MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
72, 2 (2020), 154–164
June 2020

research paper
оригинални научни рад

STABILITY OF ADDITIVE-QUADRATIC ρ-FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN INTUITIONISTIC FUZZY BANACH SPACES

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Abstract. In this paper, a Hyers-Ulam-Rassias stability result for additive-quadratic ρ-functional equations is established. The framework of the study is non-Archimedean intuitionistic fuzzy Banach spaces. These spaces are generalizations of fuzzy Banach spaces. Several studies of functional analysis have been extended to this space.

1. Introduction

In our present work we establish that an additive-quadratic functional equation in the context of non-Archimedean intuitionistic fuzzy Banach spaces is stable in the sense of Hyers-Ulam-Rassias stability. These types of stabilities have originated from the works of Hyers [8], Ulam [19] and Rassias [14]. Ulam formulated this problem for group homomorphisms [19] which was partly solved by Hyers for Cauchy functional equations [8] and thereafter it was extended by Rassias to the case of linear mappings [14]. Problems of such stabilities also arise from number theory and from considerations of certain determinants [6].

It is well known that fuzzy concept introduced by Zadeh in 1965 [21] is a new tenets of modern mathematics which has made inroads in almost all branches of mathematical studies.

Particularly, fuzzy linear algebra and fuzzy functional analysis have developed in a large way in subsequent times. The related concept of fuzzy linear spaces has been studied in a large number of papers [5, 17].

The fuzzy set theory itself has been extended in different lines leading to several such concepts like L-fuzzy sets [7], etc. Intuitionistic fuzzy sets [1] is one such extension where a non-membership function exists side by side with the membership function. Fuzzy linear spaces have been further extended to intuitionistic fuzzy linear spaces

2010 Mathematics Subject Classification: 03E72, 97I70, 39B82, 32P05
Keywords and phrases: Hyers-Ulam stability; non-Archimedean intuitionistic fuzzy normed space; generalized metric space.

in the works [2, 20]. In particular, we consider here non-Archimedean intuitionistic fuzzy Banach spaces which are a variant of intuitionistic fuzzy linear spaces mentioned above.

In this paper we work in the field of non-Archimedean intuitionistic fuzzy Banach spaces. We consider additive-quadratic ρ-functional equations in these spaces for the purpose of investigating their Hyers-Ulam-Rassias stability properties. We apply a fixed point result on generalized metric spaces for our purpose. Incidentally, the fuzzy stability was first investigated by Mirmostafaee and Moslehian [10]. Several types of functional equations in non-Archimedean intuitionistic fuzzy normed spaces have been discussed in [12].

2. Preliminaries

Definition 2.1 ([11]). Let K be a field. A non-Archimedean absolute value on K is a function $|\cdot| : K \to R$ such that for any $a, b \in K$ we have
(i) $|a| \geq 0$ and equality holds if and only if $a = 0$;

(ii) $|ab| = |a||b|$; (iii) $|a + b| \leq \max\{|a|, |b|\}$.

It can be noted that $|n| \leq 1$ for each integer n. We assume that $|\cdot|$ is non-trivial, that is, there exists an $a_0 \in K$ such that $|a_0| \neq 0, 1$.

Definition 2.2 ([13]). Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:
(i) $\|x\| = 0$ if and only if $x = 0$; (ii) $\|rx\| = |r|\|x\|$ $(r \in K, x \in X)$;

(iii) the strong triangle inequality $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ holds for all $x, y \in X$.
Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Definition 2.3 ([18]). A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions:
(i) $*$ is commutative and associative; (ii) $*$ is continuous; (iii) $a*1 = a,\ \forall a \in [0, 1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.4 ([18]). A binary operation $\diamond : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t co-norm if $\diamond$ satisfies the following conditions:
(i) $\diamond$ is commutative and associative; (ii) $\diamond$ is continuous; (iii) $a\diamond 0 = a,\ \forall a \in [0, 1]$;

(iv) $a \diamond b \leq c \diamond d$, whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.5 ([21]). A fuzzy subset A of a non-empty set X is characterized by a membership function μ_A which associates to each point of X a real number in the interval $[0, 1]$. The value of $\mu_A(x)$ represents the grade of membership of x in A.

Definition 2.6 ([1]). Let E be any nonempty set. An intuitionistic fuzzy subset A of E is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) : x \in E\}$, where the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ denote the degree of membership and the degree of non-membership of the element $x \in E$ respectively and for every $x \in E$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.7 ([2,12]). The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be a non-Archimedean intuitionistic fuzzy normed space, (in short, non-Archimedean IFN space) if X is a vector space over a field R, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are functions from $X \times R \to [0,1]$ satisfying the following conditions.

For every $x, y \in X$ and $s, t \in R$:

(i) $\mu(x,t) = 0, \forall t \leq 0$; (ii) $\mu(x,\ t) = 1$ if and only if $x = 0,\ t > 0$;

(iii) $\mu(cx,t) = \mu\left(x, \frac{t}{|c|}\right)$ if $c \neq 0,\ t > 0$;

(iv) $\mu(x,\ s) * \mu(y\ ,\ t) \leqslant \mu(x+y, \max\{s,t\}), \forall s,t \in R$; (v) $\lim_{t\to\infty} \mu(x,t) = 1$;

(vi) $\nu(x,t) = 1, \forall t \leq 0$; (vii) $\nu(x,\ t) = 0$ if and only if $x = 0,\ t > 0$;

(viii) $\nu(cx,\ t) = \nu\left(x,\ \frac{t}{|c|}\right)$ if $c \neq 0,\ t > 0$;

(ix) $\nu(x,\ s) \diamond \nu(y\ ,\ t) \geqslant \nu(x+y\ ,\ \max\{s,t\}), \forall s,t \in R$; (x) $\lim_{t\to\infty} \nu(x,t) = 0$.

Remark 2.8. From (ii) and (iv), it follows that $\mu(x,t)$ is a non-decreasing function of R, and from (vii) and (ix), it follows that $\nu(x,t)$ is a non-increasing function of R.

Example 2.9. Let $(X, \|\cdot\|)$ be a non-Archimedean normed space, and let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0\ ,1]$. Let $\mu(x,t) = \frac{t}{t+\|x\|}$ and $\nu(x,t) = \frac{\|x\|}{t+\|x\|}$ for all $x \in X$ and $t > 0$. Then $(X, \mu, \nu, *, \diamond)$ is a non-Archimedean fuzzy normed space.

Definition 2.10 ([12,16]). (a) Let $(X, \mu, \nu, *, \diamond)$ be a non-Archimedean intuitionistic fuzzy normed space. Then, a sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ for all $t > 0$, such that $\lim_{n\to\infty} \mu(x_n - x, t) = 1$ and $\lim_{n\to\infty} \nu(x_n - x, t) = 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $(\mu,\nu) - \lim_{n\to\infty} x_n = x$.

(b) Let $(X, \mu, \nu, *, \diamond)$ be a non-Archimedean intuitionistic fuzzy normed space. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $\mu(x_{n+p} - x_n, t) > 1 - \varepsilon$ and $\nu(x_{n+p} - x_n, t) < \varepsilon$.

(c) Let $(X, \mu, \nu, *, \diamond)$ be a non-Archimedean intuitionistic fuzzy normed space. Then $(X, \mu, \nu, *, \diamond)$ is said to be complete if every Cauchy sequence is convergent. In this case $(X, \mu, \nu, *, \diamond)$ is called a non-Archimedean intuitionistic fuzzy Banach space.

In order to establish the result of stability in this paper, we require the following generalized metric space.

Definition 2.11. Let X be a nonempty set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies
(i) $d(x, y) = 0$ if and only if $x = y$; (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then (X, d) is called a generalized metric space or a g.m.s.

Definition 2.12 ([3]). (a) Let (X, d) be a g.m.s., $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is g.m.s. convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$.

(b) Let (X, d) be a g.m.s. and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy sequence if and only if for each $\varepsilon > 0$, there exists a natural number N such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N$.

(c) Let (X, d) be a g.m.s. Then (X, d) is called a complete g.m.s. if every g.m.s. Cauchy seuqence is g.m.s. convergent in X.

The following theorem is crucial for the proof of our main result.

Theorem 2.13 ([4, 9]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$, that is, $d(Jx, Jy) \leq Ld(x, y)$, for all $x, y \in X$. Then for each $x \in X$, either*
$$d(J^n x, J^{n+1}x) = \infty, \ \forall n \geq 0 \quad or \quad d(J^n x, J^{n+1}x) < \infty, \ \forall n \geq n_0$$
for some non-negative integers n_0 . Moreover, if the second alternative holds then
(i) the sequence $\{J^n x\}$ converges to a fixed point $y^\star$ of J ;

(ii) $y^\star$ is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$;

(iii) $d(y, y^\star) \leq (\frac{1}{1-L})d(y, Jy)$ for all $y \in Y$.

For our purpose we take the following additive and quadratic functional equations
$$D_1 f(x, y) := \frac{3}{4}f(x+y)-\frac{1}{4}f(-x-y)+\frac{1}{4}f(x-y)+\frac{1}{4}f(y-x)-f(x)-f(y) \tag{1}$$
and
$$D_2 f(x, y) := 2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)-f(x)-f(y)) \tag{2}$$
and consider the following additive-quadratic ρ-functional equation
$$D_1(x, y) - \rho D_2(x, y) = 0 \tag{3}$$
with $\rho \neq 1$ in non-Archimedean intuitionistic fuzzy Banach spaces. We prove their Hyers-Ulam-Rassias stabilities in this space using fixed point technique.

Lemma 2.14 ([15]). *Let (Z, μ', ν') be a non-Archimedean IFN-space and $\phi : X \times X \to Z$ be a function. Let $E = \{g : X \to Y ; g(0) = 0\}$ and define d by*
$$d(g, h) = \inf \left\{ k \in R^+ : \begin{cases} \mu(g(x) - h(x), kt) \geq \mu'(\phi(x, x), t) \\ \nu(g(x) - h(x), kt) \leq \nu'(\phi(x, x), t), \end{cases} \forall x \in X, t > 0 \right\}$$
where $g, h \in E$. Then (E, d) is a complete generalized metric space.

3. Hyers-Ulam-Rassias stability of additive-quadratic ρ-functional equation (3) in non-Archimedean intuitionistic fuzzy Banach spaces

Throughout this paper X is considered to be a non-Archimedean linear space, (Y,μ,ν) a non-Archimedean IF-real Banach space, (Z,μ',ν') a non-Archimedean IFN-space.

THEOREM 3.1. *Let $\phi : X\times X \to [0,\infty)$ be a function such that $\phi(x,y) = \left\{\frac{\alpha}{|2|}\phi(2x,2y)\right\}$ for some real α with $0<\alpha<1, \forall x \in X$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{cases}\mu(D_1f(x,y)-\rho D_2f(x,y),t) \geq \frac{t}{t+\phi(x,y)}, & \textit{and}\\ \nu(D_1f(x,y)-\rho D_2f(x,y),t) \leq \frac{\phi(x,y)}{t+\phi(x,y)}\end{cases} \quad (x,y\in X, t>0), \tag{4}$$

where $\rho \neq 1$ and $D_1f(x,y)$, $D_2f(x,y)$ be given by (1) *and* (2)*, respectively. Then there exists a unique additive mapping $A : X \to Y$ defined by $A(x) := (\mu,\nu)-\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ for all $x \in X, t>0$ satisfying*

$$\begin{cases}\mu(A(x)-f(x),t) \geq \frac{|2|(1-\alpha)t}{|2|(1-\alpha)t+\alpha\phi(x,x)} & \textit{and}\\ \nu(A(x)-f(x),t) \leq \frac{\alpha\phi(x,x)}{|2|(1-\alpha)t+\alpha\phi(x,x)}\end{cases} \tag{5}$$

Proof. Putting $y=x$ in (4) we get

$$\begin{cases}\mu(f(2x)-2f(x),t) \geq \frac{t}{t+\phi(x,x)} & \text{and}\\ \nu(f(2x)-2f(x),t) \leq \frac{\phi(x,x)}{t+\phi(x,x)}\end{cases} \tag{6}$$

Now consider the set $E := \{g : X \to Y\}$ and introduce a complete generalized metric on E where $g,h \in E$ as per Lemma 2.14 by

$$d(g,h) = \inf\left\{k\in R^+ : \begin{cases}\mu(g(x)-h(x),kt) \geq \frac{t}{t+\phi(x,y)}\\ \nu(g(x)-h(x),kt) \leq \frac{\phi(x,y)}{t+\phi(x,y)}\end{cases} \forall x\in X, t>0\right\}.$$

Also consider a mapping $J : E \to E$ such that $Jg(x) := 2g\left(\frac{x}{2}\right)$ for all $g \in E$ and $x \in X$. We now prove that J is a strictly contracting mapping of E with the Lipschitz constant α.

Let $g,h \in E$ and $\epsilon > 0$. Then there exists $k' \in R^+$ satisfying

$$\begin{cases}\mu(g(x)-h(x),k't) \geq \frac{t}{t+\phi(x,x)} & \text{and}\\ \nu(g(x)-h(x),k't) \leq \frac{\phi(x,x)}{t+\phi(x,x)}\end{cases}$$

such that $d(g,h) \leq k' < d(g,h)+\epsilon$. That is,

$$\inf\left\{k\in R^+ : \begin{cases}\mu(g(x)-h(x),kt) \geq \frac{t}{t+\phi(x,x)}\\ \nu(g(x)-h(x),kt) \leq \frac{\phi(x,x)}{t+\phi(x,x)},\end{cases} \forall x\in X, t>0\right\} \leq k' < d(g,h)+\epsilon$$

$$\text{or,}\quad \inf\left\{k\in R^+ : \begin{cases}\mu(2g(\frac{x}{2})-2h(\frac{x}{2}),|2|kt) \geq \frac{t}{t+\phi(\frac{x}{2},\frac{x}{2})}\\ \nu(2g(\frac{x}{2})-2h(\frac{x}{2}),|2|kt) \leq \frac{\phi(\frac{x}{2},\frac{x}{2})}{t+\phi(\frac{x}{2},\frac{x}{2})},\end{cases} \forall x\in X, t>0\right\} \leq k' < d(g,h)+\epsilon$$

$$\text{or,}\quad \inf\left\{k\in R^+ : \begin{cases}\mu(Jg(x)-Jh(x),|2|kt) \geq \frac{t}{t+\frac{\alpha}{|2|}\phi(x,x)}\\ \nu(Jg(x)-Jh(x),|2|kt) \leq \frac{\frac{\alpha}{|2|}\phi(x,x)}{t+\frac{\alpha}{|2|}\phi(x,x)},\end{cases} \forall x\in X, t>0\right\} < d(g,h)+\epsilon$$

$$\text{or,}\quad \inf\left\{k\in R^+ : \begin{cases}\mu(Jg(x)-Jh(x),|2|kt\times\frac{\alpha}{|2|})\geq \frac{t\times\frac{\alpha}{|2|}}{t\times\frac{\alpha}{|2|}+\frac{\alpha}{|2|}\phi(x,x)} \\ \nu(Jg(x)-Jh(x),|2|kt\times\frac{\alpha}{|2|})\leq \frac{\frac{\alpha}{|2|}\phi(x,x)}{t\times\frac{\alpha}{|2|}+\frac{\alpha}{|2|}\phi(x,x)},\end{cases}\forall x\in X, t>0\right\}<d(g,h)+\epsilon$$

$$\text{or,}\quad \inf\left\{k\in R^+ : \begin{cases}\mu(Jg(x)-Jh(x),\alpha kt)\geq \frac{t}{t+\phi(x,x)} \\ \nu(Jg(x)-Jh(x),\alpha kt)\leq \frac{\phi(x,x)}{t+\phi(x,x)},\end{cases}\forall x\in X, t>0\right\}<d(g,h)+\epsilon$$

$$\text{or,}\quad d\left\{\frac{1}{\alpha}(Jg,Jh)\right\}<d(g,h)+\epsilon\quad\text{or,}\quad d\left\{(Jg,Jh)\right\}<\alpha\left\{d(g,h)+\epsilon\right\}.$$

Taking $\epsilon\to 0$ we get $d\left\{(Jg,Jh)\right\}\leq\alpha\left\{d(g,h)\right\}$. Therefore J is a strictly contractive mapping with Lipschitz constant $\alpha<1$. Also from (6),

$$\begin{cases}\mu(f(x)-2f(\frac{x}{2}),t)\geq\frac{t}{t+\phi(\frac{x}{2},\frac{x}{2})}=\frac{t}{t+\frac{\alpha}{|2|}\phi(x,x)}\\ \nu(f(x)-2f(\frac{x}{2}),t)\leq\frac{\phi(\frac{x}{2},\frac{x}{2})}{t+\phi(\frac{x}{2},\frac{x}{2})}=\frac{\frac{\alpha}{|2|}\phi(x,x)}{t+\frac{\alpha}{|2|}\phi(x,x)}\end{cases}\quad\text{or,}\quad\begin{cases}\mu\left(f(x)-2f(\frac{x}{2}),\frac{\alpha}{|2|}t\right)\geq\frac{t}{t+\phi(x,x)}\\ \nu\left(f(x)-2f(\frac{x}{2}),\frac{\alpha}{|2|}t\right)\leq\frac{\phi(x,x)}{t+\phi(x,x)}\end{cases}$$

Therefore
$$(f,Jf)\leq\frac{\alpha}{|2|}\tag{7}$$

Also, replacing x by $2^{-(n+1)}x$ in (6) we get

$$\begin{cases}\mu\left(f\left(\frac{x}{2^n}\right)-2f\left(\frac{x}{2^{n+1}}\right),t\right)\geq\frac{t}{t+\phi(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}})}=\frac{t}{t+\left(\frac{\alpha}{|2|}\right)^{n+1}\phi(x,x)}\quad\text{and}\\ \nu\left(f\left(\frac{x}{2^n}\right)-2f\left(\frac{x}{2^{n+1}}\right),t\right)\leq\frac{\phi(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}})}{t+\phi(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}})}=\frac{\left(\frac{\alpha}{|2|}\right)^{n+1}\phi(x,x)}{t+\left(\frac{\alpha}{|2|}\right)^{n+1}\phi(x,x)}\end{cases}$$

$$\text{or,}\quad\begin{cases}\mu\left(2^nf\left(\frac{x}{2^n}\right)-2^{n+1}f\left(\frac{x}{2^{n+1}}\right),|2|^nt\right)\geq\frac{t}{t+\left(\frac{\alpha}{|2|}\right)^{n+1}\phi(x,x)}\quad\text{and}\\ \nu\left(2^nf\left(\frac{x}{2^n}\right)-2^{n+1}f\left(\frac{x}{2^{n+1}}\right),|2|^nt\right)\leq\frac{\left(\frac{\alpha}{|2|}\right)^{n+1}\phi(x,x)}{t+\left(\frac{\alpha}{|2|}\right)^{n+1}\phi(x,x)}\end{cases}$$

$$\text{or,}\quad\begin{cases}\mu\left(J^nf(x)-J^{n+1}f(x),t\frac{\alpha^{n+1}}{|2|}\right)\geq\frac{t}{t+\phi(x,x)}\quad\text{and}\\ \nu\left(J^nf(x)-J^{n+1}f(x),t\frac{\alpha^{n+1}}{|2|}\right)\leq\frac{\phi(x,x)}{t+\phi(x,x)}\end{cases}$$

Hence $d(J^{n+1}f,J^nf)\leq\frac{\alpha^{n+1}}{|2|}<\infty$ as Lipschitz constant $\alpha<1$ for $n\geq n_0=1$. Therefore by Theorem 2.13 there exists a mapping $A:X\to Y$ satisfying the following: **1.** A is a fixed point of J, that is, $A(\frac{x}{2})=\frac{1}{2}A(x)$ for all $x\in X$. Since $f:X\to Y$ is an odd mapping, therefore $A:X\to Y$ is also an odd mapping and the mapping A is a unique fixed point of J in the set $E_1=\{g\in E:d(J^{n_0}f,g)=d(Jf,g)<\infty\}$. Therefore $d(Jf,A)<\infty$. Also from (7), $d(Jf,f)\leq\frac{\alpha}{|2|}<\infty$. Thus $f\in E_1$. Now, $d(f,A)\leq\max\{d(f,Jf),d(Jf,A)\}<\infty$. Thus there exists $k\in(0,\infty)$ satisfying

$$\begin{cases}\mu(f(x)-A(x),kt)\geq\frac{t}{t+\phi(x,x)}\quad\text{and}\\ \nu(f(x)-A(x),kt)\leq\frac{\phi(x,x)}{t+\phi(x,x)}\forall x\in X,t>0.\end{cases}\tag{8}$$

Also, from (8) we have

$$\begin{cases}\mu\left(f\left(\frac{x}{2^n}\right)-A\left(\frac{x}{2^n}\right),kt\right)\geq\frac{t}{t+\phi\left(\frac{x}{2^n},\frac{x}{2^n}\right)}\quad\text{and}\\ \nu\left(f\left(\frac{x}{2^n}\right)-A\left(\frac{x}{2^n}\right),kt\right)\leq\frac{\phi\left(\frac{x}{2^n},\frac{x}{2^n}\right)}{t+\phi\left(\frac{x}{2^n},\frac{x}{2^n}\right)}\end{cases}$$

or, $\begin{cases} \mu\left(2^n f\left(\frac{x}{2^n}\right)-2^n A\left(\frac{x}{2^n}\right),|2|^n kt\right)\geq \frac{t}{t+\phi\left(\frac{x}{2^n},\frac{x}{2^n}\right)}=\frac{t}{t+\left(\frac{\alpha}{|2|}\right)^n\phi(x,x)} \quad \text{and} \\ \nu\left(2^n f\left(\frac{x}{2^n}\right)-2^n A\left(\frac{x}{2^n}\right),|2|^n kt\right)\leq \frac{\phi\left(\frac{x}{2^n},\frac{x}{2^n}\right)}{t+\phi\left(\frac{x}{2^n},\frac{x}{2^n}\right)}=\frac{\left(\frac{\alpha}{|2|}\right)^n\phi(x,x)}{t+\left(\frac{\alpha}{|2|}\right)^n\phi(x,x)}\end{cases}$

or, $\begin{cases} \mu\left(2^n f\left(\frac{x}{2^n}\right)-2^n A\left(\frac{x}{2^n}\right),|2|^n kt\times(\frac{\alpha}{|2|})^n\right)\geq \frac{t\times(\frac{\alpha}{|2|})^n}{t\times(\frac{\alpha}{|2|})^n+\left(\frac{\alpha}{|2|}\right)^n\phi(x,x)} \quad \text{and} \\ \nu\left(2^n f\left(\frac{x}{2^n}\right)-2^n A\left(\frac{x}{2^n}\right),|2|^n kt\times(\frac{\alpha}{|2|})^n\right)\leq \frac{\left(\frac{\alpha}{|2|}\right)^n\phi(x,x)}{t\times(\frac{\alpha}{|2|})^n+\left(\frac{\alpha}{|2|}\right)^n\phi(x,x)}\end{cases}$

or, $\begin{cases} \mu\left(J^n f(x)-2^n A\left(\frac{x}{2^n}\right),\alpha^n kt\right)\geq \frac{t}{t+\phi(x,x)} \quad \text{and} \\ \nu\left(J^n f(x)-2^n A\left(\frac{x}{2^n}\right),\alpha^n kt\right)\leq \frac{\phi(x,x)}{t+\phi(x,x)}.\end{cases}$

or, $\begin{cases} \mu\left(J^n f(x)-A(x),\alpha^n kt\right)\geq \frac{t}{t+\phi(x,x)} \quad \text{and} \\ \nu\left(J^n f(x)-A(x),\alpha^n kt\right)\leq \frac{\phi(x,x)}{t+\phi(x,x)},\end{cases}$

since $A(x)=2A(\frac{x}{2})=2^2A(\frac{x}{2^2})=\ldots=2^nA(\frac{x}{2^n})$.

2. $d(J^nf,A)=\inf\left\{k\in R^+ : \begin{cases}\mu(J^nf(x)-A(x),\alpha^nkt)\geq \frac{t}{t+\phi(x,x)} \\ \nu(J^nf(x)-A(x),\alpha^nkt)\leq \frac{\phi(x,x)}{t+\phi(x,x)},\end{cases} \quad \forall x\in X, t>0\right\}$, since $\alpha<1$, therefore $d(J^nf,A)\leq k\alpha^n\to 0$ as $n\to\infty$. This implies the equality $A(x):=(\mu,\nu)-\lim_{n\to\infty}J^nf(x)=(\mu,\nu)-\lim_{n\to\infty}2^nf\left(\frac{x}{2^n}\right)$, for all $x\in X$.

3. $d(f,A)\leqslant\frac{1}{1-L}d(f,Jf)$ with $f\in E_1$ which implies the inequality $d(f,A)\leq\frac{1}{1-\alpha}\times\frac{\alpha}{|2|}=\frac{\alpha}{|2|(1-\alpha)}$. This implies the results (5). Now replacing x and y by $2^{-n}x$ and $2^{-n}y$ in (4) we have

$$\begin{cases} \mu\left(2^nf\left(\frac{x+y}{2^n}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right. \\ \left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),|2|^nt\right)\geq\frac{t}{t+(\frac{\alpha}{|2|})^n\phi(x,y)} \text{ and} \\ \nu\left(2^nf\left(\frac{x+y}{2^n}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right. \\ \left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),|2|^nt\right)\leq\frac{(\frac{\alpha}{|2|})^n\phi(x,y)}{t+(\frac{\alpha}{|2|})^n\phi(x,y)}\end{cases}$$

or, $$\begin{cases} \mu\left(2^nf\left(\frac{x+y}{2^n}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right. \\ \left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),\alpha^nt\right)\geq\frac{t}{t+\phi(x,y)} \text{ and} \\ \nu\left(2^nf\left(\frac{x+y}{2^n}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right. \\ \left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),\alpha^nt\right)\leq\frac{\phi(x,y)}{t+\phi(x,y)}\end{cases}$$

or, $$\begin{cases} \mu\left(2^nf\left(\frac{x+y}{2^n}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right. \\ \left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),t\right)\geq\frac{t}{t+\alpha^n\phi(x,y)} \text{ and} \\ \nu\left(2^nf\left(\frac{x+y}{2^n}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right. \\ \left.-\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)\right),t\right)\leq\frac{\alpha^n\phi(x,y)}{t+\alpha^n\phi(x,y)}.\end{cases} \tag{9}$$

Taking the limit as $n\to\infty$ in (9) and using the conditions $\mu(x,t)=1$ if and only if $x=0,t>0$, $\nu(x,\ t)=0$ if and only if $x=0,t>0$ we obtain,

$$\begin{cases}\mu\left(A(x+y)-A(x)-A(y)-\rho\left(2A\left(\frac{x+y}{2}\right)-A(x)-A(y)\right),t\right)=1 \quad \text{and} \\ \nu\left(A(x+y)-A(x)-A(y)-\rho\left(2A\left(\frac{x+y}{2}\right)-A(x)-A(y)\right),t\right)=0.\end{cases}$$

Hence, $A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$. Therefore $A(x+y) = A(x) + A(y)$. That is, $A : X \to Y$ is additive, since $\rho \neq 1$ and $2A\left(\frac{x+y}{2}\right) = A(x+y)$. The uniqueness of A follows from the fact that A is the unique fixed point of J. □

Corollary 3.2. *Let $p > 1$ be a non-negative real number, X be a non-Archimedean normed linear space with norm $\|\cdot\|$ and $z_0 \in Z$ and let $f : X \to Y$ be an odd mapping such that*

$$\begin{cases} \mu(D_1 f(x,y) - \rho D_2 f(x,y), t) \geq \frac{t}{t + z_0(\|x\|^p + \|y\|^p)} & \text{and} \\ \nu(D_1 f(x,y) - \rho D_2 f(x,y), t), t) \leq \frac{z_0(\|x\|^p + \|y\|^p)}{t + z_0(\|x\|^p + \|y\|^p)} \end{cases} \quad (x, y \in X, t > 0), \tag{10}$$

where $D_1 f(x,y)$ and $D_2 f(x,y)$ are given by (1) *and* (2). *Then there exists a unique additive mapping $A : X \to Y$ for all $x \in X, t > 0$ satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \frac{(|2|^p - |2|)t}{(|2|^p - |2|) + 2z_0\|x\|^p} & \text{and} \\ \nu(A(x) - f(x), t) \leq \frac{2z_0\|x\|^p}{(|2|^p - |2|) + 2z_0\|x\|^p}. \end{cases}$$

Proof. Define $\phi(x,y) = z_0(\|x\|^p + \|y\|^p)$ and the proof follows from Theorem 3.1 by taking $\alpha = |2|^{1-p}$. □

Theorem 3.3. *Let $\phi : X \times X \to [0, \infty)$ be a function such that $\phi(x,y) = \left\{\frac{\alpha}{|4|}\phi(2x, 2y)\right\}$ for some real $0 < \alpha < 1$ and for all $x \in X$. If $f : X \to Y$ be an even mapping with $f(0) = 0$ satisfying* (4) *then there exists a unique quadratic mapping $Q : X \to Y$ defined by $Q(x) := (\mu, \nu) - \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ for all $x \in X$, satisfying*

$$\begin{cases} \mu(Q(x) - f(x), t) \geq \frac{|2|(1-\alpha)t}{|2|(1-\alpha)t + \alpha\phi(x,x)} & \text{and} \\ \nu(Q(x) - f(x), t) \leq \frac{\alpha\phi(x,x)}{|2|(1-\alpha)t + \alpha\phi(x,x)} \end{cases} \tag{11}$$

Proof. Similarly as before, by putting $y = x$ in (4) we get

$$\begin{cases} \mu\left(\frac{1}{2}f(2x) - 2f(x), t\right) \geq \frac{t}{t + \phi(x,x)} \\ \nu\left(\frac{1}{2}f(2x) - 2f(x), t\right) \leq \frac{\phi(x,x)}{t + \phi(x,x)} \end{cases} \quad \text{or,} \quad \begin{cases} \mu\left(f(x) - 4f(\frac{x}{2}), \frac{\alpha}{|2|}t\right) \geq \frac{t}{t + \phi(x,x)} \\ \nu\left(f(x) - 4f(\frac{x}{2}), \frac{\alpha}{|2|}t\right) \leq \frac{\phi(x,x)}{t + \phi(x,x)} \end{cases}$$

Now consider the set $E := \{g : X \to Y\}$ and introduce a complete generalized metric on E as per Lemma 2.14. Also consider the mapping $J : E \to E$ such that $Jg(x) := 4g\left(\frac{x}{2}\right)$ for all $g \in E$ and $x \in X$. Similarly as before we can prove that J is a strictly contracting mapping on E with the Lipschitz constant $\alpha < 1$. Also, we have $d(f, Jf) \leq \frac{\alpha}{|2|}$ and $d\left(J^{n+1}f, J^n f\right) \leq \frac{\alpha^{n+1}}{|2|} < \infty$. Therefore by Theorem 2.13 there exists a mapping $Q : X \to Y$ satisfying the following:

1. Q is a fixed point of J, that is, $Q(\frac{x}{2}) = \frac{1}{4}Q(x)$ for all $x \in X$. Since $f : X \to Y$ is an even mapping, therefore $Q : X \to Y$ is also an even mapping.

2. $Q(x) := (\mu, \nu) - \lim_{n\to\infty} J^n f(x) = (\mu, \nu) - \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ for all $x \in X$.

3. $d(f, Q) \leqslant \frac{1}{1-L} d(f, Jf)$ with $f \in E_1$ which implies the inequality

$$d(f, Q) \leq \frac{1}{1-\alpha} \times \frac{\alpha}{|2|} = \frac{\alpha}{|2|(1-\alpha)}.$$

This implies the results (11). Also, we have

$$\begin{cases}\mu\left(4^n\times\frac{1}{2}f\left(\frac{x+y}{2^n}\right)+4^n\times\frac{1}{2}f\left(\frac{x-y}{2^n}\right)-4^nf\left(\frac{x}{2^n}\right)-4^nf\left(\frac{y}{2^n}\right)\right.\\ \left.-\rho\left(2\times 4^nf\left(\frac{x+y}{2^{n+1}}\right)+2\times 4^nf\left(\frac{x-y}{2^{n+1}}\right)+4^nf\left(\frac{x}{2^n}\right)-4^nf\left(\frac{y}{2^n}\right)\right),t\right)\geq\frac{t}{t+\alpha^n\phi(x,y)} \quad \text{and}\\ \nu\left(4^n\times\frac{1}{2}f\left(\frac{x+y}{2^n}\right)+4^n\times\frac{1}{2}f\left(\frac{x-y}{2^n}\right)+4^nf\left(\frac{x}{2^n}\right)-4^nf\left(\frac{y}{2^n}\right)\right.\\ \left.-\rho\left(2\times 4^nf\left(\frac{x+y}{2^{n+1}}\right)+2\times 4^nf\left(\frac{x-y}{2^{n+1}}\right)-4^nf\left(\frac{x}{2^n}\right)-4^nf\left(\frac{y}{2^n}\right)\right),t\right)\leq\frac{\alpha^n\phi(x,y)}{t+\alpha^n\phi(x,y)}.\end{cases}$$

Taking the limit $n\to\infty$, we obtain

$$\begin{cases}\mu\left(\frac{1}{2}Q(x+y)+\frac{1}{2}Q(x-y)-Q(x)-Q(y)\right.\\ \left.-\rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right),t\right)=1 \quad \text{and}\\ \nu\left(\frac{1}{2}Q(x+y)+\frac{1}{2}Q(x-y)-Q(x)-Q(y)\right.\\ \left.-\rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right),t\right)=0.\end{cases}$$

Hence,

$$\frac{1}{2}Q(x+y)+\frac{1}{2}Q(x-y)-Q(x)-Q(y) = \rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right).$$

Therefore, $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$, that is, $Q:X\to Y$ is quadratic, since $\rho\neq 1$ and $4Q\left(\frac{x+y}{2}\right)=Q(x+y)$. This completes the proof of the theorem. □

COROLLARY 3.4. *Let $p>2$ be a non-negative real number, X be a non-Archimedean normed linear space with norm $\|\cdot\|$, $z_0\in Z$ and let $f:X\to Y$ be an even mapping satisfying* (10). *Then there exists a unique quadratic mapping $Q:X\to Y$ for all $x\in X, t>0$ satisfying*

$$\begin{cases}\mu(Q(x)-f(x),t)\geq\frac{(|2|^p-4)t}{(|2|^p-4)t+|4|z_0\|x\|^p} \quad \textit{and}\\ \nu(Q(x)-f(x),t)\leq\frac{|4|z_0\|x\|^p}{(|2|^p-4)t+|4|z_0\|x\|^p}.\end{cases}$$

Proof. Define $\phi(x,y)=z_0(\|x\|^p+\|y\|^p)$ and the proof follows from Theorem 3.3 by taking $\alpha=|2|^{2-p}$. □

THEOREM 3.5. *Let $\phi:X\times X\to[0,\infty)$ be a function such that $\phi(x,y)=|2|\alpha\phi\left(\frac{x}{2},\frac{y}{2}\right)$ for some real α with $0<\alpha<1, \forall x,y\in X$. Let $f:X\to Y$ be an odd mapping satisfying* (4). *Then there exists a unique additive mapping $A:X\to Y$ defined by $A(x):=(\mu,\nu)-\lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$ for all $x\in X$ satisfying*

$$\begin{cases}\mu(A(x)-f(x),t)\geq\frac{|2|(1-\alpha)t}{|2|(1-\alpha)t+\phi(x,x)} \quad \textit{and}\\ \nu(A(x)-f(x),t)\leq\frac{\phi(x,y)}{|2|(1-\alpha)t+\phi(x,x)}.\end{cases}$$

Proof. Putting $y=x$ in (4) we get

$$\begin{cases}\mu(f(x)-\frac{1}{2}f(2x),\frac{t}{|2|})\geq\frac{t}{t+\phi(x,x)} \quad \text{and}\\ \nu(f(x)-\frac{1}{2}f(2x),\frac{t}{|2|})\leq\frac{\phi(x,x)}{t+\phi(x,x)}.\end{cases}$$

The rest of the proof is similar to the proof of the Theorem 3.1. □

Corollary 3.6. *Let $p < 1$ be a non-negative real number, X be a non-Archimedean normed linear space with norm $\|\cdot\|, z_0 \in Z$ and let $f : X \to Y$ be an odd mapping satisfying* (4)*. Then there exists a unique additive mapping $A : X \to Y$ for all $x \in X$ satisfying*

$$\begin{cases} \mu(A(x) - f(x), t) \geq \frac{(|2|-|2|^p)t}{(|2|-|2|^p)t+2z_0\|x\|^p} \quad \text{and} \\ \nu(A(x) - f(x), t) \leq \frac{2z_0\|x\|^p}{(|2|-|2|^p)t+2z_0\|x\|^p}. \end{cases}$$

Proof. Define $\phi(x, y) = z_0(\|x\|^p + \|y\|^p)$ and the proof follows from Theorem 3.5 by taking $\alpha = |2|^{p-1}$. □

Theorem 3.7. *Let $\phi : X \times X \to [0, \infty)$ be a function such that $\phi(x, y) = |4|\alpha\phi\left(\frac{x}{2}, \frac{y}{2}\right)$ for some real α with $0 < \alpha < 1, \forall x \in X$. Let $f : X \to Y$ be an even mapping satisfying* (4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ defined by $Q(x) := (\mu, \nu) - \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ for all $x \in X, t > 0$ satisfying*

$$\begin{cases} \mu(Q(x) - f(x), t) \geq \frac{|2|(1-\alpha)t}{|2|(1-\alpha)t+\phi(x,x)} \quad \text{and} \\ \nu(Q(x) - f(x), t) \leq \frac{\phi(x,y)}{|2|(1-\alpha)t+\phi(x,y)}. \end{cases}$$

Proof. Putting $y = x$ in (4) we get

$$\begin{cases} \mu(f(x) - \frac{1}{4}f(2x), \frac{t}{|2|}) \geq \frac{t}{t+\phi(x,y)} \quad \text{and} \\ \nu(f(x) - \frac{1}{4}f(2x), \frac{t}{|2|}) \leq \frac{\phi(x,y)}{t+\phi(x,y)}. \end{cases}$$

The rest of the proof is similar to the proof of the Theorem 3.3. □

Corollary 3.8. *Let $p < 2$ be a non-negative real number, X be a non-Archimedean normed linear space with norm $\|\cdot\|, z_0 \in Z$ and let $f : X \to Y$ be an even mapping satisfying* (4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ for all $x \in X$ satisfying*

$$\begin{cases} \mu(Q(x) - f(x), t) \geq \frac{|2|(4-|2|^p)t}{|2|(4-|2|^p)t+8z_0\|x\|^p} \quad \text{and} \\ \nu(Q(x) - f(x), t) \leq \frac{8z_0\|x\|^p}{|2|(4-|2|^p)t+8z_0\|x\|^p}. \end{cases}$$

Proof. Define $\phi(x, y) = z_0(\|x\|^p + \|y\|^p)$ and the proof follows from Theorem 3.5 by taking $\alpha = |2|^{p-2}$. □

Acknowledgement. The authors gratefully acknowledge the suggestions made by the anonymous referee.

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(received 28.11.2018; in revised form 03.04.2019; available online 10.10.2019)

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