+ : \overline{\lim}_{n\to\infty} a_n^\bullet < 1\right\}$ and $\widehat{\Gamma} = \{a \in U^+ : \lim_{n\to\infty} a_n^\bullet < 1\}$. For any given sequence $a = (a_n)_{n\geq 1} \in U^+$, we write D_a for the diagonal matrix defined by $[D_a]_{nn} = a_n$ for all n. In the following we also use the notation $D_a E = E_a$. Here we consider the set $W_a(\lambda) = D_a w_\infty(\lambda)$, for a, $\lambda \in U^+$. We have

$$W_a(\lambda) = \left\{x \in \omega : \|x\|_{W_a(\lambda)} = \sup_n \left(\frac{1}{\lambda_n}\sum_{k=1}^n \frac{|x_k|}{a_k}\right) < \infty\right\}.$$

It can easily be seen that $W_a(\lambda) = (w_\infty(\lambda))_a$ is a BK space with the norm the $\|\cdot\|_{W_a(\lambda)}$. Similarly we define $W_a^0(\lambda) = (w_0(\lambda))_a$. We write $W_{(r^n)_{n\geq 1}}(\lambda) = W_r(\lambda)$ for any $r > 0$. When $\lambda_n = n$ we put $W_a = W_a(\lambda)$ and $W_a^0 = W_a^0(\lambda)$ for $a \in U^+$, see [15]. We then have $W_a = \left\{x : \|x\|_{W_a} = \sup_n \left(n^{-1}\sum_{k=1}^n |x_k|/a_k\right) < \infty\right\}$.

Now we recall a result, where we have $\Delta_\rho \in (w_\infty(\lambda), w_\infty(\lambda))$, if ρ, $\lambda^\bullet \in \ell_\infty$.

Lemma 2.1 ([14, Theorem 3.12, p. 210]). *Let $\lambda \in U^+$ be a non-decreasing exponentially bounded squence, and let E be either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$. Assume $\overline{\lim}_{n\to\infty}|\rho_n| < 1/\overline{\lim}_{n\to\infty}\lambda_n^\bullet$. Then for any given $b \in E$ the equation $\Delta_\rho x = b$ has a unique solution in E, which is determined by $x_1 = b_1$ and $x_n = b_n + \sum_{k=1}^{n-1}\left(\prod_{j=k}^{n-1}\rho_j\right)b_k$ for all $n \geq 2$.*

As an immediate consequence of the preceding result we obtain the next lemma.

Lemma 2.2. *Assume that $\lambda \in U^+$ is a non-decreasing exponentially bounded sequence, and assume ρ, $\lambda^\bullet \in c$. If $\lim_{n\to\infty}(|\rho_n|\,\lambda_n^\bullet) < 1$, then Δ_ρ is bijective from E to itself, where E is either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$.*

2.2 Necessary conditions for Δ_ρ to be bijective from E to E, where $E = w_\infty(\lambda)$, or $w_0(\lambda)$

We need the next lemmas.

Lemma 2.3 ([12, Lemma 4]). *Let $u \in U$, and assume $(u_n/u_{n-1})_{n\geq 2} \in c$. Then $u \in \ell_\infty$ implies $\lim_{n\to\infty}|u_n/u_{n-1}| \leq 1$.*

Lemma 2.4. *Let $\lambda \in U^+$. Then we have:*
(i) Assume $1/\lambda \in \ell_\infty$. Then $A \in (w_\infty(\lambda), w_\infty(\lambda))$ implies $Ae^{(1)} \in s_\lambda$.
(ii) Assume $1/\lambda \in c_0$. Then $A \in (w_0(\lambda), w_0(\lambda))$ implies $Ae^{(1)} \in s_\lambda^0$.

Proof. (i) The proof comes from the fact that $w_\infty(\lambda) \subset s_\lambda$. Indeed $x \in w_\infty(\lambda)$ implies that there is $K > 0$ such that $\lambda_n^{-1}|x_n| \leq \lambda_n^{-1}\sum_{k=1}^n |x_k| \leq K$ for all n, and $x \in s_\lambda$. Now we have $e^{(1)} \in w_\infty(\lambda)$, since $1/\lambda \in \ell_\infty$. We conclude $A \in (w_\infty(\lambda), w_\infty(\lambda))$ implies $Ae^{(1)} \in w_\infty(\lambda)$ and $Ae^{(1)} \in s_\lambda$. (ii) It can easily be shown that $w_0(\lambda) \subset s_\lambda^0$. Now we have $e^{(1)} \in w_0(\lambda)$, since $1/\lambda \in c_0$. We conclude $A \in (w_0(\lambda), w_0(\lambda))$ implies $Ae^{(1)} \in w_0(\lambda)$ and $Ae^{(1)} \in s_\lambda^0$. □

Lemma 2.5. *Let $\lambda \in U^{+}$ and assume $(\rho_{n-1}\lambda_n^{\bullet})_{n\geq 1} \in c$.*
(i) Let $1/\lambda \in \ell_\infty$. If Δ_ρ is bijective from $w_\infty(\lambda)$ to itself, then

$$\lim_{n\to\infty}\left(\left|\rho_{n-1}\right|\lambda_n^{\bullet}\right) \leq 1. \tag{1}$$

(ii) Let $1/\lambda \in c_0$. If Δ_ρ is bijective from $w_0(\lambda)$ to itself, then (1) *holds.*

Proof. (i) Since Δ_ρ is bijective from $w_\infty(\lambda)$ to itself we have

$$\Delta_\rho^{-1} \in \left(w_\infty(\lambda), w_\infty(\lambda)\right). \tag{2}$$

Now we have $\left[\Delta_\rho^{-1}\right]_{nk} = \rho_k \dots \rho_{n-1}$ for $k \leq n-1$ and $\left[\Delta_\rho^{-1}\right]_{nn} = 1$ for all n, and by Lemma 2.4 the condition in (2) implies $\Delta_\rho^{-1}e^{(1)} \in s_\lambda$ and there is $K > 0$ such that

$$u_n = \frac{\left|\rho_1 \dots \rho_{n-1}\right|}{\lambda_n} \leq K \text{ for all } n \geq 2. \tag{3}$$

Since $(\rho_{n-1}\lambda_n^{\bullet})_{n\geq 1} \in c$, we conclude by Lemma 2.3 that $\lim_{n\to\infty}(u_n/u_{n-1}) = \lim_{n\to\infty}\left|\rho_{n-1}\right|\lambda_n^{\bullet} \leq 1$, and (1) holds.

(ii) Since Δ_ρ is bijective from $w_0(\lambda)$ to itself we have $\Delta_\rho^{-1} \in (w_0(\lambda), w_0(\lambda))$, and by Lemma 2.4 we have $\Delta_\rho^{-1}e^{(1)} \in s_\lambda^0$. Then there is $K > 0$ such that (3) holds and we conclude using similar arguments to those in (i). □

3. Some properties of the band matrix $B(\widetilde{r},\widetilde{s})$ considered as an operator on E_a, where E is either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$

In all that follows we assume that $\lambda \in U^{+}$ is a non-decreasing exponentially bounded sequence that satisfies $\lambda^{\bullet} \in c$. Notice that any sequence of the form $\left(n^{\xi}\right)_{n\geq 1}$ with $\xi > 0$ satisfies the previous hypotheses. Now we consider the sequence defined by $\lambda_1 = \lambda_2 = 1$, $\lambda_{3k} = 2k$, and $\lambda_{3k+1} = \lambda_{3k+2} = 2k+1$, for all $k \geq 1$. This sequence is exponentially bounded since for $n_i = 3.2^i$, we have

$$\frac{\lambda_{n_i}}{\lambda_{n_{i+1}}} = \frac{2.2^i}{2.2^{i+1}} \to \frac{1}{2} \ (n \to \infty).$$

It can easily be seen that $\lambda_n^{\bullet} \to 1$ $(n \to \infty)$. So this sequence also satisfies the previous conditions.

Now let $\widetilde{r} = (r_n)_{n\geq 1}$ and $\widetilde{s} = (s_n)_{n\geq 1}$ be two complex sequences, and denote by $B(\widetilde{r},\widetilde{s})$ the triangle defined by

$$B(\widetilde{r},\widetilde{s}) = \begin{pmatrix} r_1 & & & & \\ s_1 & r_2 & & & 0 \\ & \cdot & \cdot & & \\ & & s_{n-1} & r_n & \\ 0 & & & \cdot & \cdot \\ & & & & \cdot \quad \cdot \end{pmatrix}.$$

We consider $B(\widetilde{r},\widetilde{s})$ as an operator mapping E to itself, where E is either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$. We assume $\widetilde{r} \in U$, $\widetilde{r}$, $\widetilde{s} \in c$, $\lim_{n\to\infty} r_n = r \neq 0$ and

$\lim_{n\to\infty} s_n = s$. We use the following notations. First we have $B(\widetilde{r},\widetilde{s}) = D_{\widetilde{r}}\Delta_\rho$, with $\rho_{n-1} = -s_{n-1}/r_n$, for $n \geq 2$, and $\lim_{n\to\infty} |\rho_n| = |s/r|$. Now consider the next inequalities

$$|r| > |s| \lim_{n\to\infty} \lambda_n^\bullet, \tag{4}$$

$$|r| \geq |s| \lim_{n\to\infty} \lambda_n^\bullet. \tag{5}$$

3.1 A sufficient condition for $B(\widetilde{r},\widetilde{s})$ to be bijective from E to itself, where E is either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$

LEMMA 3.1. *If the condition in* (4) *holds, then the operator $B(\widetilde{r},\widetilde{s})$ is bijective from E to itself, where E is either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$.*

Proof. We have $B(\widetilde{r},\widetilde{s}) \in (E,E)$ in each of the cases $E = w_\infty(\lambda)$, or $w_0(\lambda)$. We prove the lemma for $E = w_\infty(\lambda)$. The proof for $w_0(\lambda)$ is similar. Let $x \in w_\infty(\lambda)$. Since $\widetilde{r}$, $\widetilde{s} \in c \subset \ell_\infty$, and λ is non-decreasing, there is $C > 0$ such that $\sup_k(|r_k|, |s_k|) = C$, and

$$\frac{1}{\lambda_n}\sum_{k=1}^{n} |[B(\widetilde{r},\widetilde{s})x]_k| \leq C\left(\frac{1}{\lambda_n}\sum_{k=1}^{n}|x_k| + \frac{1}{\lambda_{n-1}}\lambda_n^\bullet \sum_{k=1}^{n-1}|x_k|\right) \leq C' \|x\|_{w_\infty(\lambda)}$$

for some $C' > 0$, and so $B(\widetilde{r},\widetilde{s})x \in w_\infty(\lambda)$. Hence $B(\widetilde{r},\widetilde{s}) \in (w_\infty(\lambda), w_\infty(\lambda))$.

Now we show $[B(\widetilde{r},\widetilde{s})]^{-1} \in (w_\infty(\lambda), w_\infty(\lambda))$. By the condition in (4) we have $\lim_{n\to\infty}|\rho_n| \lim_{n\to\infty}\lambda_n^\bullet = \left|\frac{s}{r}\right| \lim_{n\to\infty}\lambda_n^\bullet < 1$, and by Lemma 2.2 the operator represented by Δ_ρ is bijective from $w_\infty(\lambda)$ to itself. Since $r_n \neq 0$ for all n, and $\lim_{n\to\infty} r_n = r \neq 0$, there are k_1, $k_2 > 0$ such that $k_1 \leq |r_n| \leq k_2$ for all n, and it can easily be shown the operator $D_{\widetilde{r}}$ is bijective from E to itself. Finally, $D_{\widetilde{r}}\Delta_\rho$ is bijective from E to itself. □

REMARK 3.2. Let E be either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$. If there is an integer k for which $r_k = 0$, then $B(\widetilde{r},\widetilde{s}) \in (E,E)$ is not bijective. Indeed let k_0 be the smallest integer for which $r_{k_0} = 0$, and consider the equation

$$B(\widetilde{r},\widetilde{s})x = e^{(k_0)}. \tag{6}$$

It can easily be seen that if x satisfies the previous equation, then $x_k = 0$ for $k = 1, 2, \ldots, k_0 - 1$, and $s_{k_0-1}x_{k_0-1} = 1$, which is a contradiction, and equation (6) has no solution in E. Since $e^{(k_0)} \in E$, we conclude that $B(\widetilde{r},\widetilde{s})$ is not surjective.

3.2 A necessary condition for $B(\widetilde{r},\widetilde{s})$ to be bijective

From the previous results we deduce the following.

LEMMA 3.3. *If the operator represented by $B(\widetilde{r},\widetilde{s})$ is bijective from E to itself, where E is either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$, then the condition in* (5) *holds.*

Proof. First we consider the case $E = w_\infty(\lambda)$. Since $B(\widetilde{r},\widetilde{s})$ is bijective from $w_\infty(\lambda)$ to itself, we have $(B(\widetilde{r},\widetilde{s}))^{-1} = \Delta_\rho^{-1}D_{1/\widetilde{r}} \in (w_\infty(\lambda), w_\infty(\lambda))$. As we have seen above $D_{\widetilde{r}}$ is bijective from $w_\infty(\lambda)$ to $w_\infty(\lambda)$, and $\Delta_\rho^{-1} = (B(\widetilde{r},\widetilde{s}))^{-1} D_{\widetilde{r}} \in$

$(w_\infty(\lambda), w_\infty(\lambda))$, is also bijective from $w_\infty(\lambda)$ to $w_\infty(\lambda)$. Since $\lambda \in U^+$ is a non-decreasing sequence we have $1/\lambda \in \ell_\infty$. Then by Lemma 2.5 the condition $(\rho_{n-1}\lambda_n^\bullet)_{n\geq 1} \in c$ implies

$$\lim_{n\to\infty}(|\rho_{n-1}|\,\lambda_n^\bullet) = \left|\frac{s}{r}\right| \lim_{n\to\infty}\lambda_n^\bullet \leq 1, \tag{7}$$

and the condition in (5) holds.

Now we consider the case $E = w_0(\lambda)$. By similar arguments as those above, the operator $B(\widetilde{r}, \widetilde{s})$ is bijective and $\Delta_\rho^{-1} = (B(\widetilde{r}, \widetilde{s}))^{-1} D_{\widetilde{r}} \in (w_0(\lambda), w_0(\lambda))$ is also bijective from $w_0(\lambda)$ to $w_0(\lambda)$. Since $\lambda \in U^+$ tends to infinity we have $1/\lambda \in c_0 \subset \ell_\infty$, and $(\rho_{n-1}\lambda_n^\bullet)_{n\geq 1} \in c$, by Lemma 2.5 we conclude that the condition in (7) holds. □

4. Applications

In this section we apply the results obtained in the previous sections to explicitly calculate the spectrum of the operator $B(\widetilde{r}, \widetilde{s})$ on E_a, where E is either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$.

4.1 An application to the spectrum of the operator $B(\widetilde{r}, \widetilde{s})$ on E_a, where E is either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$

In this section we focus our study on the spectrum of the operator $B(\widetilde{r}, \widetilde{s})$ on E_a, where E is either of the sets $w_\infty(\lambda)$, or $w_0(\lambda)$. Let E be a BK space and A be an operator mapping E to itself, (note that A is continuous since E is a BK space). We denote by $\sigma(A, E)$ the set of all complex numbers α such that $A - \alpha I$ considered as an operator from E to itself is not invertible. Then we write $\boldsymbol{\rho}(A, E) = [\sigma(A, E)]^c$ for the resolvent set, which is the set of all complex numbers α such that $\alpha I - A$ considered as an operator from E to itself is invertible. Recall that the resolvent set of a linear operator on E is an open subset of the complex plane $\mathbb{C}$. We use the notation $\overline{D}(\alpha_0, r) = \{\alpha \in \mathbb{C} : |\alpha - \alpha_0| \leq r\}$ for $\alpha_0 \in \mathbb{C}$ and $r > 0$.

Recall that the spectrum and the fine spectrum of the linear operators defined by infinite matrices over certain sequence spaces have been studied by many authors. We only give a short survey of those studies. In [6, 13] are given some results on the spectral theory of unbounded operators, that are used in the theory of the sum of operators (cf. [4]). Recently the fine spectra of the operator of the first difference over the sequence spaces ℓ_p and bv_p, were studied in [1], where bv_p is the space of all sequences of p-bounded variation, with $1 \leq p < \infty$. In [3] there is a study on the fine spectrum of the generalized difference operator $B(r, s)$ on each of the sets ℓ_p and bv_p. The fine spectrum of the operator represented by the triple band matrix $B(r, s, t)$ over the spaces ℓ_p and bv_p, $(1 < p < \infty)$ was studied in [5]. In [18] Srivastava and Kumar dealt with the fine spectrum of the generalized difference operator Δ_v over ℓ_1, where Δ_v is the triangle whose the nonzero entries are defined by $(\Delta_v)_{nn} = v_n$ and $(\Delta_v)_{n+1,n} = -v_n$. Recently Akhmedov and El-Sabrawy [2] determined the spectrum of the generalized difference operator $\Delta_{a,b}$ defined as a double band matrix mapping in c. In [12] there is a study on the spectrum of Δ on the space W_a for $a^\bullet \in \ell_\infty$ and an application to matrix transformations mapping in $(W_a)_{(\Delta-\alpha I)^h}$ for $h \in \mathbb{C}$. In [11] there is a study of the spectrum of the operator $B(\widetilde{r},\widetilde{s})$ on the sets E_a, where E is any of the symbols s, s^0, $s^{(c)}$, ℓ_p, W^0, or W, for $1 \leq p < \infty$ and $a^\bullet \in c$. We also obtain the spectrum of $B(r,s)$ over the space $(bv_p)_a = \left((\ell_p)_\Delta\right)_a$ of all sequences of a, *p-bounded variation of order* 1 (cf. [10]).

We state the main result where we write $\mathcal{R} = \{r_k : k \geq 1\}$. We still assume $\lambda \in U^+$ is a non-decreasing exponentially bounded sequence that satisfies $\lambda^\bullet \in c$.

THEOREM 4.1. *We have*

$$\sigma\left(B(\widetilde{r},\widetilde{s}), w_\infty(\lambda)\right) = \sigma\left(B(\widetilde{r},\widetilde{s}), w_0(\lambda)\right) = \overline{D}\left(r, |s| \lim_{n\to\infty} \lambda_n^\bullet\right) \cup \mathcal{R}.$$

Proof. First we consider the case $E = w_\infty(\lambda)$. In Lemma 3.1 and Lemma 3.3, we replace $\widetilde{r}$ by the sequence $(r_k - \alpha)_{k\geq 1}$ with $\alpha \neq r_k$ for all k. We have that $|r-\alpha| > |s| \lim_{n\to\infty} \lambda_n^\bullet$ implies $\alpha \in \boldsymbol{\rho}\left(B(\widetilde{r},\widetilde{s}), w_\infty(\lambda)\right)$ and $\alpha \in \boldsymbol{\rho}\left(B(\widetilde{r},\widetilde{s}), w_\infty(\lambda)\right)$ implies $|r-\alpha| \geq |s| \lim_{n\to\infty} \lambda_n^\bullet$. Since we have $\sigma\left(B(\widetilde{r},\widetilde{s}), w_\infty(\lambda)\right) = \left[\boldsymbol{\rho}\left(B(\widetilde{r},\widetilde{s}), w_\infty(\lambda)\right)\right]^c$ and using Remark 3.2 we have $r_k \in \sigma\left(B(\widetilde{r},\widetilde{s}), w_\infty(\lambda)\right)$ for all k, we conclude

$$D\left(r, |s| \lim_{n\to\infty} \lambda_n^\bullet\right) \cup \mathcal{R} \subset \sigma\left(B(\widetilde{r},\widetilde{s}), w_\infty(\lambda)\right) \subset \overline{D}\left(r, |s| \lim_{n\to\infty} \lambda_n^\bullet\right) \cup \mathcal{R}.$$

Now since $\sigma\left(B(\widetilde{r},\widetilde{s}), w_\infty(\lambda)\right)$ is a closed subset of $\mathbb{C}$, it is equal to the smallest closed set containing $\overline{D}\left(r, |s| \lim_{n\to\infty} \lambda_n^\bullet\right) \cup \mathcal{R}$, which itself is closed. The case $E = w_0(\lambda)$ can be shown using similar arguments. This completes the proof. □

In this part we use the next elementary lemma.

LEMMA 4.2. *Let a, $b \in U^+$, and E, $F \subset \omega$. Then $A \in (E_a, F_b)$ if and only if $D_{1/b} A D_a \in (E,F)$.*

We immediately deduce the following.

THEOREM 4.3. *Let $a \in U^+$ and assume $a^\bullet \in c$. We have*

$$\sigma\left(B(\widetilde{r},\widetilde{s}), W_a(\lambda)\right) = \sigma\left(B(\widetilde{r},\widetilde{s}), W_a^0(\lambda)\right) = \overline{D}\left(r, |s| \lim_{n\to\infty} \left(a_n^\bullet \lambda_n^\bullet\right)\right) \cup \mathcal{R}. \quad (8)$$

Proof. First we consider the case of the spectrum of $B(\widetilde{r},\widetilde{s})$ over $W_a(\lambda)$. We have $\alpha \in \boldsymbol{\rho}\left(B(\widetilde{r},\widetilde{s}), W_a(\lambda)\right)$ if and only if $\alpha I - B(\widetilde{r},\widetilde{s}) \in (W_a(\lambda), W_a(\lambda))$, is bijective. But we have $D_{1/a}\left(\alpha I - B(\widetilde{r},\widetilde{s})\right) D_a = \alpha I - D_{1/a} B(\widetilde{r},\widetilde{s}) D_a$, and so $\alpha \in \boldsymbol{\rho}\left(B(\widetilde{r},\widetilde{s}), W_a(\lambda)\right)$ if and only if $\alpha \in \boldsymbol{\rho}\left(D_{1/a} B(\widetilde{r},\widetilde{s}) D_a, w_\infty(\lambda)\right)$. We have $D_{1/a} B(\widetilde{r},\widetilde{s}) D_a = B\left(\widetilde{r}, \widetilde{s'}\right)$, with $s'_{n-1} = s_{n-1} a_n^\bullet$ for all $n \geq 2$. Then we have $\lim_{n\to\infty} s'_{n-1} = s \lim_{n\to\infty} a_n^\bullet$ and we obtain (8) by Theorem 4.1 with $B(\widetilde{r},\widetilde{s})$ replaced by $B\left(\widetilde{r}, \widetilde{s'}\right)$.

The case of the spectrum of $B(\widetilde{r},\widetilde{s})$ over $W_a^0(\lambda)$ can be shown similarly. This completes the proof. □

REMARK 4.4. Notice that if $\lambda \in \Gamma$, then $W_a(\lambda) = s_{a\lambda}$. Indeed, the condition $x \in W_a(\lambda)$, means $C(\lambda) D_{1/a} |x| \in \ell_\infty$, and is equivalent to $|x| \in D_a \Delta(\lambda) \ell_\infty$. But by [9, Proposition 2,p. 159], the condition $\lambda \in \Gamma$ implies $\Delta(\lambda) \ell_\infty = s_\lambda$, and $W_a(\lambda) = s_{a\lambda}$. We conclude $\sigma\left(B(\widetilde{r},\widetilde{s}), W_a(\lambda)\right) = \sigma\left(B(\widetilde{r},\widetilde{s}), s_{a\lambda}\right) = \overline{D}\left(r, |s| \lim_{n\to\infty} \left(a_n^\bullet \lambda_n^\bullet\right)\right) \cup \mathcal{R}$.

Corollary 4.5. *Let $a \in U^+$ and assume ρ and $a^\bullet \in c$. Then we have*

$$\sigma(\Delta_\rho, W_a(\lambda)) = \sigma\left(\Delta_\rho, W_a^0(\lambda)\right) = \overline{D}\left(1, \lim_{n\to\infty}\left(|\rho_n|\, a_n^\bullet \lambda_n^\bullet\right)\right).$$

Remark 4.6. Under the conditions of Theorem 4.3 we have

$$\sigma(B(\widetilde{r},\widetilde{s}), W_a(\lambda)) = \overline{D}\left(r, |s| \lim_{n\to\infty}(a_n^\bullet \lambda_n^\bullet)\right) \cup \left\{r_k : |r - r_k| > |s| \lim_{n\to\infty}(a_n^\bullet \lambda_n^\bullet)\right\}.$$

Remark 4.7. Under the conditions of Theorem 4.3 we have $(W_a(\lambda))_{B(\widetilde{r},\widetilde{s})} = W_a(\lambda)$ if and only if $0 \in \boldsymbol{\rho}(B(\widetilde{r},\widetilde{s}), W_a(\lambda))$, that is, $|s| \lim_{n\to\infty}(a_n^\bullet \lambda_n^\bullet) < |r|$.

If $r_n = r$ and $s_n = s$ for all n, with $r,\ s \neq 0$, then we write $B(r,s)$ for $B(\widetilde{r},\widetilde{s})$. The matrix $\Delta = B(1,-1)$ is called the operator of the first difference.

Corollary 4.8. *Let $a \in U^+$ and assume $a^\bullet \in c$. Then we have:*
(i) $\sigma(B(r,s), W_a(\lambda)) = \sigma\left(B(r,s), W_a^0(\lambda)\right) = \overline{D}(r, |s| \lim_{n\to\infty}(a_n^\bullet \lambda_n^\bullet))$.

(ii) $\sigma(B(r,s), w_\infty(\lambda)) = \sigma(B(r,s), w_0(\lambda)) = \overline{D}(r, |s| \lim_{n\to\infty} \lambda_n^\bullet)$.

(iii) $\sigma(\Delta, W_a(\lambda)) = \sigma\left(\Delta, W_a^0(\lambda)\right) = \overline{D}(1, \lim_{n\to\infty}(a_n^\bullet \lambda_n^\bullet))$.

(iv) $\sigma(\Delta, w_\infty(\lambda)) = \sigma(\Delta, w_0(\lambda)) = \overline{D}(1, \lim_{n\to\infty} \lambda_n^\bullet)$.

Finally when $\lambda_n = n$ for all n, we obtain the next proposition which is a direct consequence of [12, Theorem 6].

Proposition 4.9. *Let $a \in U^+$ and assume $a^\bullet \in \ell_\infty$. Then we have $\sigma(\Delta, W_a) = \sigma\left(\Delta, W_a^0\right) = \overline{D}\left(1, \overline{\lim}_{n\to\infty} a_n^\bullet\right)$.*

This proposition is an extension of Theorem 4.3 in a certain sense. Indeed, if we define the sequence $a \in U^+$ by $a_{2n} = 1$ and $a_{2n+1} = 2$ for all n, then we trivially have $a^\bullet \in \ell_\infty \setminus c$, and $\sigma(\Delta, W_a) = \sigma\left(\Delta, W_a^0\right) = \overline{D}\left(1, \overline{\lim}_{n\to\infty} a_n^\bullet\right) = \overline{D}(1,2)$.

In this way we obtain the next corollaries.

Corollary 4.10. *Let $a \in U^+$ and assume $a^\bullet \in c$. Then we have:*
(i) $\sigma(B(r,s), W_a) = \sigma\left(B(r,s), W_a^0\right) = \overline{D}(r, |s| \lim_{n\to\infty} a_n^\bullet)$.

(ii) $\sigma(B(r,s), w_\infty) = \sigma(B(r,s), w_0) = \overline{D}(r, |s|)$.

(iii) $\sigma(\Delta, W_a) = \sigma\left(\Delta, W_a^0\right) = \overline{D}(1, \lim_{n\to\infty} a_n^\bullet)$.

(iv) $\sigma(\Delta, w_\infty) = \sigma(\Delta, w_0) = \overline{D}(1,1)$.

Corollary 4.11. *Let $R > 0$. Then we have:*
(i) $\sigma(B(r,s), W_R) = \sigma\left(B(r,s), W_R^0\right) = \overline{D}(r, |s|/R)$.

(ii) $\sigma(\Delta, W_R) = \sigma\left(\Delta, W_R^0\right) = \overline{D}(1, 1/R)$.

4.2 Applications to equations of the form $(W_a(\lambda))_{B(r,s)} = W_b(\lambda)$

4.2.1 On the identity $(W_a(\lambda))_{B(r,s)} = W_b(\lambda)$.

In this section, assuming that $r,\ s \neq 0$, and $a,\ b \in U^+$, we characterize the next statement. The condition that $\sup_n\left(\lambda_n^{-1}\sum_{k=1}^n |x_k|/b_k\right) < \infty$ holds if and only if $\sup_n\left(\lambda_n^{-1}\sum_{k=1}^n |rx_k + sx_{k-1}|/a_k\right) < \infty$ for all x can be written in the form

$$(W_a(\lambda))_{B(r,s)} = W_b(\lambda). \tag{9}$$

To simplify we focus on identity (9), but we obtain similar results for the identity $\left(W_a^0(\lambda)\right)_{B(r,s)} = W_b^0(\lambda)$. First we need the next lemma.

LEMMA 4.12. *Let $a,\ b \in U^+$. Then $W_a(\lambda) = W_b(\lambda)$ if and only if $s_a = s_b$.*

Proof. We show that $W_a(\lambda) \subset W_b(\lambda)$ if and only if $s_a \subset s_b$. Necessity. Let $x \in W_a(\lambda)$. Then $y = C(\lambda) D_{1/a}|x| \in \ell_\infty$ and $|x| = D_a\Delta(\lambda) y$. So the condition $x \in W_b(\lambda)$ means $C(\lambda) D_{1/b}|x| = C(\lambda) D_{1/b} D_a \Delta(\lambda) y \in \ell_\infty$ for all $y \in \ell_\infty$. So the inclusion $W_a(\lambda) \subset W_b(\lambda)$ is equivalent to $C(\lambda) D_{a/b}\Delta(\lambda) \in (\ell_\infty, \ell_\infty)$ and to

$$\sup_n\left(\frac{1}{\lambda_n}\sum_{k=1}^{n-1}\left|\frac{a_k}{b_k} - \frac{a_{k+1}}{b_{k+1}}\right|\lambda_k + \frac{a_n}{b_n}\right) < \infty,$$

see [13, Lemma 3.2, p. 596]. This implies $\sup_n(a_n/b_n) < \infty$ and $s_a \subset s_b$. In the same way we obtain that $W_b(\lambda) \subset W_a(\lambda)$ implies $s_b \subset s_a$. Conversely, we assume $s_a \subset s_b$. Let $x \in W_a(\lambda)$. Since $a/b \in s_1$ there is $K > 0$ such that

$$\frac{1}{\lambda_n}\sum_{k=1}^n \frac{|x_k|}{b_k} \le \left(\sup_k \frac{a_k}{b_k}\right)\left(\frac{1}{\lambda_n}\sum_{k=1}^n \frac{|x_k|}{a_k}\right) \le K \text{ for all } n.$$

So we have shown $x \in W_b(\lambda)$ and $W_a(\lambda) \subset W_b(\lambda)$. We conclude $s_a \subset s_b$ if an only if $W_a(\lambda) \subset W_b(\lambda)$. Similarly we obtain $W_a(\lambda) \supset W_b(\lambda)$ if and only if $s_a \supset s_b$. This completes the proof. □

PROPOSITION 4.13. *Let $a,\ b \in U^+$ and $r,\ s \neq 0$. Assume b/a and $a^\bullet \in c$. Then the identity in* (9) *holds if and only if*

(i) $\lim_{n\to\infty}(b_n/a_n) > 0$ *and* *(ii)* $\lim_{n\to\infty}(a_n^\bullet \lambda_n^\bullet) < |r/s|$.

Proof. First we show the necessity of the conditions (i) and (ii). The identity in (9) means that $B(r,s)$ is bijective from $W_b(\lambda)$ to $W_a(\lambda)$. Then the operator $D_{1/a}B(r,s)D_b \in (w_\infty(\lambda), w_\infty(\lambda))$ is bijective. But we have $D_{1/a}B(r,s)D_b = B\left(\widetilde{r'}, \widetilde{s'}\right)$, where $r'_n = rb_n/a_n$, and $s'_n = sb_n/a_{n+1}$ for all n, and since each of the sequences $(r'_n)_{n\ge1}$ and $(s'_n)_{n\ge1}$, where $s'_n = s(b_n/a_n)a^\bullet_{n+1}$, converges, we may apply Theorem 4.3. So we have $0 \notin \sigma\left(D_{1/a}B(r,s)D_b, w_\infty(\lambda)\right)$ implies

$$|r|\lim_{n\to\infty}\frac{b_n}{a_n} > |s|\lim_{n\to\infty}\frac{b_n}{a_{n+1}}\lim_{n\to\infty}\lambda_n^\bullet = |s|\lim_{n\to\infty}\left(\frac{b_n}{a_n}a^\bullet_{n+1}\right)\lim_{n\to\infty}\lambda_n^\bullet.$$

But the conditions b/a and $a^\bullet \in c$ imply

$$|r|\lim_{n\to\infty}\frac{b_n}{a_n} > |s|\lim_{n\to\infty}\frac{b_n}{a_n}\lim_{n\to\infty}a_n^\bullet\lim_{n\to\infty}\lambda_n^\bullet, \text{ and } |r|\lim_{n\to\infty}\frac{b_n}{a_n}\left(1 - \left|\frac{s}{r}\right|\lim_{n\to\infty}(a_n^\bullet\lambda_n^\bullet)\right) > 0.$$

Since $|r| \lim_{n\to\infty} (b_n/a_n) > 0$, we have $\lim_{n\to\infty} (a_n^\bullet \lambda_n^\bullet) < \left|\frac{r}{s}\right|$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = L > 0$, and (i) and (ii) hold.

Conversely assume that (i) and (ii) hold. Then (ii) implies $0 \in \boldsymbol{\rho}(B(r,s), W_a(\lambda))$, by Corollary 4.5 and $B(r,s)$ is bijective from $W_a(\lambda)$ to itself, so we obtain $(W_a(\lambda))_{B(r,s)} = W_a(\lambda)$. Then (i) implies $s_a = s_b$ and by Lemma 4.12 we conclude $W_a(\lambda) = W_b(\lambda)$. This completes the proof. □

Example 4.14. For instance the equation $(W_2(\lambda))_{B(r,s)} = W_{(n)_{n\geq 1}}(\lambda)$ has no solution, since $\lim_{n\to\infty} n/2^n = 0$.

Example 4.15. The equation $(W_{R_1}(\lambda))_{B(r,s)} = W_{R_2}(\lambda)$ for $0 < R_2 \leq R_1$ is equivalent to $R_1 = R_2$ and $\lim_{n\to\infty} \lambda_n^\bullet < R_1 |r/s|$. This result comes from Proposition 4.13 where we have $b/a = ((R_2/R_1)^n)_{n\geq 1} \in c$ and $a^\bullet \in c$.

As a direct consequence of the preceding we obtain the next corollary.

Corollary 4.16. *Let a, $b \in U^+$ and assume b/a and $a^\bullet \in c$. Then*

$$(W_a(\lambda))_\Delta = W_b(\lambda) \tag{10}$$

if and only if $\lim_{n\to\infty} a_n/b_n > 0$ and $a\lambda \in \widehat{\Gamma}$. Especially, if $\lambda_n = n$ for all n, then the identity in (10) *holds if and only if $\lim_{n\to\infty} (a_n/b_n) > 0$ and $a \in \widehat{\Gamma}$.*

Remark 4.17. If b/a, $a^\bullet \in c$, then the identity $(W_a)_\Delta = W_b$ implies $a \notin c$. Indeed, if $\lim_{n\to\infty} a_n > 0$, then we have $\lim_{n\to\infty} a_n^\bullet = 1$ and $a \notin \widehat{\Gamma}$. If $\lim_{n\to\infty} a_n = 0$, then we have $a \notin \widehat{\Gamma}$, since the condition $a \in \widehat{\Gamma}$, implies $\sum_{n=1}^{\infty} 1/a_n < \infty$ and $a_n \to \infty$ $(n \to \infty)$. In this way it can easily be seen that each of the equations $(w_\infty)_\Delta = W_{(1/n)_{n\geq 1}}$, and $(w_\infty)_\Delta = W_r$ for $0 < r < 1$, has no solution.

For $a = b \in U^+$, we obtain the next corollary which is a direct consequence of Proposition 4.13.

Corollary 4.18. *Let $a \in U^+$ and assume $a^\bullet \in c$ and r, $s \neq 0$. Then we have:*

(i) $(W_a(\lambda))_{B(r,s)} = W_a(\lambda)$ *if and only if* $\lim_{n\to\infty} (a_n^\bullet \lambda_n^\bullet) < |r/s|$.

(ii) $(W_a(\lambda))_{B(r,s)} = W_a$ *if and only if* $\lim_{n\to\infty} a_n^\bullet < |r/s|$.

(iii) $(w_\infty(\lambda))_{B(r,s)} = w_\infty(\lambda)$ *if and only if* $\lim_{n\to\infty} \lambda_n^\bullet < |r/s|$.

(iv) $(W_a(\lambda))_\Delta = W_a(\lambda)$ *if and only if* $a\lambda \in \widehat{\Gamma}$.

(v) $(w_\infty(\lambda))_\Delta = w_\infty(\lambda)$ *if and only if* $\lambda \in \widehat{\Gamma}$.

(vi) $(W_a)_\Delta = W_a$ *if and only if* $a \in \widehat{\Gamma}$.

4.2.2 Some other properties of the identity $(W_a)_\Delta = W_b$

In the next proposition we give a result in which we do not assume b/a, $a^\bullet \in c$.

PROPOSITION 4.19. *Let a, $b \in U^+$. If the equation $(W_a)_\Delta = W_b$ holds, then we have*

$$\sup_n \left(\frac{b_n}{na_n} \right) < \infty \tag{11}$$

and

$$\sup_n \left(\frac{1}{nb_n} \sum_{k=1}^{n} a_k \right) < \infty. \tag{12}$$

Proof. The condition $W_b \subset (W_a)_\Delta$ implies $\Delta \in (W_b, W_a)$, hence $D_{1/a}\Delta D_b \in (w_\infty, w_\infty)$. As we have seen in the proof of Lemma 2.4 we have $(w_\infty, w_\infty) \subset \left(\ell_\infty, (\ell_\infty)_{(n)_{n\geq 1}}\right)$, so $D_{(1/na_n)_{n\geq 1}}\Delta D_b \in (\ell_\infty, \ell_\infty)$. Then the nonzero entries of the matrix $D_{(1/ka_k)_{k\geq 1}}\Delta D_b$ are given by $\left[D_{(1/ka_k)_{k\geq 1}}\Delta D_b\right]_{n,n-1} = -b_{n-1}/na_n$, for all $n \geq 2$, and $\left[D_{(1/ka_k)_{k\geq 1}}\Delta D_b\right]_{nn}$ $= b_n/na_n$ for all $n \geq 1$. We conclude $\sup_n \left\{ \frac{1}{na_n}(b_{n-1} + b_n) \right\} < \infty$ and the condition in (11) holds. Now the condition $(W_a)_\Delta \subset W_b$ implies $D_{1/b}\Sigma D_a \in (w_\infty, w_\infty)$ and as we have just seen $D_{(1/nb_n)_{n\geq 1}}\Sigma D_a \in (\ell_\infty, \ell_\infty)$. Then (12) holds since the nonzero entries of the triangle $D_{(1/mb_m)_{m\geq 1}}\Sigma D_a$ are given by $\left[D_{(1/mb_m)_{m\geq 1}}\Sigma D_a\right]_{nk} = a_k/nb_n$, for all $k \leq n$ and for all n. □

EXAMPLE 4.20. For R_1, $R_2 > 0$ we consider the identity

$$(W_{R_1})_\Delta = W_{R_2}. \tag{13}$$

The identity in (13) holds if and only if $R_1 = R_2 > 1$. Indeed, by Proposition 4.19 (i) the identity in (13) implies $\sup_n \left\{ n^{-1} (R_2/R_1)^n \right\} < \infty$ and $R_2 \leq R_1$. Then we may apply Proposition 4.13 where $b/a = \left((R_2/R_1)^n\right)_{n\geq 1} \in c$ and $a^\bullet \in c$, and we conclude $R_1 = R_2 > 1$. Conversely if $R_1 = R_2 > 1$ we deduce $(W_{R_1})_\Delta = W_{R_1} = W_{R_2}$.

EXAMPLE 4.21. As a direct consequence of Proposition 4.19 it can easily be seen that here is no $R > 0$ for which $\left(W_{(n)_{n\geq 1}}\right)_\Delta = W_R$.

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