MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
72, 2 (2020), 124–137
June 2020

research paper
оригинални научни рад

THE RICCI-BOURGUIGNON FLOW ON HEISENBERG AND QUATERNION LIE GROUPS

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Abstract. In this paper, we study the Ricci-Bourguignon flow on higher dimensional classical Heisenberg nilpotent Lie groups and construct a solution of this flow on Heisenberg and quaternion nilpotent Lie groups. In the end, we investigate the deformation of spectrum and length spectrum on compact nilmanifolds obtained of Heisenberg and quaternion nilpotent Lie groups.

1. Introduction and preliminaries

Geometric flow is an evolution of a geometric structure under a differential equation associated with some curvature and it is an important topic in many branches of mathematics and physics. A geometric flow is related to dynamical systems in the infinite-dimensional space of all metrics on a given manifold.

Let M be an n-dimensional manifold with a Riemannian metric g_0, the family $g(t)$ of Riemannian metrics on M is called a Ricci-Bourguignon flow when it satisfies the equations

$$\frac{d}{dt}g(t) = -2Ric(g(t)) + 2\rho R(g(t))g(t) = -2(Ric - \rho Rg), \quad g(0) = g_0 \tag{1}$$

where Ric is the Ricci tensor of $g(t)$, R is the scalar curvature and ρ is a real constant. In fact the Ricci-Bourguignon flow is a system of partial differential equations which was introduced by Bourguignon for the first time in 1981 (see [3]). For closed manifolds, short time existence and uniqueness for solution to the Ricci-Bourguignon flow on $[0, T)$ have been shown by Catino et al. in [5] for $\rho < \frac{1}{2(n-1)}$. When $\rho = 0$, the Ricci-Bourguignon flow is the Ricci flow. Also, when $\rho = \frac{1}{2}$, $\rho = \frac{1}{n}$ and $\rho = \frac{1}{2(n-1)}$, the tensor $Ric-\rho Rg$ is the Einstein tensor, the traceless Ricci tensor and the Schouten tensor, respectively.

A Riemannian metric g on the Lie group N is left invariant if the left translations L_p's are isometries for all $p \in N$. We will use $\langle \cdot, \cdot \rangle$ to denote both the inner product

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2010 Mathematics Subject Classification: 53C44, 22E25
Keywords and phrases: Ricci-Bouguignon flow; Heisenberg type; Lie group.

on $\mathcal{N} = T_e N$ and the corresponding left invariant metric on N. Let $\mathcal{Z}$ be the center of $\mathcal{N}$; we denote the orthogonal complement of $\mathcal{Z}$ in $\mathcal{N}$ by $\mathcal{V}$ and we write $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$. Define a linear transformation $j : \mathcal{Z} \to SO(\mathcal{V})$ by $j(Z)X = (adX)^*Z$ for $Z \in \mathcal{Z}$ and $X \in \mathcal{V}$. Equivalently, for each $Z \in \mathcal{Z}$, $j(Z) : \mathcal{V} \to \mathcal{V}$ is the skew-symmetric linear transformation defined by $\langle (adX)^*Z, Y\rangle = \langle Z, (adX)Y\rangle$, for all $X, Y \in \mathcal{V}$. Here $adX(Y) = [X, Y]$ for all $X, Y \in \mathcal{N}$, and $(adX)^*$ denotes the (metric) adjoint of adX. A 2-step nilpotent Lie algebra $\mathcal{N}$ is said to be of Heisenberg type if $j(Z)^2 = -|Z|^2 Id$ for all $Z \in \mathcal{Z}$, for instance, the classical Heisenberg Lie group H_n and quaternion Lie group Q_n with special metrics are of Heisenberg type (see [7,8,12]).

The collection of lengths of smoothly closed geodesics in Riemannian manifold (M, g) are called length spectrum and the collection of eigenvalues of the Laplace operator are called the Laplace spectrum of M. A major open question in spectral geometry is whether there can exist examples of two Riemannian manifolds with different periods in the length spectrum in which their Laplace spectra coincide. In [6], Verdière using the heat kernel showed that the Laplace spectrum determines the length spectrum. In [4,13,14], it was shown that two closed Riemannian surface have same Laplace spectra if and only if they have the same length spectrum.

Lauret in [15], studied the Ricci soliton on homogenous nilmanifolds and then Payne in [17,18] investigated the Ricci flow and the soliton metrics on nilmanifollds and nilpotent Lie groups. Also, Williams in [19] funded the explicit solution for the Ricci flow on some nilpotent Lie groups, for instance, the classical Heisenberg Lie group H_n of dimension $(2n + 1)$. The author and Razavi studied in [1] the eigenvalue variations of Heisenberg and quaternion Lie groups under the Ricci flow and investigated the deformation of some characteristics of compact nilmanifolds $\Gamma \backslash N$ under the Ricci flow, where N is a simply connected 2-step nilpotent Lie group with a left invariant metric and Γ is a discrete cocompact subgroup of N, in particular Heisenberg and quaternion Lie groups.

Motivated by the above works, in this paper, the Ricci-Bourguignon flow on higher dimensional classical Heisenberg and quaternion nilpotent Lie groups will be investigated and specially, the deformation of spectrum and length spectrum of compact nilmanifold will be found.

1.1 Curvature of Lie groups

We recall some properties about the geometry of Lie groups with left-invariant metrics, and derive the formula for the Ricci tensor (see [2,12,16]). Suppose that $\langle \cdot, \cdot \rangle$ is a left-invariant metric on a Lie group N, which is equivalent to an inner product on the Lie algebra $\mathcal{N}$. Let ∇ denote the Levi-Civita connection for the metric, and let $X, Y, Z \in \mathcal{N}$. We shall recall the following useful theorems and propositions about the Ricci tensor of a Lie group (see [2]).

Proposition 1.1. *Let $\langle \cdot, \cdot \rangle$ be a left-invariant metric on a Lie group N and ∇ the connection for this metric. For $X, Y, Z, W \in \mathcal{N}$, we have:*

(i) $\nabla_X Y = \frac{1}{2}\left\{(adX)\, Y - (adX)^*\, Y - (adY)^*\, X\right\}$,

(ii) $\langle R(X, Y)Z, W\rangle = \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle$.

Besides, the maps $(X,Y)\mapsto (adX)\,Y$, $\quad (X,Y)\mapsto (adX)^*\,Y$, from $\mathcal{N}\times\mathcal{N}$ to $\mathcal{N}$ are bilinear maps. We define

$$U:\mathcal{N}\times\mathcal{N}\to\mathcal{N}\quad U(X,Y)=-\frac{1}{2}\left\{(adX)^*\,Y+(adY)^*\,X\right\},$$

which is bilinear and symmetric.

Proposition 1.2. *The Riemannian curvature tensor on N is given by*

$$\begin{aligned}4\langle R(X,Y)Z,W\rangle &= 2\langle [X,Y]\,,[Z,W]\rangle+\langle [X,Z]\,,[Y,W]\rangle-\langle [X,W]\,,[Y,Z]\rangle\\ &\quad-\langle [[X,Y],Z],W\rangle+\langle [[X,Y],W],Z\rangle-\langle [[Z,W],X],Y\rangle\\ &\quad+\langle [[Z,W],Y],X\rangle+4\langle U(X,Z),U(Y,W)\rangle-4\langle U(X,W),U(Y,Z)\rangle.\end{aligned}$$

In particular,

$$\begin{aligned}\langle R(X,Y)Z,X\rangle &= \frac{1}{4}\left\|(adX)^*\,Y+(adY)^*\,X\right\|^2-\langle (adX)^*\,X,(adY)^*\,Y\rangle\\ &\quad-\frac{3}{4}\left\|[X,Y]\right\|^2-\frac{1}{2}\langle [[X,Y],Y],X\rangle-\frac{1}{2}\langle [[Y,X],X],Y\rangle.\end{aligned}$$

Now, suppose that $\{e_i\}$ is a basis for the Lie algebra $\mathcal{N}$; then we write:

$$(ade_i)\,e_j=C_{ij}^k e_k,\quad (ade_i)^*\,e_j=a_{ij}^k e_k,\quad \langle e_i,e_j\rangle=g_{ij}.$$

This yields the following corollary.

Corollary 1.3. *(i)* $a_{ij}^k=C_{il}^m g_{jm}g^{kl}$,

(ii) If $\nabla_{e_i}e_j=\gamma_{ij}^k e_k$ *then* $\gamma_{ij}^k=\frac{1}{2}g^{kl}\left(C_{ij}^m g_{lm}-C_{il}^m g_{jm}-C_{jl}^m g_{im}\right)$.

(iii) The components of the Riemann curvature tensor satisfy

$$\begin{aligned}4R_{ijkl} &= 2C_{ij}^pC_{kl}^q g_{pq}+C_{ik}^pC_{jl}^q g_{pq}-C_{il}^pC_{jk}^q g_{pq}-C_{ij}^pC_{pk}^q g_{ql}+C_{ij}^pC_{pl}^q g_{pk}-C_{kl}^pC_{pi}^q g_{qj}\\ &\quad+C_{kl}^pC_{pj}^q g_{qi}+(a_{ik}^p+a_{ki}^p)\left(a_{jl}^q+a_{lj}^q\right)g_{pq}-(a_{il}^p+a_{li}^p)\left(a_{jk}^q+a_{kj}^q\right)g_{pq}.\end{aligned}$$

(iv) The components of the Ricci curvature tensor satisfy

$$\begin{aligned}4R_{ij} &= \Big\{2C_{ki}^pC_{jm}^q g_{pq}+C_{kj}^pC_{im}^q g_{pq}-C_{km}^pC_{ij}^q g_{pq}-C_{ki}^pC_{pj}^q g_{qm}\\ &\quad+C_{ki}^pC_{pm}^q g_{qj}-C_{jm}^pC_{pk}^q g_{qi}+C_{jm}^pC_{pj}^q g_{qk}\\ &\quad+\left(a_{jk}^p+a_{kj}^p\right)(a_{im}^q+a_{mi}^q)\,g_{pq}-(a_{km}^p+a_{mk}^p)\left(a_{ij}^q+a_{ji}^q\right)\,g_{pq}\}\,g^{km}.\end{aligned}$$

1.2 Heisenberg Lie group

We now recall the construction and properties of the higher-dimensional, classical Heisenberg Lie group. Let H_n be a $(2n+1)$-dimensional Heisenberg Lie group. Let $x=(x^1,\dots,x^n)$, $y=(x^{n+1},\dots,x^{2n})$. If $q=(x,y,z)\in H_n$ and $q=(x',y',z')\in H_n$ then the group multiplication is $(x,y,z)\circ(x',y',z')=(x+x',y+y',z+z'+x\cdot y')$, where $x\cdot y'$ is the usual inner product of vectors $x\in\mathbb{R}^n$ and $y'\in\mathbb{R}^n$. With respect to this multiplication, we have the following frame of left invariant vector fields,

$$e_i=\partial_i,\quad e_{n+i}=\partial_{n+i}+x^i\partial_{2n+1},\quad e_{2n+1}=\partial_{2n+1},\quad \text{for all } 1\le i\le n,$$

and the only nontrivial Lie bracket relation is $[e_i, e_{n+i}] = e_{2n+1}$, for all $1 \leq i \leq n$. The dual coframe is $\theta^i = dx^i, \quad \theta^{n+i} = dx^{n+i}, \quad \theta^{2n+1} = dx^{2n+1}$, for all $1 \leq i \leq n$.

Set $\mathcal{V} = \text{span}\{e_i, e_{n+i} | \ 1 \leq i \leq n\}$ and $\mathcal{Z} = \text{span}\{e_{2n+1}\}$. With the above multiplication $\mathcal{V} \cup \mathcal{Z}$ is an orthonormal basis for $\mathcal{H}_n$, then $\mathcal{H}_n = \mathcal{V} \oplus \mathcal{Z}$ and the Heisenberg Lie group is of Heisenberg type.

1.3 The Ricci-Bourguignon flow on the Heisenberg Lie group

In this section, we study solutions of the Ricci-Bourguignon flow (1) starting at some initial metric g_0 on Heisenberg Lie group. Any one-parameter family of left invariant metrics $g(t)$ on H_n which is a solution of the Ricci-Bourguignon flow, can be written as $g(t) = g_{IJ}(t)\theta^I \otimes \theta^J$.

In [19], Williams, using Propositions 1.1, 1.2 and Corollary 1.3, showed that the Ricci tensor of H_n is as follows:

$$\begin{cases} R_{ij}(t) = -\frac{1}{2}g^{i+n,j+n}(t)g_{NN}(t) + \frac{1}{2}g_{iN}(t)g_{jN}(t)\sum, & \text{if } 1 \leq i,j \leq n; \\ R_{i,j+n}(t) = \frac{1}{2}g^{i+n,j}(t)g_{NN}(t) + \frac{1}{2}g_{iN}(t)g_{j+n,N}(t)\sum, & \text{if } 1 \leq i,j \leq n; \\ R_{iN}(t) = \frac{1}{2}g_{iN}(t)g_{NN}(t)\sum, & \text{if } 1 \leq i \leq n; \\ R_{i+n,j+n}(t) = -\frac{1}{2}g^{ij}(t)g_{NN}(t) + \frac{1}{2}g_{i+n,N}(t)g_{j+n,N}(t)\sum, & \text{if } 1 \leq i,j \leq n; \\ R_{i+N,N}(t) = \frac{1}{2}g_{i+n,N}(t)g_{NN}(t)\sum, & \text{if } 1 \leq i \leq n; \\ R_{NN}(t) = \frac{1}{2}g^2_{NN}(t)\sum, \end{cases}$$

where $\displaystyle\sum = \sum_{k,m=1}^{n} g^{km}(t)g^{k+n,m+n}(t) - \sum_{k=1}^{n}\sum_{m=n+1}^{2n} g^{km}(t)g^{k+n,m-n}(t)$, and $N = 2n+1$.

We assume that the Riemannian metric initial is diagonal. From now on, we only use single subscripts for the metric components: $g_1(t), \ldots, g_N(t)$. This implies that the Ricci tensor stays diagonal under the Ricci-Bourguignon flow, and the Ricci tensor is as follows:

$$\begin{cases} R_{ij}(t) = \begin{cases} -\frac{1}{2}g^{i+n}(t)g_N(t) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, & \text{if } 1 \leq i,j \leq n; \\ R_{i,j+n}(t) = 0, & \text{if } 1 \leq i,j \leq n; \\ R_{iN}(t) = 0, & \text{if } 1 \leq i \leq n; \\ R_{i+n,j+n}(t) = \begin{cases} -\frac{1}{2}g^{i}(t)g_N(t) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, & \text{if } 1 \leq i,j \leq n; \\ R_{i+N,N}(t) = 0, & \text{if } 1 \leq i \leq n; \\ R_{NN}(t) = \frac{1}{2}g^2_{N}(t)\sum, \end{cases}$$

where $\displaystyle\sum = \sum_{k=1}^{n} \frac{1}{g_k(t)g_{k+n}(t)}$.

By direct computation we obtain the scalar curvature as follows: $R(t) = -\frac{1}{2}g_N\sum$. Then the Ricci-Bourguignon flow equation on H_n with a diagonal left-invariant metric

g_0 has the following form

$$\begin{cases} \frac{d}{dt}g_i(t) = \frac{g_N(t)}{g_{i+n}(t)} - \rho g_i(t) g_N(t)\sum, & \text{for } 1 \leq i \leq n; \\ \frac{d}{dt}g_{i+n}(t) = \frac{g_N(t)}{g_i(t)} - \rho g_{i+n}(t) g_N(t)\sum, & \text{for } 1 \leq i \leq n; \\ \frac{d}{dt}g_N(t) = -(1+\rho) g_N^2(t)\sum. \end{cases} \tag{2}$$

Let $g_1, g_2, \dots, g_{2n}, g_N$ be a solution of the Ricci-Bourguignon flow. As diagonal components of a metric, they are positive function of t.

THEOREM 1.4. *Consider the Heisenberg Lie group H_n with a diagonal left-invariant metric g_0. Let $g(t)$ be a solution to the Ricci-Bourguignon flow with initial metric g_0; then*
(i) $\frac{d}{dt}\frac{g_i(t)}{g_{i+n}(t)} = 0$, *if* $1 \leq i \leq n$*;*

(ii) $\frac{d}{dt}\big(g_1(t)\dots g_n(t) g_N^{\frac{1-n\rho}{1+\rho}}(t)\big) = \frac{d}{dt}\big(g_{1+n}(t)\dots g_{2n}(t) g_N^{\frac{1-n\rho}{1+\rho}}(t)\big) = 0,$

(iii) If $\rho < 0$ *and* $G_N(t) = \int_0^t g_N(t)dt$ *then* $\lim_{t\to+\infty} G_N(t) = +\infty.$

(iv) Moreover, if $g_i(0)g_{n+i}(0) = g_1(0)g_{1+n}(0)$*, for* $1 \leq i \leq n$ *then a solution* $g(t)$ *has the following form*

$$\begin{cases} g_j(t) = g_j(0)\Big(1+bt\Big)^{\frac{1-n\rho}{n+2-n\rho}}, & \text{if} \quad 1 \leq j \leq 2n \\ g_N(t) = g_N(0)\Big(1+bt\Big)^{\frac{n+n\rho}{n\rho-n-2}} \end{cases} \tag{3}$$

where $b = (n+2-n\rho)\frac{g_N(0)}{g_1(0)g_{1+n}(0)}$.

Proof. (i) Using (2) and direct computation we have $\frac{d}{dt}\frac{g_i(t)}{g_{i+n}(t)} = 0$.

(ii) By differentiation with respect to variable time t and using (2) we obtain

$$\begin{aligned} &\frac{d}{dt}\big(g_1(t)\dots g_n(t)(g_N(t))^{\frac{1-n\rho}{1+\rho}}\big) \\ &\quad = \Big(\sum_{k=1}^{n}\frac{1}{g_k(t)}\frac{dg_k(t)}{dt} + \frac{1-n\rho}{1+\rho}\frac{1}{g_N(t)}\frac{dg_N(t)}{dt}\Big)g_1(t)\dots g_n(t)g_N^{\frac{1-n\rho}{1+\rho}} \\ &\quad = \Big(\sum_{k=1}^{n}\big(\frac{g_N(t)}{g_k(t)g_{n+k}(t)} - \rho g_N(t)\sum\big)\Big)g_1(t)\dots g_n(t)(g_N(t))^{\frac{1-n\rho}{1+\rho}} \\ &\quad\quad - \Big((1-n\rho)g_N(t)\sum\Big)g_1(t)\dots g_n(t)(g_N(t))^{\frac{1-n\rho}{1+\rho}} = 0, \end{aligned}$$

the part (i) implies that $\frac{g_i(t)}{g_{i+n}(t)}$ is constant for $1 \leq i \leq n$, so we can set $A_i = \frac{g_i(t)}{g_{i+n}(t)} = \frac{g_i(0)}{g_{i+n}(0)}$, therefore $g_{i+n}(t) = \frac{g_i(t)}{A_i}$. Hence $\frac{d}{dt}\big(g_1(t)\dots g_n(t)(g_N(t))^{\frac{1-n\rho}{1+\rho}}\big) = 0$ results in $\frac{d}{dt}\big(g_{1+n}(t)\dots g_{2n}(t)g_N^{\frac{1-n\rho}{1+\rho}}(t)\big) = 0$.

(iii) For $\rho < 0$ the equations (2) implies that g_j, $1 \leq j \leq 2n$ is an increasing function, so $\sum$ is positive and decreasing. Since $g_N(t)$ is positive, then last equation in (2) yields $\frac{d}{dt}g_N(t) = -(1+\rho)g_N^2(t)\sum \geq -g_N^2(t)\sum \geq -\sum(0)g_N^2(t)$, which by direct computation results in $g_N(t) \geq \frac{1}{\sum(0)t+g_N^{-1}(0)}$; by this it holds that

$$\lim_{t\to+\infty} G_N(t) = \lim_{t\to+\infty} \int_0^t g_N(r)dr \geq \lim_{t\to+\infty} \int_0^t \frac{1}{\sum(0)r+g_N^{-1}(0)}dr = +\infty.$$

(iv) $g_j(t)$ and $g_N(t)$ for $1 \leq j \leq 2n$ given in (3) satisfy the Ricci-Bourguignon flow (2). □

Consider $\mathcal{Z} = \text{span } \{e_{2n+1}\}$, $\mathcal{V} = \text{span } \{e_1, e_2, \ldots, e_{2n}\}$, $\mathcal{H}_n = \mathcal{V} \oplus \mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathcal{H}_n$ and $\mathcal{V}$ is the orthogonal complement of $\mathcal{Z}$ in $\mathcal{H}_n$. If $Z = e_{2n+1}$ then $Z = e_{2n+1}$, $j(Z)e_i = e_{n+i}$, $j(Z)e_{n+i} = -e_i$. Hence

$$j(Z) = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad (j(aZ))^2 = \begin{bmatrix} -a^2 I_n & 0 \\ 0 & -a^2 I_n \end{bmatrix},$$

where I_n is an $n \times n$ identity matrix and it yields to $(j(aZ))^2 = -|aZ|^2 Id$, therefore $\mathcal{H}_n$ with this structure is of Heisenberg type.

Proposition 1.5. *Heisenberg type of Lie group H_n is not preserved along the Ricci-Bourguignon flow with solution* (3) *with additional condition $g_{2n+1}(0) = g_i(0)g_{n+i}(0)$, for $1 \leq i \leq n$.*

Proof. In $(\mathcal{H}_n, g_t = \langle , \rangle_t)$, we have

$$j(Z)e_i = \sum_{j=1}^{2n+1} \frac{\langle Z, [e_i, e_j]\rangle_t}{\langle e_j, e_j\rangle_t} e_j = \frac{|Z|_t^2}{g_{n+i}(t)} e_{n+i}, \quad \text{and} \quad j(Z)e_{n+i} = -\frac{|Z|_t^2}{g_i(t)} e_i, \ 1 \leq i \leq n.$$

Hence
$$j(Z) = A \begin{bmatrix} 0 & -B_1 \\ B_2 & 0 \end{bmatrix},$$

where $B_1 = diag(\frac{1}{g_1(t)}, \ldots, \frac{1}{g_n(t)})$, $B_2 = diag(\frac{1}{g_{n+1}(t)}, \ldots, \frac{1}{g_{2n}(t)})$, $A = |Z|_t^2$. Now for any real constant a we obtain

$$j(aZ) = aA \begin{bmatrix} 0 & -B_1 \\ B_2 & 0 \end{bmatrix}, \quad (j(aZ))^2 = -a^2 A^2 \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix},$$

where $D = diag(\frac{1}{g_1(t)g_{n+1}(t)}, \ldots, \frac{1}{g_n(t)g_{2n}(t)})$. But for $1 \leq i \leq n$ we have $g_i(0)g_{n+i}(0) = g_{2n+1}(0)$, then (3) results in $(j(Z))^2 = -\frac{1}{(n+2-n\rho)t+1} |Z|_t^2 I_{2n}$. So, Heisenberg type of H_n is not preserved under the evolution of the Ricci-Bourguignon flow. □

Definition 1.6. (i) Let $\mu(Z)$ denote the number of distinct eigenvalues of $j(Z)^2$ and $-\theta_1(Z)^2, -\theta_2(Z)^2, \ldots, -\theta_\mu(Z)^2$ denote the μ distinct eigenvalues of $j(Z)^2$, with the assumption that $0 \leq \theta_1(Z) < \theta_2(Z) < \ldots < \theta_\mu(Z)$.

(ii) A two-step nilpotent metric Lie algebra $(\mathcal{N}, \langle\cdot,\cdot\rangle)$ is Heisenberg-like if $[j(Z)X_m, X_m] \in span_{\mathbb{R}} Z$ for all $Z \in \mathcal{Z}$ and all $X_m \in W_m(Z), m = 1, \ldots, \mu(Z)$, where W_m denotes the invariant subspace of $j(Z)$ corresponding to $\theta_m(Z)$, $m = 1, \ldots, \mu(Z)$.

If $\mathcal{N}$ is of Heisenberg type, then for all $Z \in \mathcal{Z}$ and $X \in \mathcal{V}$, $[X, j(Z)X] = | X |^2 Z$. If $\mathcal{N}$ is Heisenberg-like, then for all $Z \in \mathcal{Z}$ and every $X_m \in W_m(Z), m = 1, \ldots, \mu(Z)$, $[X_m, j(Z)X_m] = (\frac{\theta_m(Z)|X_m|}{|Z|})^2 Z$. Therefore, with the above notation H_n under the evolution of the Ricci-Bourguignon flow from Heisenberg type convert to Heisenberg-like type.

1.4 Quaternion Lie groups

We now recall the construction of the higher-dimensional, classical quaternion Lie groups. Let $N = Q_n$ be a $(4n+3)-$dimensional quaternion group. Let $x = (x_{11}, x_{21}, \dots, x_{4n})$, $z = (z_1, z_2, z_3)$. Assume that $q = (x, z) \in N$ and $q' = (x', z') \in N$. Multiplication on N is defined as follows:

$$
\begin{aligned}
L_q(q') &= L_{(x,z)}(x', z') = (x, z) \circ (x', z') \\
&= \left(x + x', z_1 + z_1' + \frac{1}{2}(M_1 x, x'), z_2 + z_2' + \frac{1}{2}(M_2 x, x'), z_3 + z_3' + \frac{1}{2}(M_3 x, x') \right),
\end{aligned}
$$

$$
\text{where} \quad M_k = \begin{bmatrix} A_k & 0 & \cdots & 0 \\ 0 & A_k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{bmatrix}, \quad \text{for } = 1, 2, 3
$$

$$
\text{and} \quad A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},
$$

$(M_k x, x')$ is the usual inner product of vectors $M_k x \in \mathbb{R}^{4n}$ and $x' \in \mathbb{R}^{4n}$. With respect to this multiplication, we have the following vector fields

$$
\begin{aligned}
X_{1l} &= \frac{\partial}{\partial x_{1l}} + \frac{1}{2}\left(x_{2l}\frac{\partial}{\partial z_1} - x_{4l}\frac{\partial}{\partial z_2} - x_{3l}\frac{\partial}{\partial z_3} \right), \\
X_{2l} &= \frac{\partial}{\partial x_{2l}} + \frac{1}{2}\left(-x_{1l}\frac{\partial}{\partial z_1} - x_{3l}\frac{\partial}{\partial z_2} + x_{4l}\frac{\partial}{\partial z_3} \right), \\
X_{3l} &= \frac{\partial}{\partial x_{3l}} + \frac{1}{2}\left(x_{4l}\frac{\partial}{\partial z_1} + x_{2l}\frac{\partial}{\partial z_2} + x_{1l}\frac{\partial}{\partial z_3} \right), \\
X_{4l} &= \frac{\partial}{\partial x_{4l}} + \frac{1}{2}\left(-x_{3l}\frac{\partial}{\partial z_1} + x_{1l}\frac{\partial}{\partial z_2} + x_{2l}\frac{\partial}{\partial z_3} \right), \\
Z_m &= \frac{\partial}{\partial z_m},
\end{aligned}
$$

for $l = 1, 2, \dots, n$ and $m = 1, 2, 3$. The nonzero Lie brackets of vector fields are

$$
\begin{aligned}
&[X_{1l}, X_{2l}] = -Z_1, \qquad &&[X_{1l}, X_{3l}] = Z_3, \qquad &&[X_{1l}, X_{4l}] = Z_2, \\
&[X_{2l}, X_{3l}] = Z_2, \qquad &&[X_{2l}, X_{4l}] = -Z_3, \qquad &&[X_{3l}, X_{4l}] = -Z_1.
\end{aligned}
$$

Given the above definitions, Q_n is two-step nilpotent. Note the dual of the above vector fields are as follows: dx_{kl}, $k = 1, 2, 3, 4$, $\quad 1 \le l \le n$, and $\theta_r = dz_r - \frac{1}{2}(M_r x, dx)$, $r = 1, 2, 3$.

Set $\mathcal{V} = span\{X_{1l}, X_{2l}, X_{3l}, X_{4l} | 1 \le l \le n\}$, $\mathcal{Z} = span\{Z_1, Z_2, Z_3\}$. If we choose an inner product on Q_n such that $\mathcal{V} \cup \mathcal{Z}$ is an orthonormal basis for Q_n then quaternion

Lie group is of Heisenberg type.

1.5 The Ricci-Burguignon flow on quaternion Lie group

Assume that

$$e_i = X_{1i}, \quad e_{n+i} = X_{2i}, \quad e_{2n+i} = X_{3i}, \quad e_{3n+i} = X_{4i}, \quad i = 1, 2, \ldots, n,$$
$$e_{4n+r} = Z_r, \quad r = 1, 2, 3.$$

Let g be diagonal and $g_\alpha = g_{\alpha\alpha}$. With the above symbols and $[e_i, e_j] = C_{ij}^k e_k$, and Propositions 1.1, 1.2 and Corollary 1.3 we conclude that the Ricci tensor stay diagonal under the Ricci-Burguignon flow, also as follows

$$R_i = -\frac{1}{2}\left(\frac{g_{4n+1}}{g_{n+i}}+\frac{g_{4n+3}}{g_{2n+i}}+\frac{g_{4n+2}}{g_{3n+i}}\right), \quad R_{4n+1} = \frac{g_{4n+1}^2}{2}\sum_{i=1}^{n}\left(\frac{1}{g_i g_{n+i}}+\frac{1}{g_{2n+i}g_{3n+i}}\right),$$

$$R_{n+i} = -\frac{1}{2}\left(\frac{g_{4n+1}}{g_i}+\frac{g_{4n+2}}{g_{2n+i}}+\frac{g_{4n+3}}{g_{3n+i}}\right), \quad R_{4n+2} = \frac{g_{4n+2}^2}{2}\sum_{i=1}^{n}\left(\frac{1}{g_i g_{3n+i}}+\frac{1}{g_{n+i}g_{2n+i}}\right),$$

$$R_{2n+i} = -\frac{1}{2}\left(\frac{g_{4n+3}}{g_i}+\frac{g_{4n+2}}{g_{n+i}}+\frac{g_{4n+1}}{g_{3n+i}}\right), \quad R_{4n+3} = \frac{g_{4n+3}^2}{2}\sum_{i=1}^{n}\left(\frac{1}{g_i g_{2n+i}}+\frac{1}{g_{n+i}g_{3n+i}}\right),$$

$$R_{3n+i} = -\frac{1}{2}\left(\frac{g_{4n+2}}{g_i}+\frac{g_{4n+3}}{g_{n+i}}+\frac{g_{4n+1}}{g_{2n+i}}\right),$$

and $$R = -\frac{1}{2}\left(g_{4n+1}\sum_{i=1}^{n}\left(\frac{1}{g_i g_{n+i}} + \frac{1}{g_{2n+i}g_{3n+i}}\right)\right.$$
$$\left. + g_{4n+2}\sum_{i=1}^{n}\left(\frac{1}{g_i g_{3n+i}} + \frac{1}{g_{n+i}g_{2n+i}}\right) + g_{4n+3}\sum_{i=1}^{n}\left(\frac{1}{g_i g_{2n+i}} + \frac{1}{g_{n+i}g_{3n+i}}\right)\right).$$

Therefore, the Ricci-Bourguignon flow equation, $\frac{\partial g}{\partial t} = -2Ric + 2\rho R g$, on Q_n has the form

$$\begin{cases} \frac{d}{dt}g_i = \frac{g_{4n+1}}{g_{n+i}} + \frac{g_{4n+3}}{g_{2n+i}} + \frac{g_{4n+2}}{g_{3n+i}} - \rho g_i \sum', & \text{for } 1 \leq i \leq n; \\ \frac{d}{dt}g_{n+i} = \frac{g_{4n+1}}{g_i} + \frac{g_{4n+2}}{g_{2n+i}} + \frac{g_{4n+3}}{g_{3n+i}} - \rho g_{n+i} \sum', & \text{for } 1 \leq i \leq n; \\ \frac{d}{dt}g_{2n+i} = \frac{g_{4n+3}}{g_i} + \frac{g_{4n+2}}{g_{n+i}} + \frac{g_{4n+1}}{g_{3n+i}} - \rho g_{2n+i} \sum', & \text{for } 1 \leq i \leq n; \\ \frac{d}{dt}g_{3n+i} = \frac{g_{4n+2}}{g_i} + \frac{g_{4n+3}}{g_{n+i}} + \frac{g_{4n+1}}{g_{2n+i}} - \rho g_{3n+i} \sum', & \text{for } 1 \leq i \leq n; \\ \frac{d}{dt}g_{4n+1} = -\left(\sum_{i=1}^n \frac{g_{4n+1}^2}{g_i g_{n+i}} + \sum_{i=1}^n \frac{g_{4n+1}^2}{g_{2n+i}g_{3n+i}}\right) - \rho g_{4n+1} \sum', \\ \frac{d}{dt}g_{4n+2} = -\left(\sum_{i=1}^n \frac{g_{4n+2}^2}{g_i g_{3n+i}} + \sum_{i=1}^n \frac{g_{4n+2}^2}{g_{n+i}g_{2n+i}}\right) - \rho g_{4n+2} \sum', \\ \frac{d}{dt}g_{4n+3} = -\left(\sum_{i=1}^n \frac{g_{4n+3}^2}{g_i g_{2n+i}} + \sum_{i=1}^n \frac{g_{4n+3}^2}{g_{n+i}g_{3n+i}}\right) - \rho g_{4n+3} \sum'. \end{cases} \tag{4}$$

THEOREM 1.7. *Consider the quaternion Lie group Q_n with a diagonal left-invariant metric g_0. Let $g(t)$ be a solution to the Ricci-Bourguignon flow with initial metric g_0, then* *(i)* $\frac{d}{dt}\left(g_1(t)g_2(t)\dots g_{4n}(t)\big(g_{4n+1}(t)g_{4n+2}(t)g_{4n+3}(t)\big)^{\frac{2(1-2n\rho)}{1+3\rho}}\right) = 0$,

(ii) If $\rho < 0$ *and* $G_k(t) = \int_0^t g_k(t)dt$ *for* $k = 4n+1, \dots, 4n+3$ *then* $\lim\limits_{t\to+\infty} G_k(t) = +\infty$.

(iii) If moreover $g_j(0) = g_1(0), \quad g_{4n+1}(0) = g_{4n+2}(0) = g_{4n+3}(0)$, *for* $1 \le j \le 4n$ *then a sloution* $g(t)$ *has the following form*

$$\begin{cases} g_j(t) = g_1(0)\Big(1+ct\Big)^{\frac{3(1-2n\rho)}{6+2n-6n\rho}}, & \textit{for } 1 \le j \le 4n, \\ g_{4n+k}(t) = g_{4n+1}(0)\Big(1+ct\Big)^{\frac{-n(1+3\rho)}{3+n-3n\rho}}, & \textit{for } 1 \le k \le 3 \end{cases} \tag{5}$$

where $c = \frac{g_{4n+1}(0)}{g_1^2(0)}(6+2n-6n\rho)$.

Proof. (i) Assume that $G(t) = g_1(t)g_2(t)\dots g_{4n}(t)\big(g_{4n+1}(t)g_{4n+2}(t)g_{4n+3}(t)\big)^{\frac{2(1-2n\rho)}{1+3\rho}}$, then using (4), we get

$$\begin{aligned} \frac{d}{dt}G(t) &= \left(\sum_{r=1}^{4n}\frac{1}{g_r}\frac{dg_r}{dt} + \frac{2(1-2n\rho)}{1+3\rho}\sum_{k=1}^{3}\frac{1}{g_{4n+k}}\frac{dg_{4n+k}}{dt}\right)G(t) \\ &= \sum_{i=1}^{n}\left(\frac{1}{g_i}\frac{dg_i}{dt} + \frac{1}{g_{n+i}}\frac{dg_{n+i}}{dt} + \frac{1}{g_{2n+i}}\frac{dg_{2n+i}}{dt} + \frac{1}{g_{3n+i}}\frac{dg_{3n+i}}{dt}\right)G(t) \\ &\quad + \frac{2(1-2n\rho)}{1+3\rho}G(t)\sum_{k=1}^{3}\frac{1}{g_{4n+k}}\frac{dg_{4n+k}}{dt} = 0. \end{aligned}$$

(ii) For $\rho < 0$ the first four equations (4) yield that g_j, $1 \le j \le 4n$ is an increasing function, so

$$\sum\nolimits_1 = \sum_{i=1}^{n}\left(\frac{1}{g_i g_{n+i}} + \frac{1}{g_{2n+i}g_{3n+i}}\right), \quad \sum\nolimits_2 = \sum_{i=1}^{n}\left(\frac{1}{g_i g_{3n+i}} + \frac{1}{g_{n+i}g_{2n+i}}\right),$$
$$\sum\nolimits_3 = \sum_{i=1}^{n}\left(\frac{1}{g_i g_{2n+i}} + \frac{1}{g_{n+i}g_{3n+i}}\right)$$

are positive and decreasing functions of t. Since $g_{4n+k}(t)$, $1 \le k \le 3$, are positive we have

$$\begin{aligned} \frac{d}{dt}g_{4n+1}(t) &= -g_{4n+1}^2\sum_{i=1}^{n}\left(\frac{1}{g_i g_{n+i}} + \frac{1}{g_{2n+i}g_{3n+i}}\right) - \rho g_{4n+1}\sum\nolimits' \\ &\ge -g_{4n+1}^2\sum_{i=1}^{n}\left(\frac{1}{g_i g_{n+i}} + \frac{1}{g_{2n+i}g_{3n+i}}\right) \ge -g_{4n+1}^2\sum\nolimits_1(0) \end{aligned}$$

which by direct computation implies that

$$g_{4n+1}(t) \ge \frac{1}{\sum_1(0)t + g_{4n+1}^{-1}(0)},$$

therefore

$$\lim_{t\to+\infty} G_{4n+1}(t) = \lim_{t\to+\infty}\int_0^t g_{4n+1}(r)dr \geq \lim_{t\to+\infty}\int_0^t \frac{1}{\sum_1(0)r + g^{-1}_{4n+1}(0)}dr = +\infty;$$

similarly $\lim\limits_{t\to+\infty} G_{4n+2}(t) = +\infty$ and $\lim\limits_{t\to+\infty} G_{4n+3}(t) = +\infty$.

(iii) $g_j(t)$ and $g_{4n+k}(t)$ for $1 \leq j \leq 4n$, $1 \leq k \leq 3$ given in (5) satisfy the Ricci-Bourguignon flow (4). ☐

Proposition 1.8. *Heisenberg type of Lie group Q_n is not preserved under the evolution of the Ricci-Bourguignon flow with solution* (5) *and additional condition* $g_1^2(0) = g_{4n+1}(0)$.

Proof. The proof is similar to proof of Proposition 1.5. In $(Q_n, g_t = \langle,\rangle_t)$, we have

$$j(Z_1)e_i = -\frac{g_{4n+1}(t)}{g_{n+i}(t)}e_{n+i}, \quad j(Z_1)e_{n+i} = \frac{g_{4n+1}(t)}{g_i(t)}e_i, \quad j(Z_1)e_{2n+i} = -\frac{g_{4n+1}(t)}{g_{3n+i}(t)}e_{3n+i},$$

$$j(Z_1)e_{3n+i} = \frac{g_{4n+1}(t)}{g_{2n+i}(t)}e_{2n+i}, \quad j(Z_2)e_i = \frac{g_{4n+2}(t)}{g_{3n+i}(t)}e_{3n+i}, \quad j(Z_2)e_{n+i} = \frac{g_{4n+2}(t)}{g_{2n+i}(t)}e_{2n+i},$$

$$j(Z_2)e_{2n+i} = -\frac{g_{4n+2}(t)}{g_{n+i}(t)}e_{n+i}, \quad j(Z_2)e_{3n+i} = -\frac{g_{4n+2}(t)}{g_i(t)}e_i, \quad j(Z_3)e_i = \frac{g_{4n+3}(t)}{g_{2n+i}(t)}e_{2n+i},$$

$$j(Z_3)e_{n+i} = -\frac{g_{4n+3}(t)}{g_{3n+i}(t)}e_{3n+i}, \quad j(Z_3)e_{2n+i} = -\frac{g_{4n+3}(t)}{g_i(t)}e_i, \quad j(Z_3)e_{3n+i} = -\frac{g_{4n+3}(t)}{g_{n+i}(t)}e_{n+i}.$$

By Theorem 1.7 for $1 \leq i \leq 4n$ and $1 \leq k \leq 3$ we have $g_i(t) = g_1(t)$ and $g_{4n+k}(t) = g_{4n+1}(t)$. Therefore, if we set $E = \frac{g_{4n+1}(t)}{g_i(t)}$, then

$$j(Z_1) = E\begin{bmatrix} 0 & I_n & 0 & 0 \\ -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \end{bmatrix}, \qquad j(Z_2) = E\begin{bmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & -I_n & 0 \\ 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix},$$

$$j(Z_3) = E\begin{bmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_n \\ I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \end{bmatrix},$$

hence for any real constants c_1, c_2 and c_3, we find that

$$j(c_1Z_1 + c_2Z_2 + c_3Z_3) = E\begin{bmatrix} 0 & c_1I_n & -c_3I_n & -c_2I_n \\ -c_1I_n & 0 & -c_2I_n & c_3I_n \\ c_3I_n & c_2I_n & 0 & c_1I_n \\ c_2I_n & -c_3I_n & -c_1I_n & 0 \end{bmatrix}.$$

If $Z = c_1Z_1 + c_2Z_2 + c_3Z_3$ then

$$(j(Z))^2 = -E^2(c_1^2 + c_2^2 + c_3^2)I_{4n} = -E^2\frac{|Z|_t^2}{g_{4n+1}(t)}I_{4n} = -\frac{1}{(6 + 2n - 6n\rho)t + 1}|Z|_t^2 I_{4n}.$$

Hence, it is not of Heisenberg type. ☐

Remark 1.9. By Definition 1.6, along the Ricci-Bourguignon flow, Q_n converts from Heisenberg type to Heisenberg-like type.

2. Deformation of marked length spectrum

Suppose that the Lie group N is H_n or Q_n and $g(t)$ is the solution of the Ricci-Bourguignon flow in (3) and (5) respectively, with some conditions given in Propositions 1.5 and 1.8. Then for $t=0$ we have $j(Z)^2=-|Z|^2Id$ for all $Z\in\mathcal{Z}$, that is the group N is Heisenberg type. But if in Heisenberg Lie group $(H_n,g(t))$ suppose that $\eta_t=\frac{1}{(n+2-n\rho)t+1}$, then in $(H_n,g(t))$ from the proof of Proposition 1.5 we have $j(Z)^2=-\eta_t|Z|_t^2Id$ for all $Z\in\mathcal{Z}$. Also if in the quaternion Lie group $(Q_n,g(t))$ we suppose that $\zeta_t=\frac{1}{(6+2n-6n\rho)t+1}$ then in $(Q_n,g(t))$ from the proof of Proposition 1.8 we obtain $j(Z)^2=-\zeta_t|Z|_t^2Id$ for all $Z\in\mathcal{Z}$.

Let $P_t=\eta_t$ or ζ_t. For H_n or Q_n we have $j(Z)^2=-P_t|Z|_t^2Id$. Similarly to the argument of [1], we conclude the following statements about the deformation of spectrum and length spectrum. The spectrum and the length spectrum have relationship with each other (see [9–11]).

Proposition 2.1. *Let $(\mathcal{N},\langle,\rangle_t)$ is the Lie algebra of N where N is H_n or Q_n . Then we have*

(i) $\langle j(Z)X,j(Z^*)X\rangle_t=P_t\langle Z,Z^*\rangle_t\langle X,X\rangle_t$ *for all* $Z,Z^*\in\mathcal{Z}$ *and* $X\in\mathcal{V}$*;*

(ii) $\langle j(Z)X,j(Z)Y\rangle_t=P_t\langle Z,Z\rangle_t\langle X,Y\rangle_t$ *for all* $Z\in\mathcal{Z}$ *and* $X,Y\in\mathcal{V}$*;*

(iii) $|j(Z)X|_t=P_t^{\frac{1}{2}}|Z|_t|X|_t$ *for all* $Z\in\mathcal{Z}$ *and* $X\in\mathcal{V}$*;*

(iv) $j(Z)\circ j(Z^*)+j(Z^*)\circ j(Z)=-2P_t\langle Z,Z^*\rangle_tId$ *for all* $Z,Z^*\in\mathcal{Z}$*;*

(v) $[X,j(Z)X]=P_t\langle X,X\rangle_tZ$ *for all* $Z\in\mathcal{Z}$ *and* $X\in\mathcal{V}$*.*

Proposition 2.2. *Let $\sigma(s,t)=\exp(X(s,t)+Z(s,t))$ be a curve in 2-step nilpotent Lie group with left invariant metric $(N,g(t))$ where N is H_n or Q_n, such that $\sigma(0,t)=e$ and $\sigma'(0,t)=X_0(t)+Z_0(t)$, where $X_0(t)\in\mathcal{V}(t)$, $Z_0(t)\in\mathcal{Z}(t)$ and e is the identity in N. Let $g(t)$ is the solution of the Ricci-Bourguignon flow on H_n and Q_n in* (3) *and* (5) *respectively. Then*

$$\begin{cases} X(s,t)=(\cos s\theta-1)J^{-1}X_0(t)+\frac{\sin s\theta}{\theta}X_0(t)\\ Z(s,t)=\left(s\left(1+\frac{|X_0(t)|_t^2}{2|Z_0(t)|_t^2}\right)+\frac{\sin s\theta}{\theta}\frac{|X_0(t)|_t^2}{2|Z_0(t)|_t^2}\right)Z_0(t)\end{cases}\tag{6}$$

where $J=j(Z_0(t))$, $\theta=\sqrt{P_t}|Z_0(t)|_t$.

Definition 2.3. A nonidentity element $\varphi(t)$ of $(N,g(t))$ translates a unit speed geodesic $\sigma(s,t)$ in $(N,g(t))$ by an amount $\omega(t)>0$ if $\varphi(t)\cdot\sigma(s,t)=\sigma(s+\omega(t),t)$ for all $s\in\mathbb{R}$. The amount $\omega(t)$ is called a period of $\varphi(t)$.

Definition 2.4. Let N be a simply connected, nilpotent Lie group with a left invariant metric, and let $\Gamma\subseteq N$ be a discrete subgroup of N. The group Γ is said to be a lattice in N if the quotient manifold $\Gamma\setminus N$ obtained by letting Γ act on N by left translation is compact.

Proposition 2.5. *Let* $(N, g(t))$ *be* $(H_n, g(t))$ *or* $(Q_n, g(t))$*,* $g(t)$ *is the solution of the Ricci-Bourguignon flow in 3 and 5 respectively, and* Γ *be a discrete subgroup of* N*. Let* $\varphi(t) \in \Gamma$ *be a family of nonidentity elements of the center of* N*, such that* $\log \varphi(t) \in \mathcal{Z}$*. Then* $\varphi(t) = \exp(V^*(t) + Z^*(t))$ *has the following periods.*

$$\left\{ |Z^*(t)|_t, \sqrt{(4\pi k)(|Z^*(t)|_t - \pi k)}; \textit{where } k \textit{ is an integer and } \{1 \leq k \leq \frac{1}{2\pi}|Z^*(t)|_t\} \right\}$$

Proof. Every unit speed geodesic of N is translated by some element $\varphi(t)$ of N (see [7]). Thus (6) proves the proposition. □

Definition 2.6. Let M be a compact Riemannian manifold. For each nontrivial free homotopy class C of closed curves in M we define $l(C)$ to be the collection of all lengths of smoothly closed geodesics that belong to C.

Definition 2.7. The length spectrum of a compact Riemannian manifold M is the collection of all ordered pairs (L, m), where L is the length of a closed geodesic in M and m is the multiplicity of L, i.e. m is the number of free homotopy classes C of closed curves in M that contain a closed geodesic of length L.

Lemma 2.8. *Let* $g(t)$ *be the solution of the Ricci-Bourguignon flow in* (3) *and* (5)*. Then* $(\Gamma \setminus H_n, g(t))$ *and* $(\Gamma \setminus H_n, g_0)$ *have the same length spectrum, also* $(\Gamma \setminus Q_n, g(t))$ *and* $(\Gamma \setminus Q_n, g_0)$ *have the same length spectrum.*

Proof. Let $(N, g(t))$ be $(H_n, g(t))$ or $(Q_n, g(t))$. If $\varphi(t)$ belongs to a discrete group $\Gamma \subseteq N$, then the periods of $\varphi(t)$ are precisely the lengths of the closed geodesic in $\Gamma \setminus N$ that belong to the free homotopy class of closed curves in $\Gamma \setminus N$ determined by $\varphi(t)$. Therefore a free homotopy class of closed curves in $\Gamma \setminus N$ corresponds to a conjugate class of an element φ in Γ and the collection $l(C)$ is then precisely the set of periods of φ. For any nonidentity element $\varphi(t) = \exp(V^*(t) + Z^*(t)) \in N$ that does not lie in the center of N, by Lemma 3.2 in [1] it has a unique period $\omega(t) = |V^*(t)|_t$. Therefore in Heisenberg Lie group $(H_n, g(t)))$, if we suppose that $V^*(t) = \Sigma_{i=1}^n a_i e_i + b_i e_{n+i}$ for some $a_i, b_i \in \mathbb{R}$, then we obtain

$$|V^*(t)|_t^2 = g_1(t) \sum_{i=1}^n (a_i^2 + b_i^2) = (1 + bt)^{\frac{1-n\rho}{n+2-n\rho}} |V^*(t)|_0^2,$$

where $b = (n + 2 - n\rho)\frac{g_N(0)}{g_1(0)g_{n+1}(0)}$ and in quaternion Lie group we suppose that $V^*(t) = \Sigma_{i=1}^n a_i X_{1i} + b_i X_{2i} + c_i X_{3i} + d_i X_{4i}$ for some $a_i, b_i, c_i, d_i \in \mathbb{R}$, then

$$|V^*(t)|_t^2 = \Sigma_{i=1}^n a_i^2 |X_{1i}|_t^2 + b_i^2 |X_{2i}|_t^2 + c_i^2 |X_{3i}|_t^2 + d_i^2 |X_{4i}|_t^2 = (1 + ct)^{\frac{3(1-2n\rho)}{6+2n-6n\rho}} |V^*(t)|_0^2.$$

where $c = (6 + 2n - 6n\rho)\frac{g_{4n+1}(0)}{g_1^2(0)}$. Let $W^*(t) = (1 + ct)^{-\frac{3(1-2n\rho)}{12+4n-12n\rho}} V^*(t)$ and $\psi(t) = \exp(W^*(t) + Z^*(t))$ then $|W^*(t)|_t = |V^*(t)|_0$ in $(Q_n, g(t))$. Similarly, if $W^*(t) = (1 + bt)^{-\frac{1-n\rho}{2n+4-2n\rho}} V^*(t)$ then in $(H_n, g(t))$ we have $|W^*(t)|_t = |V^*(t)|_0$. Hence the period of $\psi(t)$ is $\omega(t)$. Also, since for arbitrary nonidentity elements $\varphi(t) = \exp(V^*(t) + Z^*(t)) \in N$ which are in the center of N, we have the following periods.

$$\left\{ |Z^*(t)|_t, \sqrt{(4\pi k)(|Z^*(t)|_t - \pi k)}; \text{where } k \text{ is an integer and } \{1 \leq k \leq \frac{1}{2\pi}|Z^*(t)|_t\} \right\}.$$

Therefore in Heisenberg Lie group $(H_n, g(t)))$ we see that $Z^*(t) = ae_{2n+1}$ for some $a \in \mathbb{R}$. We obtain $|Z^*(t)|_t^2 = a^2|e_{2n+1}|_t^2 = (1+bt)^{-\frac{n+n\rho}{n+2-n\rho}}|Z^*(t)|_0^2$ and in quaternion Lie group we suppose that $Z^*(t) = \Sigma_{i=1}^3 a_i Z_{4n+i}$ for some $a_i \in \mathbb{R}$, then

$$|Z^*(t)|_t^2 = \Sigma_{i=1}^3 a_i^2 |Z_{4n+i}|_t^2 = (1+ct)^{\frac{-n(1+3\rho)}{n+3-3n\rho}}|Z^*(t)|_0^2.$$

Then in any case the set of periods of $\varphi(t)$ is similar and this implies that length spectrum on (H_n, g_0) or (Q_n, g_0) is preserved under the metric in (3) and (5). $\square$

Definition 2.9. Two Riemannian manifolds M_1 and M_2 are said to have the same marked length spectrum if there exists an isomorphism $T : \pi_1(M_1) \to \pi_1(M_2)$ (called a marking) such that, for each $\gamma \in \pi_1(M_1)$, the collection of lengths (counting multiplicities) of closed geodesics in the free homotopy class $[\gamma]$ of M_1 coincides with the analogous collection in the free homotopy class $[T(\gamma)]$ of M_2, i.e. $l(T_*(C)) = l(C)$ for all nontrivial free homotopy classes of closed curves in M_1, where T_* denotes the induced map on free homotopy classes.

Definition 2.10. Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to have C^k-conjugate geodesic flows if there is a C^k diffeomorphism $F : S(M_1, g_1) \to S(M_2, g_2)$ between their unit tangent bundles that intertwines their geodesic flows i.e., $F \circ G^s_{M_1} = G^s_{M_2} \circ F$ where $G^s_{M_1}$ and $G^s_{M_2}$ are geodesic flows of M_1 and M_2 respectively.

Definition 2.11. A compact Riemannian manifold M is said to be C^k-geodesically rigid within a given class $\mathcal{M}$ of Riemannian manifolds if any Riemannian manifold M_1 in $\mathcal{M}$ whose geodesic flow is C^k-conjugate to that of M is isometric to M.

Definition 2.12. The solution $g(t)$ of the Ricci-Bourguignon flow with the initial condition $g(0) = g_0$ is called a Ricci-Bourguignon soliton if there exist a smooth function $u(t)$ and a 1-parameter family of diffeomorphisms ψ_t of M^n such that $g(t) = u(t)\psi_t^*(g_0)$, $u(0) = 1$, $\psi_0 = id_{M^n}$.

Similarly to the proof of [1, Theorems 3.1 and 3.2], we have the following lemma.

Lemma 2.13. *The spectrum and marked length spectrum on a compact nilmanifold is preserved under the Ricci-Bourguignon soliton.*

The geodesically rigidity on compact nilmanifold of Heisenberg type is invariant under the Ricci-Bourguignon soliton.

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(received 06.10.2018; in revised form 27.11.2018; available online 19.09.2019)

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