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S-PSEUDOSPECTRA AND S-ESSENTIAL PSEUDOSPECTRA

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Abstract. In the present paper, we introduce and study the $\mathcal{S}$-pseudospectra and the essential $\mathcal{S}$-pseudospectra of linear relations. We start by giving the definition and we investigate the characterization and some properties of these $\mathcal{S}$-pseudospectra.

1. Introduction

The concept of pseudospectra was introduced independently by J. M. Varah [13] and has been subsequently employed by other authors for example, H. Landau [10], L. N. Trefethen [12], D. Hinrichsen, A. J. Pritchard [9] and E. B. Davies [8]. This concept was especially due to L. N. Trefethen, who developed this idea for matrices and operators, and who applied it to several highly interesting problems. This notion of pseudospectra arose as a result of realizing that several pathological properties of highly non-self-adjoint operators were closely related. These include the existence of approximate eigenvalues far from the spectrum, the instability of the spectrum even under small perturbations. The analysis of pseudospectra has been performed in order to determine and localize the spectrum of operators, hence leading to many applications of pseudospectra. For example, in aeronautics, eigenvalues may determine whether the flow over a wing is laminar or turbulent. In ecology, eigenvalues may determine whether a food web will settle into a steady equilibrium. In electrical engineering, they may determine the frequency response of an amplifier or the reliability of a national power system. Moreover, in probability theory, eigenvalues may determine the convergence rate of a Markov process and, in other fields, we can find the eigenvalues allowing us to examine the respective properties. The definition of pseudospectra of a closed densely linear operator T for every $\varepsilon > 0$ is given by:

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{ \lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\}.$$

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By convention we write $\|(\lambda - T)^{-1}\| = \infty$ if $(\lambda - T)^{-1}$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(T)$. Inspired by the notion of pseudospectra, A. Ammar and A. Jeribi in their works [1,3,4,6], extended these results for the essential spectra of closed, densely defined, and linear operators on a Banach space. They introduced a new concept of essential pseudospectra of closed, densely defined, and linear operators on a Banach space. Because of the existence of several essential spectra, they were interested to focus their study on the pseudo-Browder essential spectrum. This set was shown to be characterized in the way one would expect by analogy with the essential numerical range.

Recently, A. Ammar, H. Daoud and A. Jeribi in [5], have introduced and studied the pseudospectra and the essential pseudospectra of linear relations, in order to extend these results for the essential spectra of linear relations by

$$\sigma_{w,\varepsilon}(T) = \bigcap_{K \in \mathcal{K}_T(X)} \sigma_\varepsilon(T + K) = \bigcup_{\substack{\|D\| < \varepsilon \\ \mathcal{D}(D) \supset \mathcal{D}(T)}} \sigma_w(T + D),$$

where $\mathcal{K}_T(X) := \{K \in KR(X)$ such that $\mathcal{D}(K) \supset \mathcal{D}(T)$ and $K(0) \subset T(0)\}$, and $\sigma_w(T)$ is the Weyl essential spectrum of the linear relation T defined by $\sigma_w(T) := \bigcap_{K \in \ \mathcal{K}_T(X)} \sigma_\varepsilon(T + K)$.

Our aim in this work is to show some properties of S-pseudospectra and S-essential pseudospectra of closed linear relations in Banach spaces.

We organize the paper in the following way. Section 2 contains preliminary and auxiliary properties that are needed to prove the main results of the other sections. We begin by giving some entirely algebraic results about linear relations in vector spaces. The main aim of Section 3 is to characterize the S-pseudospectra of a closed multivalued linear operator and investigate some properties. In Section 4, we characterize the S-essential pseudospectra of a closed multivalued linear operator, we apply some results obtained in Section 3 to investigate the S-essential pseudospectrum and we establish some results for perturbation and properties.

2. $\mathcal{S}$-spectra of linear relation in normed space

We will introduce a definition of S-spectrum in a normed space X.

Definition 2.1. Let $T \in LR(X)$, S be a continuous linear relation such that $S(0) \subset \overline{T(0)}$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$. Then we define the $\mathcal{S}$-resolvent set of T by $\rho_S(T) := \{\lambda \in \mathbb{C}$ such that $(\lambda S - T)$ is injective, open with dense range on $X\}$. We define the S-spectra set of T as $\sigma_S(T) = \mathbb{C} \backslash \rho_S(T)$.

In the sequel of this section, X will denote a normed space.

Lemma 2.2. *Let $S \in LR(X)$. Then $\|S\| = 0$ if and only if $R(S) \subset \overline{S(0)}$.*

Proof. $\|S\| = \sup\limits_{\substack{x \in \mathcal{D}(S) \\ x \neq 0}} \dfrac{\|Sx\|}{\|x\|}$. Then $\|S\| = 0$ if and only if $\|Sx\| = 0$ for all $x \in \mathcal{D}(S)$,

$x \neq 0$. This is equivalent to the condition $x \in \mathcal{D}(S)$, $y \in Sx$, $d(y, S(0)) = 0$, i.e. $x \in \mathcal{D}(S)$, $y \in Sx$, $y \in \overline{S(0)}$. From this we get that $Sx \subset \overline{S(0)}$ and $S(\mathcal{D}(S)) \subset \overline{S(0)}$ which implies the result. □

LEMMA 2.3. *Let $T \in LR(X)$ and S a continuous linear relation such that $S(0) \subset \overline{T(0)}$ and $\mathcal{D}(S) \supset D(T)$. Then $\|S\| = 0$ implies $\rho_S(T) = \emptyset$ or $\mathbb{C}$.*

Proof. We will show that $\lambda \in \rho_S(T)$ if and only if T is injective, open with dense range. Let $\lambda \in \rho_S(T)$, then $T - \lambda S$ is injective, open with dense range. But $\|\lambda S\| = |\lambda|\|S\| = 0 < \gamma(T - \lambda S)$. Moreover, $\lambda S(0) \subset \overline{T(0)}$ and $\mathcal{D}(\lambda S) = \mathcal{D}(S) \supset \mathcal{D}(T)$, hence $T - \lambda S + \lambda S = T$ is injective, open with dense range. Conversely, let T is injective, open with dense range. Since $\| - \lambda S\| = |\lambda|\|S\| = 0 < \gamma(T)$, then $T - \lambda S$ is injective, open with dense range. Hence $\lambda \in \rho_S(T)$. □

THEOREM 2.4. *Let $T \in LR(X)$ be injective with dense range and S be a continuous linear relation such that $S(0) \subset \overline{T(0)}$, $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $\|S\| \neq 0$. Then $\sigma_S(T) \subset \left\{\lambda \in \mathbb{C} \text{ such that } |\lambda| \geq \frac{\gamma(T)}{\|S\|}\right\}$.*

Proof. It suffices to show that $\left\{\lambda \in \mathbb{C} \text{ such that } |\lambda| < \frac{\gamma(T)}{\|S\|}\right\} \subset \rho_S(T)$. Let $|\lambda| < \frac{\gamma(T)}{\|S\|}$, if $\gamma(T) = 0$, then $\{\lambda \in \mathbb{C} \text{ such that } |\lambda| < \frac{\gamma(T)}{\|S\|}\} = \emptyset$, and there is nothing to prove.

Now, if $\gamma(T) > 0$, then T is open, injective with dense range. We have $\lambda S(0) \subset \overline{T(0)}$ and $\mathcal{D}(\lambda S) = \mathcal{D}(S) \supset \mathcal{D}(T)$. Moreover $\|\lambda S\| = |\lambda|\|S\| < \gamma(T)$. Then, we obtain that $\lambda S - T$ is open, injective with dense range. Hence $\lambda \in \rho_S(T)$. □

THEOREM 2.5. *Let $T \in LR(X)$ and S be a continuous linear relation such that $S(0) \subset \overline{T(0)}$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$. Then $\rho_S(T)$ is an open set.*

Proof. We will discuss two cases.
Case 1: If $\|S\| = 0$, then $\rho_S(T) = \emptyset$ or $\rho_S(T) = \mathbb{C}$. Thus $\rho_S(T)$ is open.
Case 2: If $\|S\| \neq 0$. Let $\lambda \in \rho_S(T)$. Then $\gamma(\lambda S - T) > 0$. Let $|\mu - \lambda| < \dfrac{\gamma(\lambda S - T)}{\|S\|}$, then $\|(\mu - \lambda)S\| = |\mu - \lambda|\|S\| < \gamma(\lambda S - T)$. Furthermore, we have $(\mu - \lambda)S(0) \subset \overline{T(0)} = \overline{(\lambda S - T)(0)}$ and $\mathcal{D}((\mu - \lambda)S) = \mathcal{D}(S) \supset \mathcal{D}(T) = \mathcal{D}(\lambda S - T)$ then, $\mu S - T$ is injective, open with dense range. Then $\mu \in \rho_S(T)$. Therefore $\rho_S(T)$ is open. □

LEMMA 2.6. *Let $T \in LR(X)$ and S be a continuous linear relation such that $S(0) \subset \overline{T(0)}$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$. Then S' is continuous, $S'(0) \subset T'(0)$ and $\mathcal{D}(S') \supset \mathcal{D}(T')$.*

Proof. By [7, Proposition III 4.6], $\|S'\| = \|S\|$, then S' is continuous. Also, $\mathcal{D}(S') = S(0)^{\perp} \supset \overline{T(0)}^{\perp} \supset \overline{T}(0)^{\perp} = \mathcal{D}(T')^{\top\perp} = \overline{\mathcal{D}(T')}$, and using [7, Proposition III 1.4], $S'(0) = \mathcal{D}(S)^{\perp} \subset \mathcal{D}(T)^{\perp} = T'(0)$. □

THEOREM 2.7. *Let $T \in LR(X)$ and S be a continuous linear relation such that $S(0) \subset \overline{T(0)}$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$. Then $\sigma_S(T) = \sigma_{S'}(T')$.*

Proof. We will discuss two cases. If $0 \in \rho_S(T)$, it is clear that $0S - T = -T$ is injective, open with dense range. Then $0 \in \rho(T) = \rho(T')$. Thus $0S' - T' = -T'$ is injective, open with dense range. Hence $0 \in \rho_{S'}(T')$. Now, If $0 \neq \lambda \in \rho_S(T)$, then $\lambda S - T$ is injective, open with dense range. Using [7, Proposition III 1.4 (a)], we have $N((\lambda S - T)') = (R(\lambda S - T))^{\perp} = (\overline{R(\lambda S - T)})^{\perp} = X^{\perp}$, $N((\lambda S - T)') = \{0\}$, and by [7, Proposition III 4.6 (b) and (d)], it holds that $\gamma((\lambda S - T)') = \gamma(\lambda S - T) > 0$ and $R((\lambda S - T)') = N(\lambda S - T)^{\perp} = \{0\}^{\perp}$, $R((\lambda S - T)') = X'$. Therefore $(\lambda S - T)'$ is injective, open with dense range. It remains to show that $(\lambda S - T)' = \lambda S' - T'$. Using [7, Proposition III 1.5], since λS is continuous and $\mathcal{D}(\lambda S) = \mathcal{D}(S) \supset \mathcal{D}(T)$, then $(\lambda S - T)' = (\lambda S)' - T' = \lambda S' - T'$ by [7, Proposition III 1.3 (c)]. Conversely, it is clear with the same reasoning that if $0 \in \rho_{S'}(T')$ thus $0 \in \rho(T') = \rho(T)$ then $0 \in \rho_S(T)$. Now, if $0 \neq \lambda \in \rho_S(T)$, then $\lambda S' - T' = (\lambda S - T)'$ is injective, open with dense range. But $G((\lambda S - T)')$ is closed. Thus $\lambda S' - T'$ is a closed linear relation. Using [7, Theorem III 4.2], $R((\lambda S - T)')$ is closed, then $R((\lambda S - T)') = X'$.

Moreover, by [7, Proposition III 1.4], $\overline{R(\lambda S - T)} = R(\lambda S - T)^{\perp\top} = N((\lambda S - T)')^{\top} = \{0\}^{\top} = X$. On the other hand, by the same proposition, we have $N(\overline{\lambda S - T}) = R((\lambda S - T)')^{\top} = X'^{\top} = \{0\}$, and it is clear that $N(\lambda S - T) \subset N(\overline{\lambda S - T})$, in fact, if $x \in N(\lambda S - T)$ then $0 \in (\lambda S - T)x \subset \overline{(\lambda S - T)}x$, thus $x \in N(\overline{\lambda S - T})$. This gives that $N(\lambda S - T) = 0$. Hence $R((\lambda S - T)') = X' = N(\lambda S - T)^{\perp}$, by [7, Proposition III 4.6 (b)], $(\lambda S - T)$ is open, moreover it is injective with dense range. Then $\lambda \in \rho_S(T)$. □

3. $\mathcal{S}$-pseudospectra of linear relation

Throughout the next sections, X will denote a Banach space, $\varepsilon > 0$ and $S, T \in LR(X)$ such that S is continuos, T is closed with $S(0) \subset T(0)$, $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $\|S\| \neq 0$ except where stated otherwise. The purpose of this section is to define and characterise the $\mathcal{S}$-pseudospectra of multivalued linear operator and study some properties.

Definition 3.1. We define the $\mathcal{S}$-pseudospectra of T by

$$\sigma_{\varepsilon,S}(T) = \sigma_S(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \|(\lambda S - T)^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

We denote the $\mathcal{S}$-pseudoresolvent set of T by

$$\rho_{\varepsilon,S}(T) = \mathbb{C}\backslash\sigma_{\varepsilon,S}(T) = \rho_S(T) \cap \left\{\lambda \in \mathbb{C} \text{ such that } \|(\lambda S - T)^{-1}\| \leq \frac{1}{\varepsilon}\right\}.$$

Remark 3.2. If $0 < \varepsilon_1 < \varepsilon_2$, it is clear that $\sigma_{\varepsilon_1,S}(T) \subset \sigma_{\varepsilon_2,S}(T)$.

Lemma 3.3. *Let $\varepsilon > 0$. If $\lambda \notin \sigma_S(T)$ then $\lambda \in \sigma_{\varepsilon,S}(T)$ if and only if there exists $x \in X$ such that $\|(\lambda S - T)x\| < \varepsilon\|x\|$.*

Proof. Let $\lambda \in \sigma_{\varepsilon,S}(T)\backslash\sigma_S(T)$ then $\|(\lambda S - T)^{-1}\| > \frac{1}{\varepsilon}$. Since $(\lambda S - T)^{-1}$ is continuous operator, there exists a non-zero vector $y \in X$ such that

$$\|(\lambda S - T)^{-1}y\| > \frac{1}{\varepsilon}\|y\|. \tag{1}$$

Putting $x := (\lambda S - T)^{-1}y$, then $y \in (\lambda S - T)x$. On the other hand, $(\lambda S - T)(0) = \lambda S(0) - T(0) = T(0)$ (as $S(0) \subset T(0)$). Hence

$$\|(\lambda S - T)x\| = d(y, (\lambda S - T)(0)) = d(y, T(0)) \leq d(y, 0) = \|y\|. \qquad (2)$$

From (1) and (2), we have $\|x\| > \frac{1}{\varepsilon}\|y\| \geq \frac{1}{\varepsilon}\|(\lambda S - T)x\|$, hence $\|(\lambda S - T)x\| < \varepsilon\|x\|$.

Conversely, assume there exists $x \in X$ such that $\|(\lambda S - T)x\| < \varepsilon\|x\|$. Since $\lambda \in \rho_S(T)$, it is clear that $\lambda S - T$ is injective and open, then $\gamma(\lambda S - T)\|x\| \leq \|(\lambda S - T)x\| < \varepsilon\|x\|$, so $0 < \gamma(\lambda S - T) < \varepsilon$. Hence, we have $\gamma(\lambda S - T) = \|(\lambda S - T)^{-1}\|^{-1}$, therefore $\lambda \in \sigma_{\varepsilon,S}(T)$. □

Theorem 3.4. *Let $\varepsilon > 0$. Then $\lambda \in \sigma_{\varepsilon,S}(T)$ if and only if there exists a continuous linear relation B satisfying $\mathcal{D}(B) \supset \mathcal{D}(T)$, $B(0) \subset T(0)$, $\|B\| < \varepsilon$ such that $\lambda \in \sigma_S(T + B)$.*

Proof. In the first case, assume that $\lambda \in \sigma_{\varepsilon,S}(T)$. It is clear that if $\lambda \in \sigma_S(T)$, and we may put $B = 0$. So we will discuss the second case when $\lambda \notin \sigma_S(T)$. By Lemma 3.3 there exists $x_0 \in X$, $\|x_0\| = 1$ such that $\|(\lambda S - T)x_0\| < \varepsilon$; then, there exists $x' \in X'$ such that $\|x'\| = 1$ and $x'(x_0) = \|x_0\|$. We can define a relation $B : X \to X$ by $B(x) = x'(x)(\lambda S - T)x_0$. It is clear that B is everywhere defined and single valued. Since $\|Bx\| = \|x'(x)(\lambda S - T)x_0)\| \leq \|x'\|\|x\|\|(\lambda S - T)x_0\|$, for $x \neq 0$, we have $\frac{\|Bx\|}{\|x\|} \leq \|(\lambda S - T)x_0\|$, so, $\|B\| < \varepsilon$. On the other hand,

$$\begin{aligned}(\lambda S - (T + B))x_0 &= (\lambda S - T)x_0 - Bx_0 = (\lambda S - T)x_0 - x'(x_0)(\lambda S - T)x_0\\ &= (\lambda S - T)(0) = (\lambda S - T)(0) - B(0) = (\lambda S - (T + B))(0),\end{aligned}$$

therefore $0 \neq x_0 \in N(\lambda S - (T + B))$ and hence $(\lambda S - (T + B))$ is not injective, so, $\lambda \in \sigma_S(T + B)$.

Conversely, we derive a contradiction from the assumption that $\lambda \notin \sigma_{\varepsilon,S}(T)$. Then $\lambda \in \rho_S(T)$ and $\gamma(\lambda S - T) \geq \varepsilon$. Therefore $(\lambda S - T)$ is injective, open with dense range, furthermore, $B(0) \subset T(0) = (\lambda S - T)(0)$, $\mathcal{D}(B) \supset \mathcal{D}(T) = \mathcal{D}(\lambda S - T)$ and $\|B\| < \varepsilon \leq \gamma(\lambda S - T)$. Since $\lambda S - T - B$ is injective, open with dense range and so, $\lambda \in \rho_S(T + B)$ and this is a contradiction. □

Remark 3.5. (i) It follows, immediately, from Theorem 3.4, that for $T \in \mathcal{CR}(X)$ and $\varepsilon > 0$: $\sigma_{\varepsilon,S}(T) = \bigcup_{\substack{\|B\|<\varepsilon \\ B(0)\subset T(0) \\ \mathcal{D}(B)\supset\mathcal{D}(T)}} \sigma_S(T + B)$.

(ii) Theorem 3.4 generalizes [8, Theorem 9.2.13].

Proposition 3.6. *We have $\bigcap_{\varepsilon>0} \sigma_{\varepsilon,S}(T) = \sigma_S(T)$.*

Proof. It is clear that $\sigma_S(T) \subset \sigma_{\varepsilon,S}(T)$ for all $\varepsilon > 0$, then $\sigma_S(T) \subset \bigcap_{\varepsilon>0} \sigma_{\varepsilon,S}(T)$. Conversely, if $\lambda \notin \sigma_S(T)$ then $\lambda \in \rho_S(T)$, hence $(\lambda S - T)^{-1}$ is a bounded linear operator, so there exists $\varepsilon > 0$ such that $\|(\lambda S - T)^{-1}\| \leq \frac{1}{\varepsilon}$. Then $\lambda \notin \sigma_{\varepsilon,S}(T)$ and thus $\lambda \notin \bigcap_{\varepsilon>0} \sigma_{\varepsilon,S}(T)$. □

Proposition 3.7. *Let T be injective with dense range and $\|S\| \neq 0$. Then*

$$\sigma_{\varepsilon,S}(T) \subset \left\{ \lambda \in \mathbb{C} \text{ such that } |\lambda| > \frac{\gamma(T)-\varepsilon}{\|S\|} \right\}.$$

Proof. If $\gamma(T) < \varepsilon$, then $\|T^{-1}\| = \|(0S-T)^{-1}\| > \frac{1}{\varepsilon}$, hence $0 \in \sigma_{\varepsilon,S}(T)$ and there is nothing to prove. If $\gamma(T) = \varepsilon > 0$, then $T = T - 0S$ is open, hence $0 \in \rho_S(T)$ and furthermore $\|T^{-1}\| = \|(0S-T)^{-1}\| = \frac{1}{\varepsilon}$. Then $0 \notin \sigma_{\varepsilon,S}(T)$ and therefore $\sigma_{\varepsilon,S}(T) \subset \{\lambda \in \mathbb{C} \text{ such that } |\lambda| > 0\}$. Now suppose that $\gamma(T) > \varepsilon$. Then T is open, injective and surjective. On the other hand, for $\lambda \in \mathbb{C}$, such that $|\lambda| \leq \frac{\gamma(T)-\varepsilon}{\|S\|}$, then $\|\lambda S\| = |\lambda|\|S\| \leq \gamma(T) - \varepsilon$, thus $\|\lambda S\| < \gamma(T)$.

The relation $\lambda S - T$ is open, injective with dense range, i.e. $\lambda \in \rho_S(T)$ and for $x \in \mathcal{D}(T)$, since $S(0) \subset T(0)$,

$$\|(\lambda S - T)x\| = \|(T - \lambda S)x\| \geq \|Tx\| - \|\lambda Sx\| \geq (\gamma(T) - |\lambda|\|S\|)\|x\|.$$

Therefore $\gamma(\lambda S - T) \geq \gamma(T) - |\lambda|\|S\| \geq \gamma(T) - \gamma(T) + \varepsilon = \varepsilon$. So, $\|(\lambda S - T)^{-1}\| \leq \frac{1}{\varepsilon}$. Then $\lambda \in \rho_{\varepsilon,S}(T)$. Hence $\sigma_{\varepsilon,S}(T) \subset \{\lambda \in \mathbb{C} \text{ such that } |\lambda| > \frac{\gamma(T)-\varepsilon}{\|S\|}\}$. □

Theorem 3.8. *$\sigma_{\varepsilon,S}(T)$ is an open set.*

Proof. Let $\lambda \in \overline{\rho_{\varepsilon,S}(T)}$. Then for $r \in]0, \dfrac{\varepsilon}{\|S\|}[$ we have $B_f(\lambda, r) \cap \rho_{\varepsilon,S}(T) \neq \emptyset$, where $B_f(\lambda, r) = \{\mu \in \mathcal{C} \text{ such that } |\lambda - \mu| < r\}$. So there exists $\mu \in \rho_{\varepsilon,S}(T)$ such that $|\lambda - \mu| \leq r$. Then $\mu \in \rho_S(T)$ and $\gamma(\mu S - T) \geq \varepsilon$. Hence $(\mu S - T)$ is open, injective with dense range. Also, $|\lambda - \mu| \leq r < \frac{\varepsilon}{\|S\|}$, then $\|(\lambda - \mu)S\| = |\lambda - \mu|\|S\| < \varepsilon \leq \gamma(\mu S - T)$. Hence $(\lambda - \mu)S + (\mu S - T) = \lambda S - T$ is open injective with dense range. Then $\lambda \in \rho_S(T)$.

For $x \in \mathcal{D}(T)$, since $S(0) \subset T(0)$ we have

$$\begin{aligned}\|(\lambda S - T)x\| &= \|(T - \mu S) + (\mu - \lambda)Sx\| \geq \|(T - \mu S)x\| - \|(\mu - \lambda)Sx\| \\ &\geq \|(T - \mu S)x\| - |\mu - \lambda|\|S\|\|x\| \geq (\gamma(T - \mu S) - |\mu - \lambda|\,\|S\|)\|x\|,\end{aligned}$$

therefore $\gamma(\lambda S - T) \geq \gamma(\mu S - T) - |\mu - \lambda|\|S\|$. Hence $\gamma(\lambda S - T) \geq \varepsilon - r\|S\|$, $\forall\, 0 < r < \frac{\varepsilon}{\|S\|}$. Then $\gamma(\lambda S - T) \geq \varepsilon$. So, $\lambda \in \rho_{\varepsilon,S}(T)$. Therefore $\rho_{\varepsilon,S}(T)$ is a closed set. □

Theorem 3.9. *$\sigma_{\varepsilon,S}(T) = \sigma_{\varepsilon,S'}(T')$.*

Proof. At first, it is clear from the proof of Theorem 2.7 that $(\lambda S - T)' = \lambda S' - T'$. Now let $\lambda \in \rho_{\varepsilon,S'}(T')$. Then $\lambda \in \rho_{S'}(T')$ and $\|(\lambda S - T)'^{-1}\| \leq \frac{1}{\varepsilon}$. By Theorem 2.7, $\lambda \in \rho_S(T)$. So, $(\lambda S - T)^{-1}$ is continuous. From [7, Proposition III 1.3 and Proposition III 4.6 (c)], $\|(\lambda S - T)^{-1}\| = \|((\lambda S - T)^{-1})'\| \leq \frac{1}{\varepsilon}$, thus $\lambda \in \rho_{\varepsilon,S}(T)$.

Conversely, if $\lambda \in \rho_{\varepsilon,S}(T)$, then $\|(\lambda S - T)^{-1}\| \leq \frac{1}{\varepsilon}$. By [7, Proposition III 1.3 and Proposition III 4.6 (c)], $\|(\lambda S - T)^{-1}\| = \|(\lambda S' - T')^{-1}\| \leq \frac{1}{\varepsilon}$. Furthermore, $\lambda \in \rho_{S'}(T')$ by Theorem 2.7. Then $\lambda \in \rho_{\varepsilon,S'}(T')$. □

4. $\mathcal{S}$-essential pseudospectra of linear relation

In this section, we define the $\mathcal{S}$-essential pseudospectra of a closed linear relation, study some properties and establish some results of perturbation in the context of linear relations.

Definition 4.1. Let T be a linear relation in $CR(X)$ and $\varepsilon > 0$. The *essential pseudospectrum* of T is the set $\sigma_{w,\varepsilon,S}(T) = \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon,S}(T+K)$ where $\mathcal{K}_T(X) := \{K \in KR(X)$ such that $\mathcal{D}(K) \supset \mathcal{D}(T)$ and $K(0) \subset T(0)\}$, and we define the *essential pseudoresolvent set* $\rho_{w,\varepsilon,S}(T) = \mathbb{C}\backslash\sigma_{w,\varepsilon,S}(T)$.

Theorem 4.2. *The following properties are equivalent:*
(i) $\lambda \notin \sigma_{w,\varepsilon,S}(T)$.

(ii) For all continuous linear relations $B \in LR(X)$ *such that* $\mathcal{D}(B) \supset \mathcal{D}(T)$, $B(0) \subset T(0)$ *and* $\|B\| < \varepsilon$, *we have* $T + B - \lambda S \in \Phi(X)$ *and* $i(T + B - \lambda S) = 0$.

(iii) For all continuous single valued relations $D \in LR(X)$ *such that* $\mathcal{D}(D) \supset \mathcal{D}(T)$ *and* $\|D\| < \varepsilon$, *we have* $T + D - \lambda S \in \Phi(X)$ *and* $i(T + D - \lambda S) = 0$.

Proof. (i) $\Rightarrow$ (ii) Let $\lambda \notin \sigma_{w,\varepsilon,S}(T)$. Then there exists $K \in \mathcal{K}_T(X)$ such that $\lambda \notin \sigma_{\varepsilon,S}(T + K)$. Using Theorem 3.4, for all continuous linear relations $B \in LR(X)$ such that $\mathcal{D}(B) \supset \mathcal{D}(T + K) = \mathcal{D}(T) \cap \mathcal{D}(K) = \mathcal{D}(T)$ as $\mathcal{D}(K) \supset \mathcal{D}(T)$, $B(0) \subset (T + K)(0) = T(0)$ as $K(0) \subset T(0)$ and $\|B\| < \varepsilon$, we have $\lambda \in \rho_S(T + B + K)$. Then $T + B + K - \lambda S$ is open, injective with dense range. But T is closed, K is compact then K is continuous so $B + K - \lambda S$ is continuous. Furthermore $(B + K - \lambda S)(0) \subset T(0)$, then $T + B + K - \lambda S$ is closed. Hence $R(T + B + K - \lambda S)$ is closed. So, $R(T+B+K-\lambda S) = X$. Therefore $T+B+K-\lambda S \in \Phi(X)$ and $i(T+B+K-\lambda S) = 0$, for all continuous linear relations $B \in LR(X)$ such that $\mathcal{D}(B) \supset \mathcal{D}(T)$, $B(0) \subset T(0)$ and $\|B\| < \varepsilon$. It follows from [2, Lemma 3.6] that for all continuous linear relations $B \in LR(X)$ such that $\mathcal{D}(B) \supset \mathcal{D}(T)$, $B(0) \subset T(0)$ and $\|B\| < \varepsilon$ we have $T+B-\lambda S \in \Phi(X)$ and $i(T + B - \lambda S) = 0$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) Let D be a continuous single valued relation in $LR(X)$ such that $\mathcal{D}(D) \supset \mathcal{D}(T)$ and $\|D\| < \varepsilon$; then we have $T+D-\lambda S \in \Phi(X)$ and $i(T+D-\lambda S) = 0$. By [7, Proposition III 1.4 (a)], $N((T + D - \lambda S)') = R(T + D - \lambda S)^{\perp}$.

Let $n = \alpha(T+D-\lambda S) = \beta(T+D-\lambda S)$, $\{x_1, \dots, x_n\}$ be a basis for $N(T+D-\lambda S)$ and $\{y'_1, \dots, y'_n\}$ be a basis for the $N((T + D - \lambda S)')$. Using [11, Theorems I 2.5 and 2.6], there are functionals $x'_1, \dots, x'_n \in X'$ (the adjoint space of X) and elements $y_1, \dots, y_n$ such that $x'_j(x_k) = \delta_{jk}$ and $y'_j(y_k) = \delta_{jk}$, $1 \le j, k \le n$, where $\delta_{jk} = 0$ if $j \ne k$ and $\delta_{jk} = 1$ if $j = k$. A single valued relation K is defined by $Kx = \sum_{k=1}^{n} x'_k(x)y_k$, $x \in X$. K is bounded, since $\mathcal{D}(K) = X$ and $\|Kx\| \le \|x\|(\sum_{k=1}^{n} \|x'_k\|\|y_k\|)$. Moreover, the range of K is contained in a finite subspace of X. Then K is a finite rank relation in X. By [7, Proposition V 1.3], K is compact. Let $x \in N(T + D - \lambda S)$, then $x = \sum_{k=1}^{n} \alpha_k x_k$, therefore, $x'_j(x) = \alpha_j$, $1 \le j \le n$. On the other hand, if $x \in N(K)$ then $x'_j(x) = 0$, $1 \le j \le n$. This proves that $N(T + D - \lambda S) \cap N(K) = \{0\}$. Now, if $y \in R(K)$, then $y = \sum_{k=1}^{n} \alpha_k y_k$, and hence, $y'_j(y) = \alpha_j$, $1 \le j \le n$. But, if $y \in R(T + D - \lambda S)$, then $y'_j(y) = 0$, $1 \le j \le n$. Thus we obtain that

$$N(T + D - \lambda S) \cap N(K) = \{0\} \quad \text{and} \quad R(T + D - \lambda S) \cap R(K) = \{0\}. \tag{3}$$

On the other hand, $K \in \mathcal{K}_T(X) = \mathcal{K}_{T+D-\lambda S}(X)$ since $K(0) \subset T(0) = (T + D - \lambda S)(0)$ and $D(K) \supset \mathcal{D}(T) = \mathcal{D}(T + D - \lambda S)$. We deduce from [2, Lemma 3.6] that $T + D + K - \lambda S \in \Phi(X)$ and $i(T + D + K - \lambda S) = 0$.

If $x \in N(T+D+K-\lambda S)$ then $0 \in Tx+Dx+Kx-\lambda Sx$, hence $-Kx \in (T+D-\lambda S)x$ and so, $Kx \in R(K) \cap R(T+D-\lambda S) = \{0\}$. Therefore $Kx = 0$ and $0 \in (T+D-\lambda S)x$. This implies that $x \in N(T+D-\lambda S) \cap N(K)$, hence $x = 0$. Thus $\alpha(T+D+K-\lambda S) = 0$. In the same way, one proves that $R(T + D + K - \lambda S) = X$. Since $T + D + K - \lambda S$ is closed by [2, Lemma 3.5], then $\lambda \in \rho_S(T + D + K)$ for all continuous single valued relations $D \in LR(X)$ such that $\mathcal{D}(D) \supset \mathcal{D}(T)$ and $\|D\| < \varepsilon$.

But from the proof of Theorem 3.4 ((i) $\Rightarrow$ (ii)), if $\lambda \in \sigma_{\varepsilon,S}(T + K)$ there exists a continuous single valued relation $D \in LR(X)$ satisfying $\mathcal{D}(D) \supset \mathcal{D}(T + K) = \mathcal{D}(T)$ and $\|D\| < \varepsilon$ such that $\lambda \in \sigma_S(T + K + D)$. Hence $\lambda \notin \sigma_{\varepsilon,S}(T + K)$ and since $K \in \mathcal{K}_T(X)$ $\lambda \notin \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon,S}(T + K)$, so $\lambda \notin \sigma_{w,\varepsilon,S}(T)$. □

Remark 4.3. It follows immediately from Theorem 4.2 that for $T \in CR(X)$ and $\varepsilon > 0$, $\sigma_{w,\varepsilon,S}(T) = \bigcup_{\substack{\|D\|<\varepsilon \\ \mathcal{D}(D)\supset\mathcal{D}(T)}} \sigma_{w,S}(T + D) = \bigcup_{\substack{\|B\|<\varepsilon \\ B(0)\subset T(0) \\ \mathcal{D}(B)\supset\mathcal{D}(T)}} \sigma_{w,S}(T + B)$.

Proposition 4.4. *Let $T \in CR(X)$. Then:*

(i) If $0 < \varepsilon_1 < \varepsilon_2$ then $\sigma_{w,S}(T) \subset \sigma_{w,\varepsilon_1,S}(T) \subset \sigma_{w,\varepsilon_2,S}(T)$.

(ii) For $\varepsilon > 0$, $\sigma_{w,\varepsilon,S}(T) \subset \sigma_{\varepsilon,S}(T)$.

(iii) $\bigcap_{\varepsilon>0} \sigma_{w,\varepsilon,S}(T) = \sigma_{w,S}(T)$.

Proof. (i) If $\lambda \notin \sigma_{w,\varepsilon_2,S}(T)$ then by Theorem 4.2 for all continuous linear relations $B \in LR(X)$ such that $\mathcal{D}(B) \supset \mathcal{D}(T)$, $B(0) \subset T(0)$ and $\|B\| < \varepsilon_2$, we have $T + B - \lambda S \in \Phi(X)$ and $i(T + B - \lambda S) = 0$. Hence for all continuous linear relations $B \in LR(X)$ such that $\mathcal{D}(B) \supset \mathcal{D}(T)$, $B(0) \subset T(0)$ and $\|B\| < \varepsilon_1$, we have $T + B - \lambda S \in \Phi(X)$ and $i(T + B - \lambda S) = 0$. Then $\lambda \notin \sigma_{\varepsilon_1,S}(T)$.

On other hand, if $\lambda \notin \sigma_{w,\varepsilon_1,S}(T)$ then for all continuous linear relations $B \in LR(X)$ such that $\mathcal{D}(B) \supset \mathcal{D}(T)$, $B(0) \subset T(0)$ and $\|B\| < \varepsilon_1$, we have $T + B - \lambda S \in \Phi(X)$ and $i(T + B - \lambda S) = 0$. In particular $B = 0$. Then $\lambda \notin \sigma_{w,S}(T)$.

(ii) $\sigma_{w,\varepsilon,S}(T) = \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon,S}(T + K) \subset \sigma_{\varepsilon,S}(T + K)$ for all $K \in \mathcal{K}_T(X)$. In particular, $K = 0$.

(iii) From (i), $\sigma_{w,S}(T) \subset \sigma_{w,\varepsilon,S}(T)$ for all $\varepsilon > 0$. Then $\sigma_{w,S}(T) \subset \bigcap_{\varepsilon>0} \sigma_{w,\varepsilon,S}(T)$.

Conversely, If $\lambda \notin \sigma_{w,S}(T)$ then $T - \lambda S \in \Phi(X)$ and $i(T - \lambda S) = 0$. Therefore $R(T - \lambda S)$ is closed and by [7, Theorem III 4.2] $T - \lambda S$ is open, so $\gamma(T - \lambda S) > 0$.

Let ε be such that $0 < \varepsilon \le \gamma(T - \lambda S)$ and let D be a single valued linear relation in $LR(X)$ such that $\mathcal{D}(D) \supset \mathcal{D}(T)$ and $\|D\| < \varepsilon \le \gamma(T - \lambda S)$.

By [7, Theorem III 4.6 (d)], $\gamma(T-\lambda S) = \gamma((T-\lambda S)')$ then $\|D\| < \varepsilon \le \gamma((T-\lambda S)')$; so from [7, Theorem V 5.12], $T+D-\lambda S \in \Phi_-(X)$. Now by [7, Theorem III 7.4], $\alpha(T+$

$D - \lambda S) \leq \alpha(T - \lambda S) < \infty$, and hence by [7, Proposition V 5.13] $T + D - \lambda S \in \Phi_+(X)$, so $T + D - \lambda S \in \Phi(X)$. Furthermore from [7, Corollary V 15.7], we have $i(T + D - \lambda S) = i(T - \lambda S) = 0$. Hence by Theorem 4.2, $\lambda \notin \sigma_{w,\varepsilon,S}(T)$ and s, $\lambda \notin \bigcap_{\varepsilon>0} \sigma_{w,\varepsilon,S}(T)$. It follows that $\bigcap_{\varepsilon>0}\sigma_{w,\varepsilon}(T) \subset \sigma_{w,S}(T)$. □

THEOREM 4.5. *$\sigma_{w,\varepsilon,S}(T)$ is a closed set.*

Proof. Let $\lambda \in \rho_{w,\varepsilon,S}(T)$, then $\lambda \notin \sigma_{w,\varepsilon,S}(T)$. Let D be a single valued continuous linear relation such that $\mathcal{D}(D) \supset \mathcal{D}(T)$ and $\|D\| < \varepsilon$. Hence by Theorem 4.2, $T + D - \lambda S \in \Phi(X)$ and $i(T + D - \lambda S) = 0$. So, $R(T + D - \lambda S)$ is closed and from [2, Lemma 3.5], we have $T + D - \lambda S$ is closed. Then by [7, Theorem III 4.4], $T + D - \lambda S$ is open and hence $\gamma(T + D - \lambda S) > 0$.

Furthermore by [7, Theorem III 4.6 (d)], $\gamma(T + D - \lambda S) = \gamma((T + D - \lambda S)')$.

Let $r > 0$ such that $r < \frac{\gamma(T+D-\lambda S)}{\|S\|}$, let $\mu \in B_f(\lambda, r)$; then $\|(\lambda - \mu)S\| \leq r\|S\| < \gamma(T + D - \lambda S) = \gamma((T + D - \lambda S)')$. Therefore by [7, Theorem V 5.12], $T + D - \lambda S + (\lambda - \mu)S = T + D - \mu S \in \Phi_-(X)$, also by [7, Theorem III 7.4], $\alpha(T + D - \mu S) \leq \gamma(T + D - \lambda S) < \infty$ as $T + D - \lambda S \in \Phi(X)$. Then by [7, Proposition V 5.13], $T + D - \mu S \in \Phi_+(X)$, hence $T + D - \mu S \in \Phi(X)$.

On the other hand $i(T + D - \mu S) = i(T + D - \lambda S) = 0$. Then for all continuous single valued relations $D \in LR(X)$ such that $\mathcal{D}(D) \supset \mathcal{D}(T)$ and $\|D\| < \varepsilon$, we have $T + D - \mu S \in \Phi(X)$ and $i(T + D - \mu S) = 0$. By Theorem 4.2, $\mu \in \rho_{w,\varepsilon,S}(T)$ and thus there exists $r > 0$ such that $B_f(\lambda, r) \subset \rho_{w,\varepsilon,S}(T)$, Hence $\rho_{w,\varepsilon,S}(T)$ is an open set. □

THEOREM 4.6. $\sigma_{w,\varepsilon,S}(T) = \sigma_{w,\varepsilon,S'}(T')$.

Proof. Let $K \in \mathcal{K}_T(X)$; then K is compact, therefore it is continuous and by [7, Corollary V 2.3 and Proposition V 5.3], K' is compact. Moreover, by Lemma 2.6, $K'(0) \subset T'(0)$ and $\mathcal{D}(K') \supset \mathcal{D}(T')$. Hence $\mathcal{K}_T(X) \subset \{K \in LR(X)$ such that K is continuous and $K' \in \mathcal{K}_{T'}(X')\}$. Now, let $K \in \{K \in LR(X)$ such that K is continuous $K' \in \mathcal{K}_{T'}(X')\}$; by [7, Proposition V 5.3] K is compact.

From [7, Proposition III 1.4 (b)] $K'(0) = \mathcal{D}(K)^\perp \subset T'(0) = \mathcal{D}(T)^\perp$; then $\mathcal{D}(K) \supset \mathcal{D}(T)$. Also, by [7, Proposition III 1.4 (d)], $K(0) \subset \overline{K}(0) = \mathcal{D}(K')^\top \subset \mathcal{D}(T')^\top = T(0)$ as T is closed.

Then $K \in \mathcal{K}_T(X)$. Hence,

$$\mathcal{K}_T(X) = \{K \in LR(X) \text{ such that } K \text{ is continuous and } K' \in \mathcal{K}_{T'}(X')\}. \quad (4)$$

On the other hand for $K \in \mathcal{K}_T(X)$, $T+K$ is closed. Using Theorem 3.9, $\sigma_{\varepsilon,S}(T+K) = \sigma_{\varepsilon,S'}((T + K)')$. But $\mathcal{D}(K) \supset \mathcal{D}(T)$ and K is continuous; then by [7, Proposition III 1.5 (b)] $(T + K)' = T' + K'$ hence $\sigma_{\varepsilon,S}(T + K) = \sigma_{\varepsilon,S'}(T' + K')$ for all $K \in \mathcal{K}_T(X)$. Therefore by using (4) we have

$$\sigma_{w,\varepsilon,S}(T) = \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon,S}(T + K) = \bigcap_{\substack{K'\in\mathcal{K}_{T'}(X') \\ K\in LR(X) \\ K \text{ continuous}}} \sigma_{\varepsilon,S'}(T' + K')$$

$$\supset \bigcap_{\substack{K'\in\mathcal{K}_{T'}(X') \\ K\in LR(X)}} \sigma_{\varepsilon,S'}(T' + K') = \sigma_{w,\varepsilon,S'}(T').$$

Let
$$\mathcal{O} := \bigcap_{\substack{K' \in \mathcal{K}_{T'}(X') \\ K \in LR(X) \\ K \text{ continuous}}} \sigma_{\varepsilon,S'}(T'+K'),$$
then
$$\mathcal{O} = \sigma_{w,\varepsilon,S}(T) \supset \sigma_{w,\varepsilon,S'}(T'). \tag{5}$$
On the other hand, let $\lambda \in \mathcal{O}$; hence for all $K \in LR(X)$, K continuous such that $K' \in \mathcal{K}_{T'}(X)$, $\lambda \in \sigma_{\varepsilon,S'}(T'+K')$. Let $K \in LR(X)$ such that $K' \in \mathcal{K}_{T'}(X)$; by [7, Corollary V.5.15] $\tilde{K}$ is compact; but $G(\tilde{K})$ is the completion of $G(K)$ in the complete space X, hence $G(\tilde{K}) = \overline{G(K)} = G(\overline{K})$. Hence $\tilde{K} = \overline{K}$ is compact and hence continuous. Furthermore, $G(\overline{K}') = G(-\overline{K}^{-1})^{\perp} = \overline{G(-K^{-1})}^{\perp} = G(-K^{-1})^{\perp} = G(K')$. Hence $\overline{K}' = K'$. Thus $\overline{K}$ in $LR(X)$, $\overline{K}$ is continuous and $\overline{K}' = K' \in \mathcal{K}_{T'}(X')$; since $\lambda \in \mathcal{O}$ then $\lambda \in \sigma_{\varepsilon,S'}(T'+\overline{K}') = \sigma_{\varepsilon,S'}(T'+K')$. We conclude that if $\lambda \in \mathcal{O}$, for all $K \in LR(X)$ such that $K' \in \mathcal{K}_{T'}(X)$, then $\lambda \in \sigma_{\varepsilon,S'}(T'+K')$. Then $\lambda \in \bigcap_{\substack{K' \in \mathcal{K}_{T'}(X') \\ K \in LR(X)}} \sigma_{\varepsilon,S'}(T'+K') = \sigma_{w,\varepsilon,S'}(T')$. Hence $\mathcal{O} \subset \sigma_{w,\varepsilon,S'}(T')$. Using (5) we have $\mathcal{O} = \sigma_{w,\varepsilon,S}(T) = \sigma_{w,\varepsilon,S'}(T')$. □

COROLLARY 4.7. *From Proposition 4.4 (iii), we obtain that* $\sigma_{w,S}(T) = \sigma_{w,S'}(T')$.

DEFINITION 4.8. Let $P \in LR(X,Y)$ be continuous where X, Y are normed spaces.
(i) P is called a Fredholm perturbation if $T+P \in \Phi(X,Y)$ whenever $T \in \Phi(X,Y)$.

(ii) P is called an upper semi-Fredholm perturbation if $T+P \in \Phi_+(X,Y)$ whenever $T \in \Phi_+(X,Y)$.

(iii) P is called a lower semi-Fredholm perturbation if $T+P \in \Phi_-(X,Y)$ whenever $T \in \Phi_-(X,Y)$.

The sets of Fredholm, upper and lower semi-Fredholm perturbations are denoted by $\mathcal{P}(X,Y)$, $\mathcal{P}_+(X,Y)$, and $\mathcal{P}_-(X,Y)$, respectively. If $X = Y$, we denote by $\mathcal{P}(X) := \mathcal{P}(X,X)$, $\mathcal{P}_+(X) := \mathcal{P}_+(X,X)$ and $\mathcal{P}_-(X) := \mathcal{P}_-(X,X)$.

We denote also the set $\mathcal{P}_T(X,Y) := \{P \in \mathcal{P}(X,Y)$ such that $P(0) \subset T(0)$ and $\mathcal{D}(P) \supset \mathcal{D}(T)\}$, $\mathcal{P}_{+_T}(X,Y) := \{P \in \mathcal{P}_+(X,Y)$ such that $P(0) \subset T(0)$ and $\mathcal{D}(P) \supset \mathcal{D}(T)\}$ and $\mathcal{P}_{-_T}(X,Y) := \{P \in \mathcal{P}_-(X,Y)$ such that $P(0) \subset T(0)$ and $\mathcal{D}(P) \supset \mathcal{D}(T)\}$. We write $P_{+_T}(X) := P_{+_T}(X,X)$, $P_{-_T}(X) := P_{-_T}(X,X)$ and $P_T(X) := P_T(X,X)$.

In general, by [2, Lemma 3.6], we have $\mathcal{K}_T(X,Y) \subset \mathcal{P}_{+_T}(X,Y) \subset \mathcal{P}_T(X,Y)$ and $\mathcal{K}_T(X,Y) \subset \mathcal{P}_{-_T}(X,Y) \subset \mathcal{P}_T(X,Y)$.

THEOREM 4.9. $\sigma_{w,\varepsilon,S}(T) = \bigcap_{P \in \mathcal{P}_T(X)} \sigma_{\varepsilon,S}(T+P)$.

Proof. Let $\mathcal{O} := \bigcap_{P \in \mathcal{P}_T(X)} \sigma_{\varepsilon,S}(T+P)$. Since $\mathcal{K}_T(X) \subset \mathcal{P}_T(X)$, we infer that $\mathcal{O} \subset \sigma_{w,\varepsilon,S}(T)$. Conversely, let $\lambda \notin \mathcal{O}$; then there exists $P \in \mathcal{P}_T(X)$ such that $\lambda \notin \sigma_{\varepsilon,S}(T+P)$, or P is continuous. Then, $T+P$ is closed. Thus, by Theorem 3.4 we see that $\lambda \in \rho_S(T+B+P)$ for all continuous linear relations $B \in LR(X)$ such that $B(0) \subset (T+P)(0) = T(0)$, $D(B) \supset \mathcal{D}(T+P) = \mathcal{D}(T) \cap \mathcal{D}(P) = \mathcal{D}(T)$ and $\|B\| < \varepsilon$. But $(B+P-\lambda S)(0) \subset T(0)$, $\mathcal{D}(B+P-\lambda S) = \mathcal{D}(B) \cap \mathcal{D}(P) \supset \mathcal{D}(T)$ and T is closed, and $T+B+P-\lambda S$ is closed. Hence $T+B+P-\lambda S$ is injective and surjective. Then we have $T+B+P-\lambda S \in \Phi(X)$ and $i(T+B+P-\lambda S) = 0$. Since $-P \in \mathcal{P}(X)$, $-P(0) = P(0) \subset (T+B+P-\lambda S)(0)$ and $\mathcal{D}(-P) = \mathcal{D}(P) \supset \mathcal{D}(T+B+P-\lambda S) = \mathcal{D}(T) \cap \mathcal{D}(B) \cap \mathcal{D}(P) \cap \mathcal{D}(S)$, $-P$ is in $\mathcal{P}_{T+B+P-\lambda S}(X)$. Hence for all continuous linear relations $B \in LR(X)$ such that $B(0) \subset T(0)$, $D(B) \supset \mathcal{D}(T)$ and $\|B\| < \varepsilon$, $T+B-\lambda S \in \Phi(X)$ and $i(T+B-\lambda S) = 0$. Finally, Theorem 4.2 shows that $\lambda \notin \sigma_{w,\varepsilon,S}(T)$. □

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