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COUNTING SPACES OF EXCESSIVE WEIGHTS

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Abstract. Let κ, λ be infinite cardinal numbers with $\kappa < \lambda \leq 2^\kappa$. We show that there exist precisely 2^λ T_0-spaces of size κ and weight λ up to homeomorphism. Among these non-homeomorphic spaces we track down (i) 2^λ zero-dimensional, scattered, paracompact, perfectly normal spaces (which are also extremally disconnected in case that $\lambda = 2^\kappa$); (ii) 2^λ connected and locally connected Hausdorff spaces; (iii) 2^λ pathwise connected and locally pathwise connected, paracompact, perfectly normal spaces provided that $\kappa \geq 2^{\aleph_0}$; (iv) 2^λ connected, nowhere locally connected, totally pathwise disconnected, paracompact, perfectly normal spaces provided that $\kappa \geq 2^{\aleph_0}$; (v) 2^λ scattered, compact T_1-spaces; (vi) 2^λ connected, locally connected, compact T_1-spaces; (vii) 2^λ pathwise connected *and* scattered, compact T_0-spaces; (viii) 2^λ scattered, paracompact P_α-spaces whenever $\kappa^{<\alpha} = \kappa$ and $\lambda^{<\alpha} = \lambda$ and $2^\lambda > 2^\kappa$.

1. Introduction

Write $|M|$ for the cardinal number (the *size*) of a set M and define $\mathbf{c} := |\mathbb{R}| = 2^{\aleph_0}$. We use κ, λ, μ throughout to stand for *infinite* cardinal numbers. As usual, $w(X)$ denotes the weight of a topological space X. Naturally, $w(X) \leq 2^{|X|}$ and $|X| \leq 2^{w(X)}$ for every infinite T_0-space X. It is trivial that $w(X) \leq |X|$ for every infinite, first countable space X and well-known (see [2, 3.3.6]) that $w(X) \leq |X|$ for every compact Hausdorff space X. Furthermore, $w(X) \geq |X|$ for every infinite, scattered T_0-space X (see Lemma 2.1 below).

According to the title, we are concerned with topological spaces X satisfying the strict inequality $w(X) > |X|$. While the extreme case $w(X) = 2^{|X|}$ is of natural interest, to investigate the case $|X| < w(X) < 2^{|X|}$ is reasonable in view of the following remarkable fact.

(I) *It is consistent with* ZFC *set theory that* $\mu < \lambda$ *implies* $2^\mu < 2^\lambda$ *and that for every regular* κ *there exist precisely* 2^κ *cardinals* λ *with* $\kappa < \lambda < 2^\kappa$.

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A short explanation why (I) is true is given in Section 2.

For fundamental enumeration theorems about spaces X with $w(X) \leq |X|$ see [3, 5–7, 9]. However, it would be artificial to avoid an overlap with these enumeration theorems and hence in the following we include the case $w(X) = |X|$. The benefit of this inclusion is that we will also establish several new enumeration theorems about spaces X with $w(X) = |X|$. A short proof of the following basic estimate is given in the next section.

(II) *If θ is an infinite cardinal and $\mathcal{F}$ is a family of mutually non-homeomorphic infinite* T_0*-spaces such that* $\max\{|X|, w(X)\} \leq \theta$ *for every $X \in \mathcal{F}$ then $|\mathcal{F}| \leq 2^\theta$.*

For abbreviation let us call a Hausdorff space X *almost discrete* if and only if $X \setminus \{x\}$ is a discrete subspace of X for some $x \in X$. Recall that a space is *perfectly normal* when it is normal and every closed set is a G_δ-set. Note that every subspace of a perfectly normal space is perfectly normal. Recall that a normal space is *strongly zero-dimensional* if and only if for every closed set A and every open set $U \supset A$ there is an open-closed set V with $A \subset V \subset U$. Our first goal is to prove the following enumeration theorem.

Theorem 1.1. *If $\kappa \leq \lambda \leq 2^\kappa$ then there exist 2^λ mutually non-homeomorphic scattered, strongly zero-dimensional, hereditarily paracompact, perfectly normal spaces X with $|X| = \kappa$ and $w(X) = \lambda$. In case that $\lambda \leq 2^\mu < 2^\lambda$ for some μ it can be accomplished that all these spaces are also almost discrete. Moreover, it can be accomplished that all these spaces are almost discrete and extremally disconnected in case that $\lambda = 2^\mu$ for some μ (which includes the case $\lambda = 2^\kappa$).*

Since every scattered Hausdorff space is totally disconnected, the following theorem is a noteworthy counterpart of Theorem 1.1. For abbreviation, let us call a space X *almost metrizable* if and only if X is perfectly normal and $X \setminus \{x\}$ is metrizable for some $x \in X$. In view of Lemma 3.2 in Section 3, almost metrizable space are hereditarily paracompact.

Theorem 1.2. *If $\mathbf{c} \leq \kappa \leq \lambda \leq 2^\kappa$ then there exist 2^λ mutually non-homeomorphic pathwise connected, locally pathwise connected, almost metrizable spaces of size κ and weight λ.*

The restriction $\mathbf{c} \leq \kappa$ in Theorem 1.2 is inevitable because if X is an infinite, *pathwise* connected Hausdorff space then X is *arcwise* connected (see [2, 6.3.12.a]) and hence $\mathbf{c} = |[0,1]| \leq |X|$. However, for infinite, *connected* Hausdorff spaces X the restriction $\mathbf{c} \leq |X|$ is not justified and we can prove the following theorem. Note that, by applying (I) for $\kappa = \aleph_0$, the existence of $\mathbf{c}$ infinite cardinals $\kappa < \mathbf{c}$ is consistent with ZFC.

Theorem 1.3. *If $\kappa < \mathbf{c}$ and $\kappa \leq \lambda \leq 2^\kappa$ then there exist 2^λ mutually non-homeomorphic connected and locally connected Hausdorff spaces of size κ and weight λ. In particular, up to homeomorphism there exist precisely $2^{\mathbf{c}}$ countably infinite, connected, locally connected Hausdorff spaces and precisely $\mathbf{c}$ countably infinite, connected, locally connected, second countable Hausdorff spaces.*

No space provided by Theorem 1.3 is completely regular because, naturally, every completely regular space of size smaller than $\mathbf{c}$ and greater than 1 is totally disconnected. Moreover, every countably infinite, regular space is totally disconnected (see [2, 6.2.8]). The *connected* spaces provided by Theorem 1.3 are *totally pathwise disconnected* since they are Hausdorff spaces of size smaller than $\mathbf{c}$. Therefore the following counterpart of Theorem 1.2 is worth mentioning.

THEOREM 1.4. *If* $\mathbf{c} \leq \kappa \leq \lambda \leq 2^\kappa$ *then there exist* 2^λ *mutually non-homeomorphic connected, totally pathwise disconnected, nowhere locally connected, almost metrizable spaces of size* κ *and weight* λ.

2. Some explanations and preparations

Referring to Jech's profound textbook [4], a proof of (I) can be carried out as follows. Define in Gödel's universe L for every regular cardinal κ a cardinal number $\theta(\kappa)$ by $\theta(\kappa) := \min\{\mu \mid \mu = \aleph_\mu \wedge \mathrm{cf}\mu = \kappa^+\}$. Then $|\{\lambda \mid \kappa < \lambda < \theta(\kappa)\}| = \theta(\kappa)$ holds in every generic extension of L. By applying Easton's theorem [4, 15.18] one can create an Easton universe E generically extending L such that the continuum function $\kappa \mapsto 2^\kappa = \kappa^+$ in L is changed into $\kappa \mapsto 2^\kappa = g(\kappa)$ in E with $g(\kappa) = \theta(\kappa)$ for every regular cardinal κ. So in E we have $|\{\lambda \mid \kappa < \lambda < 2^\kappa\}| = 2^\kappa$ for every regular κ. By definition, in E we have $2^\alpha < 2^\beta$ whenever α, β are regular cardinals with $\alpha < \beta$. Therefore and in view of [4, Theorem 5.22 and Exercise 15.12], if μ is singular in E then 2^μ is a successor cardinal in E while 2^κ is a limit cardinal in E for every regular κ in E. Consequently, in E we have $2^\mu < 2^\lambda$ whenever μ, λ are arbitrary cardinals with $\mu < \lambda$.

In order to verify (II), first of all it is clear that a topological space (X, τ) has a basis of size $\lambda \leq |\tau|$ if and only if $w(X) \leq \lambda$. Let S be an infinite set of size ν and let P be the power set of S, whence $|P| = 2^\nu$. Let $\mu(\nu, \lambda)$ denote the total number of all topologies τ on S such that (S, τ) has a basis B of size λ. Clearly, $\mu(\nu, \lambda) = 0$ if $\lambda > 2^\nu$. For $\lambda \leq 2^\nu$ we have $\mu(\nu, \lambda) \leq |P|^\lambda = \max\{2^\nu, 2^\lambda\}$. So if θ and $\mathcal{F}$ satisfy the assumption in (II) then $|\mathcal{F}|$ is not greater than the sum Σ of all cardinals $\mu(\nu, \lambda)$ with (ν, λ) running through the set $Q := \{\kappa \mid \kappa \leq \theta\}^2$. Thus from $\mu(\nu, \lambda) \leq 2^\theta$ for all $(\nu, \lambda) \in Q$ we derive $\Sigma \leq 2^\theta$ and this concludes the proof of (II).

In the following we write down a short proof of an important fact mentioned in the previous section.

LEMMA 2.1. *If* X *is an infinite scattered* T_0*-space then* $w(X) \geq |X|$.

Proof. Since X is infinite and T_0, no basis of X is finite. Assume that $\lambda := w(X) < |X|$ and let $\mathcal{B}$ be a basis of X with $|\mathcal{B}| = \lambda$. Let X^* denote the set of all $x \in X$ such that $|U| > \lambda$ for every neighborhood U of x. Then $X \setminus X^* \subset \bigcup\{U \in \mathcal{B} \mid |U| \leq \lambda\}$ and hence $|X \setminus X^*| \leq \lambda$. Consequently, $X^* \neq \emptyset$ and if $x \in X^*$ and U is a neighborhood of x then $|X^* \cap U| > \lambda$ (since $|U| > \lambda$). Therefore, the nonempty set X^* is dense in itself and hence the space X is not scattered. □

In order to settle the case $2^\kappa = 2^\lambda$ in Theorems 1.1, 1.2 and 1.4 we will apply the following two enumeration theorems about metrizable spaces. Note that, other than in the model E which proves (I), for $\kappa < \lambda \leq 2^\kappa$ we can rule out $2^\kappa = 2^\lambda$ only in case that $\lambda = 2^\kappa$. (Thus the following two propositions can be ignored if Theorems 1.1, 1.2 and 1.4 are only read as enumeration theorems about spaces X of *maximal possible weights* $2^{|X|}$.)

Let $X + Y$ denote the topological sum of two Hausdorff spaces X and Y. (So $X + Y$ is a space S such that $S = \tilde{X} \cup \tilde{Y}$ for disjoint open subspaces $\tilde{X}, \tilde{Y}$ of S where $\tilde{X}$ is homeomorphic to X and $\tilde{Y}$ is homeomorphic to Y.) If $Y = \emptyset$ then we put $X + Y = X$.

PROPOSITION 2.2. *For every κ there is a family $\mathcal{H}_\kappa$ of mutually non-homeomorphic scattered, strongly zero-dimensional metrizable spaces of size κ such that $|\mathcal{H}_\kappa| = 2^\kappa$ and if D is any discrete space (including the case $D = \emptyset$) then the spaces $H_1 + D$ and $H_2 + D$ are never homeomorphic for distinct $H_1, H_2 \in \mathcal{H}_\kappa$.*

By Lemma 2.1 and since $w(Y) \leq |Y|$ for every metrizable space Y, we have $w(X) = |X|$ for every $X \in \mathcal{H}_\kappa$. Proposition 2.2 can be verified by considering the spaces constructed in [7] which proves [7, Theorem 1]. Because these spaces X are revealed as mutually non-homeomorphic ones by investigating the αth Cantor derivative $X^{(\alpha)}$ for every ordinal $\alpha > 0$. And, naturally, if X is any space and D is discrete then $(X + D)^{(\alpha)} = X^{(\alpha)}$ for every $\alpha > 0$. The following proposition is proved in [5] Section 4 (if X is connected then $a \in X$ is a *noncut point* when a is not a cut point, i.e. when $X \setminus \{a\}$ remains connected.)

PROPOSITION 2.3. *For every $\kappa \geq \mathbf{c}$ there is a family $\mathcal{P}_\kappa$ of mutually non-homeomorphic pathwise connected, locally pathwise connected, complete metric spaces of size and weight κ such that $|\mathcal{P}_\kappa| = 2^\kappa$ and if $H \in \mathcal{P}_\kappa$ then H contains a noncut point and the cut points of H lie dense in H.*

3. Almost discrete and almost metrizable spaces

In accordance with [11], a space is *completely normal* when every subspace is normal. In [2] such spaces are called *hereditarily normal*.

LEMMA 3.1. *If X is a Hausdorff space and $z \in X$ such that $X \setminus \{z\}$ is a discrete subspace of X then X is scattered and completely normal and strongly zero-dimensional.*

Proof. Put $Y := X \setminus \{z\}$. Since Y is a discrete and open subspace of X, every nonempty subset of X contains an isolated point, whence X is scattered. Let $A, B \subset X$ with $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. If $z \notin A \cup B$ then $A, B \subset Y$ and hence $A \subset U$ and $B \subset V$ with the two disjoint open sets $U = A$ and $V = B$. Assume $z \in A \cup B$ and, say, $z \in A$. Then $z \notin \overline{B}$ and hence $B \subset Y$. Thus $A \subset \tilde{U}$ and $B \subset \tilde{V}$ with the two disjoint open sets $\tilde{U} = X \setminus \overline{B}$ and $\tilde{V} = B$. So X is completely normal. Finally, let $A \subset X$ be closed. If $z \notin A$ then A is open. If $z \in A$ and U is an open neighborhood of A then U is closed since $X \setminus U \subset Y$. So every closed subset of X has a basis of open-closed neighborhoods and hence X is strongly zero-dimensional. □

LEMMA 3.2. *If Z is a regular space such that $Z \setminus \{z\}$ is paracompact for some $z \in Z$ then Z is paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of Z. Trivially, $\mathcal{U}^* := \{U \setminus \{z\} \mid U \in \mathcal{U}\}$ is an open cover of the paracompact open subspace $P = Z \setminus \{z\}$ of Z. Hence we can find an open cover $\mathcal{V}^*$ of P which is a locally finite refinement of $\mathcal{U}^*$. Fix one set $U_z \in \mathcal{U}$ with $z \in U_z$ and choose a closed neighborhood C of z in the regular space Z such that $C \subset U_z$. Now put $\mathcal{V} := \{V^* \setminus C \mid V^* \in \mathcal{V}^*\} \cup \{U_z\}$. Clearly, $\mathcal{V}$ is an open cover of Z which is a refinement of $\mathcal{U}$. If $z \neq x \in Z$ then some neighborhood of x meets only finitely many members of $\mathcal{V}^*$ and hence only finitely many members of $\mathcal{V}$. And C is a neighborhood of z which meets $V \in \mathcal{V}$ if and only if $V = U_z$. Therefore, the cover $\mathcal{V}$ is locally finite in Z and hence Z is paracompact. □

Since metrizability implies paracompactness and since the union of two G_δ-sets is a G_δ-set, from Lemma 3.1 and Lemma 3.2 we derive the following two corollaries.

COROLLARY 3.3. *Let X be a Hausdorff space and $z \in X$ such that $X \setminus \{z\}$ is a discrete subspace of X and $\{z\}$ is a G_δ-set in X. Then the almost discrete space X is hereditarily paracompact and perfectly normal.*

COROLLARY 3.4. *Let X be a regular space and $z \in X$ such that the subspace $X \setminus \{z\}$ is metrizable and $\{z\}$ is a G_δ-set in X. Then X is hereditarily paracompact and perfectly normal and hence almost metrizable.*

4. The single filter topology

Let X, z be as in Lemma 3.1 and consider the family $\mathcal{U}$ of all open neighborhoods of the point z. Since $\{x\}$ is open in X whenever $z \neq x \in X$, the family $\mathcal{U}$ coincides with the neighborhood filter at z in the space X. Consequently, $\mathcal{U}^* := \{U \setminus \{z\} \mid U \in \mathcal{U}\}$ is the power set of $X \setminus \{z\}$ if z is isolated in X or, equivalently, if X is discrete. And $\mathcal{U}^*$ is a filter on the set $X \setminus \{z\}$ if z is a limit point of X or, equivalently, if the discrete subspace $X \setminus \{z\}$ is dense in X. Since X is Hausdorff, it is plain that $\bigcap \mathcal{U}^* = \emptyset$.

Conversely, let Y be an infinite set and $z \notin Y$ and let $\mathcal{F}$ be a filter on the set Y. Define a topology $\tau[\mathcal{F}]$ on the set $X := Y \cup \{z\}$ by declaring $U \subset X$ open if and only if either $z \notin U$ or $U = \{z\} \cup F$ for some $F \in \mathcal{F}$. It is plain that this is a correct definition of a topology on the set X. Furthermore, Y is a discrete and open and dense subspace of $(X, \tau[\mathcal{F}])$, whence $\{z\}$ is closed in X. It is plain that $(X, \tau[\mathcal{F}])$ is a Hausdorff space if and only if the filter $\mathcal{F}$ is free, i.e.$\bigcap \mathcal{F} = \emptyset$. So by Lemma 3.1 the almost discrete space $(X, \tau[\mathcal{F}])$ is hereditarily paracompact and scattered and strongly zero-dimensional for every free filter $\mathcal{F}$ on Y.

For abbreviation throughout the paper let us call a filter $\mathcal{F}$ *ω-free* if and only if $\bigcap \mathcal{A} = \emptyset$ for some countable $\mathcal{A} \subset \mathcal{F}$. In view of Corollary 3.3 the following statement is evident.

(III) *If $\mathcal{F}$ is a filter on Y then $(X, \tau[\mathcal{F}])$ is almost discrete and perfectly normal if and only if $\mathcal{F}$ is ω-free.*

The following observation is essential for the proof of Theorem 1.1.

(IV) *If $\mathcal{F}$ is a free filter on Y then the almost discrete space $(X, \tau[\mathcal{F}])$ is extremally disconnected if and only if $\mathcal{F}$ is an ultrafilter.*

Proof. Firstly let $\mathcal{F}$ be a free ultrafilter. Let $U \subset X$ be open. If $\overline{U} = U$ then $\overline{U}$ is open. So assume $\overline{U} \neq U$. Then $\overline{U} = U \cup \{z\}$ and $z \notin U$ since z is the only limit point in X. Thus $U \subset Y$ and z is a limit point of U. Hence every open neighborhood of z meets U. In other words, $F \cap U \neq \emptyset$ for every $F \in \mathcal{F}$. Consequently, $U \in \mathcal{F}$ since $\mathcal{F}$ is an ultrafilter. Thus $\overline{U} = U \cup \{z\}$ is open in X, whence $(X, \tau[\mathcal{F}])$ is extremally disconnected. Secondly, let $\mathcal{F}$ be a free filter and assume that $(X, \tau[\mathcal{F}])$ is extremally disconnected. Let $A \subset Y$, whence A is open in X. If $\overline{A} = A$ then $X \setminus A$ is open and hence $Y \setminus A$ lies in $\mathcal{F}$. If $\overline{A} \neq A$ then $\overline{A} = \{z\} \cup A$ is open and hence A lies in $\mathcal{F}$. This reveals $\mathcal{F}$ as an ultrafilter. □

REMARK 4.1. If $|Y| = \aleph_0$ and $\mathcal{F}$ is a free ultrafilter on Y then $\tau[\mathcal{F}]$ is the well-known *single ultrafilter topology* (see [11, Example 114]).

For a filter $\mathcal{F}$ on Y let $\chi(\mathcal{F})$ denote the least possible size of a filter base which generates $\mathcal{F}$. Trivially, $\chi(\mathcal{F}) \leq |\mathcal{F}| \leq 2^{|Y|}$. The notation $\chi(\cdot)$ corresponds with the obvious fact that $\chi(\mathcal{F})$ is the *character* of z in $(X, \tau[\mathcal{F}])$. (The character $\chi(a, A)$ of a point a in a space A is the smallest possible size of a local basis at a in the space A.) Therefore, since $\{y\}$ is open in $(X, \tau[\mathcal{F}])$ for every $y \in Y$, we obtain:
(V) *If $\mathcal{F}$ is a free filter on Y then the weight of $(X, \tau[\mathcal{F}])$ is* $\max\{|Y|, \chi(\mathcal{F})\}$.

PROPOSITION 4.2. *If $|Y| = \kappa \leq \lambda \leq 2^\kappa$ then there exist 2^λ ω-free filters $\mathcal{F}$ on Y such that $\chi(\mathcal{F}) = \lambda$.*

REMARK 4.3. The cardinal 2^λ in Proposition 4.2 is best possible. Indeed, let Y be an infinite set of size κ and let $\lambda \geq \kappa$. Since a filter base on Y is a subset of the power set of Y, there are at most 2^λ filter bases $\mathcal{B}$ on Y with $|\mathcal{B}| = \lambda$. Hence Y cannot carry more than 2^λ filters $\mathcal{F}$ with $\chi(\mathcal{F}) = \lambda$.

Proof (of Proposition 4.2). Assume $|Y| = \kappa \leq \lambda \leq 2^\kappa$ and let $\mathcal{A}$ be a family of subsets of Y such that $|\mathcal{A}| = 2^\kappa$ and
(VI) *If $\mathcal{D}, \mathcal{E} \neq \emptyset$ are disjoint finite subfamilies of $\mathcal{A}$ then $\bigcap \mathcal{D} \not\subset \bigcup \mathcal{E}$.*
A construction of such a family $\mathcal{A}$ is elementary, see [4, 7.7]. However, this is not enough for our purpose. In view of the property *ω-free*, we additionally have to make sure that the family $\mathcal{A}$ also contains a countably infinite family $\mathcal{A}_\omega$ such that $\bigcap \mathcal{A}_\omega = \emptyset$. By applying Lemma 11.3 in Section 11 for $\mu = \aleph_0$ we can assume that such a family $\mathcal{A}_\omega \subset \mathcal{A}$ exists. Now put $\mathbf{A}_\lambda := \{\mathcal{H} \mid \mathcal{A}_\omega \subset \mathcal{H} \subset \mathcal{A} \wedge |\mathcal{H}| = \lambda\}$. Clearly, $|\mathbf{A}_\lambda| = (2^\kappa)^\lambda = 2^\lambda$. By virtue of (VI), if for $\mathcal{H} \in \mathbf{A}_\lambda$ we put

$$\mathcal{B}_\mathcal{H} := \{H_1 \cap \cdots \cap H_n \mid n \in \mathbb{N} \wedge H_1, \ldots, H_n \in \mathcal{H}\}$$

then $\emptyset \notin \mathcal{B}_\mathcal{H}$ and hence $\mathcal{B}_\mathcal{H}$ is a filter base on Y. For every $\mathcal{H} \in \mathbf{A}_\lambda$ let $\mathcal{F}[\mathcal{H}]$ denote the filter on Y generated by $\mathcal{B}_\mathcal{H}$. Clearly, $|\mathcal{B}_\mathcal{H}| = |\mathcal{H}| = \lambda$ for every $\mathcal{H} \in \mathbf{A}_\lambda$.

The filter $\mathcal{F}[\mathcal{H}]$ is ω-free because $\mathcal{A}_\omega \subset \mathcal{F}[\mathcal{H}]$ by definition. Furthermore, (VI) implies that for distinct families $\mathcal{H}_1, \mathcal{H}_2 \in \mathbf{A}_\lambda$ the filters $\mathcal{F}[\mathcal{H}_1]$ and $\mathcal{F}[\mathcal{H}_2]$ must be distinct. So the family $\{\mathcal{F}[\mathcal{H}] \mid \mathcal{H} \in \mathbf{A}_\lambda\}$ consists of 2^λ ω-free filters on Y.

It remains to verify that $\chi(\mathcal{F}[\mathcal{H}]) = \lambda$ for every $\mathcal{H} \in \mathbf{A}_\lambda$. Assume indirectly that for some $\mathcal{H} \in \mathbf{A}_\lambda$ we have $\chi(\mathcal{F}[\mathcal{H}]) \neq \lambda$ and hence $\chi(\mathcal{F}[\mathcal{H}]) < \lambda$. (Clearly $\chi(\mathcal{F}[\mathcal{H}]) \leq \lambda$ since $|\mathcal{B}_\mathcal{H}| = |\mathcal{H}| = \lambda$.) Choose a filter base $\mathcal{B}$ on Y which generates the filter $\mathcal{F}[\mathcal{H}]$ such that $|\mathcal{B}| < \lambda$. Since $\mathcal{B} \subset \mathcal{F}[\mathcal{H}]$ and $\mathcal{F}[\mathcal{H}]$ is generated by the filter base $\mathcal{B}_\mathcal{H}$, we can choose for every $B \in \mathcal{B}$ a finite set $\mathcal{H}_B \subset \mathcal{H}$ such that $B \supset \bigcap \mathcal{H}_B$. Put $\mathcal{U} := \bigcup_{B\in\mathcal{B}} \mathcal{H}_B$. Then $\mathcal{U} \subset \mathcal{H}$ and $|\mathcal{U}| \leq |\mathcal{B}| < \lambda$. Consequently, $\mathcal{H} \setminus \mathcal{U} \neq \emptyset$. Choose any set $A \in \mathcal{H} \setminus \mathcal{U}$. Then $A \in \mathcal{F}[\mathcal{H}]$ and hence we can find a set $B \in \mathcal{B}$ with $A \supset B$. Then $A \supset \bigcap \mathcal{H}_B$ and hence $A \in \mathcal{H}_B$ by virtue of (VI). But then $A \in \mathcal{U}$ in contradiction with choosing A in $\mathcal{H} \setminus \mathcal{U}$. □

Proposition 4.2 can be improved in the important case $\lambda = 2^\kappa$ as follows.

Proposition 4.4. *On an infinite set of size κ there exist precisely 2^{2^κ} ω-free ultrafilters $\mathcal{F}$ such that $\chi(\mathcal{F}) = 2^\kappa$.*

Proof. Let Y be a set of size κ. As in the previous proof let $\mathcal{A}$ be a family of subsets of Y such that $|\mathcal{A}| = 2^\kappa$ and (VI) holds. Here we need not consider $\mathcal{A}_\omega \subset \mathcal{A}$. Let $\mathbf{A}$ denote the family of all subfamilies $\mathcal{G}$ of $\mathcal{A}$ such that $|\mathcal{G}| = 2^\kappa$. Clearly, $|\mathbf{A}| = 2^{2^\kappa}$. Now for every $\mathcal{G} \in \mathbf{A}$ define $\mathcal{W}[\mathcal{G}] := \mathcal{G} \cup \{Y \setminus \bigcap \mathcal{H} \mid \mathcal{H} \subset \mathcal{G} \wedge |\mathcal{H}| \geq \aleph_0\} \cup \{Y \setminus A \mid A \in \mathcal{A} \setminus \mathcal{G}\}$.

A moment's reflection suffices to see that (VI) implies that $W_1 \cap \cdots \cap W_n \neq \emptyset$ whenever $W_1, \ldots, W_n \in \mathcal{W}[\mathcal{G}]$. Hence for every $\mathcal{G} \in \mathbf{A}$ we can choose an ultrafilter $\mathcal{U}[\mathcal{G}]$ on Y such that $\mathcal{U}[\mathcal{G}] \supset \mathcal{W}[\mathcal{G}]$ (see [1, 7.1]).

If $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{A}$ are distinct and, say, $G \in \mathcal{G}_1 \setminus \mathcal{G}_2$ then $G \in \mathcal{W}[\mathcal{G}_1]$ and $Y \setminus G \in \mathcal{W}[\mathcal{G}_2]$ and hence $G \in \mathcal{U}[\mathcal{G}_1]$ and $G \notin \mathcal{U}[\mathcal{G}_2]$ and hence the ultrafilters $\mathcal{U}[\mathcal{G}_1]$ and $\mathcal{U}[\mathcal{G}_2]$ are distinct as well. Consequently, the family $\{\mathcal{U}[\mathcal{G}] \mid \mathcal{G} \in \mathbf{A}\}$ consists of 2^{2^κ} ultrafilters on Y. All these ultrafilters are ω-free because if $\mathcal{G} \in \mathbf{A}$ and $\mathcal{H}$ is a countably infinite subset of $\mathcal{G}$ then by virtue of (VI) the family $\mathcal{H}^* := \{H \setminus \bigcap \mathcal{H} \mid H \in \mathcal{H}\}$ is countably infinite and it is trivial that $\bigcap \mathcal{H}^* = \emptyset$ and from $\mathcal{H} \subset \mathcal{W}[\mathcal{G}]$ and $Y \setminus \bigcap \mathcal{H} \in \mathcal{W}[\mathcal{G}]$ we derive $\mathcal{H}^* \subset \mathcal{U}[\mathcal{G}]$. (Actually, by a deep argument from set theory it is superfluous to verify that $\mathcal{U}[\mathcal{G}]$ is ω-free, see the remark below.)

Finishing the proof, we claim that $\chi(\mathcal{U}[\mathcal{G}]) = 2^\kappa$ for every $\mathcal{G} \in \mathbf{A}$. Assume indirectly that for $\mathcal{G} \in \mathbf{A}$ the ultrafilter $\mathcal{U}[\mathcal{G}]$ is generated by a filter base $\mathcal{B}$ with $|\mathcal{B}| < 2^\kappa$. Since $\mathcal{G} \subset \mathcal{U}[\mathcal{G}]$, for every $G \in \mathcal{G}$ we have $G \supset B$ for some $B \in \mathcal{B}$. From $|\mathcal{B}| < |\mathcal{G}|$ we derive the existence of a set $B \in \mathcal{B}$ and an infinite subset $\mathcal{H} \subset \mathcal{G}$ such that $H \supset B$ for every $H \in \mathcal{H}$. Consequently, $\bigcap \mathcal{H} \supset B$ and hence $\bigcap \mathcal{H} \in \mathcal{U}[\mathcal{G}]$. This, however, is a contradiction since $Y \setminus \bigcap \mathcal{H}$ lies in $\mathcal{U}[\mathcal{G}]$ by the definition of $\mathcal{W}[\mathcal{G}]$. □

Remark 4.5. Our proof of Proposition 4.4 is elementary and purely set-theoretical. There is also a topological but much less elementary way to prove Proposition 4.4. First of all, if one can prove that any set of size κ carries 2^{2^κ} ultrafilters of character 2^κ then Proposition 4.4 must be true. Because, an ultrafilter $\mathcal{F}$ is free if and only if $\chi(\mathcal{F}) > 1$ and if a free ultrafilter $\mathcal{F}$ is not ω-free then it is plain that $\mathcal{F}$ is σ-complete. However, the existence of a σ-complete free ultrafilter is unprovable in ZFC (see [4, 10.2 and 10.4]). Now, consider the set Y of size κ equipped with the discrete topology and consider the Stone-Čech compactification βY of Y and its compact remainder $Y^* = \beta Y \setminus Y$. So the points in Y^* are the free ultrafilters on Y and if for $p \in Y^*$ we consider the subspace $Y \cup \{p\}$ of βY then it is clear that the character of the ultrafilter p equals $\chi(p, Y \cup \{p\})$. It is a nice exercise to verify that $\chi(p, Y \cup \{p\}) = \chi(p, Y^*)$ for every $p \in Y^*$. By embedding an appropriate Stone space of a Boolean algebra into Y^* it can be proved that Y^* must contain 2^{2^κ} points p with $\chi(p, Y^*) = 2^\kappa$, see [1, 7.13, 7.14, 7.15].

5. Proof of Theorem 1.1

Assume $\mu \leq \kappa \leq \lambda \leq 2^\mu$ and let Y be a set of size μ. Let $\mathbf{F}_\lambda$ denote a family of ω-free filters on Y such that $|\mathbf{F}_\lambda| = 2^\lambda$ and $\chi(\mathcal{F}) = \lambda$ for every $\mathcal{F} \in \mathbf{F}_\lambda$. Such a family exists by Proposition 4.2. We additionally assume that if $\lambda = 2^\mu$ then every member of $\mathbf{F}_\lambda$ is an ultrafilter. This additional assumption is justified by Proposition 4.4.

Now fix $z \notin Y$ and for every $\mathcal{F} \in \mathbf{F}_\lambda$ consider the single filter topology $\tau[\mathcal{F}]$ on the set $X = Y \cup \{z\}$ as in Section 4. If $\mu < \kappa$ then let D be a discrete space of size κ. If $\mu = \kappa$ then put $D = \emptyset$. In both cases define the space $(\tilde{X}, \tilde{\tau}[\mathcal{F}])$ as the topological sum of D and the space $(X, \tau[\mathcal{F}])$. (So if $\mu = \kappa$ then $\tilde{X} = X$ and $\tilde{\tau}[\mathcal{F}] = \tau[\mathcal{F}]$.) Clearly, $\tilde{X}$ is almost discrete, scattered, strongly zero-dimensional, hereditarily paracompact, and perfectly normal. Furthermore, $w(\tilde{X}) = \lambda$ and $|\tilde{X}| = \kappa$. If $\lambda = 2^\mu$ then the space $\tilde{X}$ is also extremally disconnected by virtue of (IV).

Obviously, $\tilde{\tau}[\mathcal{F}_1] \neq \tilde{\tau}[\mathcal{F}_2]$ whenever the filters $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}_\lambda$ are distinct. (For if $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}_\lambda$ and $F \in \mathcal{F}_1 \setminus \mathcal{F}_2$ then $F \cup \{z\}$ is $\tilde{\tau}[\mathcal{F}_1]$-open but not $\tilde{\tau}[\mathcal{F}_2]$-open.) Consequently, the family $\mathcal{T}_\lambda := \{\tilde{\tau}[\mathcal{F}] \mid \mathcal{F} \in \mathbf{F}_\lambda\}$ is of size 2^λ.

We distinguish the two cases $2^\lambda > 2^\mu$ and $2^\lambda \leq 2^\mu$. Assume firstly that $2^\lambda > 2^\mu$ or, equivalently, that $|\mathcal{T}_\lambda| > 2^\mu$. Define an equivalence relation $\sim$ on $\mathcal{T}_\lambda$ by $\tau_1 \sim \tau_2$ if and only if the spaces $(\tilde{X}, \tau_1)$ and $(\tilde{X}, \tau_2)$ are homeomorphic. We claim that the size of an equivalence class cannot be greater than 2^μ.

This is clearly true if $\mu = \kappa$ because there are only 2^μ permutations on X. So assume $\mu < \kappa$. If $\tau \in \mathcal{T}_\lambda$ then in the space $(\tilde{X}, \tau)$ the point z is the only limit point and every neighborhood U of z is open-closed. As a consequence, for $\tau_1, \tau_2 \in \mathcal{T}_\lambda$ the spaces $(\tilde{X}, \tau_1)$ and $(\tilde{X}, \tau_2)$ are homeomorphic if and only if there is a homeomorphism φ from the τ_1-subspace X of $\tilde{X}$ onto some τ_2-open-closed subspace of $\tilde{X}$. Indeed, if f is a homeomorphism from $(\tilde{X}, \tau_1)$ onto $(\tilde{X}, \tau_2)$ then put $\varphi(x) = f(x)$ for every $x \in X$ and φ fits since $f(z) = z$. Conversely, if φ is a homeomorphism from the τ_1-subspace X of $\tilde{X}$ onto some τ_2-open-closed subspace of $\tilde{X}$ and g is any bijection from $\tilde{X} \setminus X$ onto $\tilde{X} \setminus \varphi(X)$ then it is plain that a homeomorphism f from $(\tilde{X}, \tau_1)$ onto $(\tilde{X}, \tau_2)$ is defined by $f(x) = \varphi(x)$ for $x \in X$ and $f(x) = g(x)$ for $x \notin X$. Note that $|\tilde{X} \setminus X| = |\tilde{X} \setminus \varphi(X)|$ since $\mu < \kappa$. Therefore, since there are precisely κ^μ mappings from X into $\tilde{X}$, the size of an eqivalence class in $\mathcal{T}_\lambda$ cannot exceed κ^μ. And from $2 < \mu \leq \kappa \leq 2^\mu$ we derive $2^\mu \leq \mu^\mu \leq \kappa^\mu \leq (2^\mu)^\mu = 2^\mu$ and hence $\kappa^\mu = 2^\mu$.

So the size of an equivalence class can indeed not be greater than 2^μ. Consequently,

$|\mathcal{T}_\lambda| > 2^\mu$ implies that the total number of all equivalence classes equals $|\mathcal{T}_\lambda| = 2^\lambda$. Thus by choosing one topology in each equivalence class we obtain 2^λ mutually non-equivalent topologies $\tau \in \mathcal{T}_\lambda$ and hence the 2^λ corresponding spaces $(\tilde{X}, \tau)$ are mutually non-homeomorphic. This settles the case $2^\lambda > 2^\mu$. In particular, we have already proved the second and the third statement in Theorem 1.1 because, under the assumption $\kappa \leq \lambda \leq 2^\kappa$, if $\lambda = 2^\mu$ for some μ then $\lambda = 2^\mu$ (and hence $2^\lambda > 2^\mu$) for some $\mu \leq \kappa$ and if $\lambda \leq 2^\mu < 2^\lambda$ and $\mu > \kappa$ then $2^\kappa \leq 2^\mu < 2^\lambda$ and hence $2^\lambda > 2^{\mu'}$ for $\mu' = \kappa$.

Secondly assume that $2^\lambda \leq 2^\mu$. Then we have $2^\lambda = 2^\kappa$ since $\mu \leq \kappa \leq \lambda$ implies $2^\mu \leq 2^\kappa \leq 2^\lambda$. So in order to conclude the proof of Theorem 1.1 we assume $\kappa \leq \lambda \leq 2^\kappa = 2^\lambda$. (Then, of course, $\kappa \leq \lambda < 2^\kappa = 2^\lambda$.) Since the special case $\kappa = \lambda$ is settled by Proposition 2.2, we also assume $\kappa < \lambda$. For two spaces X_1 and X_2 let, again, $X_1 + X_2$ denote the topological sum of X_1 and X_2. Let $\mathcal{H}_\kappa$ be a family provided by Proposition 2.2. Due to metrizability, every space in $\mathcal{H}_\kappa$ is perfectly normal and hereditarily paracompact.

By considering an appropriate single filter topology on a set of size κ, we can choose a perfectly normal space Z of size κ such that for some point $z \in Z$ the subspace $Z \setminus \{z\}$ is discrete and $\chi(z, Z) = \lambda$. (Consequently, $w(Z) = \lambda$.) For every space $H \in \mathcal{H}_\kappa$ consider the topological sum $H + Z$. Of course, the topological sum of two paracompact spaces is paracompact and $(H + Z) \setminus \{z\} = H + (Z \setminus \{z\})$ for every $H \in \mathcal{H}_\kappa$. Consequently, for every $H \in \mathcal{H}_\kappa$ the space $H + Z$ is scattered and strongly zero-dimensional and perfectly normal and hereditarily paracompact and $|H + Z| = |H| = \kappa$ and $w(H + Z) = \max\{w(H), w(Z)\} = \max\{\kappa, \lambda\} = \lambda$. Therefore, since $|\mathcal{H}_\kappa| = 2^\kappa$, the case $2^\lambda = 2^\kappa$ in Theorem 1.1 is settled by showing that for two distinct (and hence non-homeomorphic) metrizable spaces $H_1, H_2 \in \mathcal{H}_\kappa$ the two spaces $H_1 + Z$ and $H_2 + Z$ are never homeomorphic. Assume that $H_1, H_2 \in \mathcal{H}_\kappa$ and that f is a homeomorphism from $H_1 + Z$ onto $H_2 + Z$. Then $f(z) = z$ since $w((H_i + Z) \setminus \{z\}) = \kappa < \lambda$ and $\chi(z, H_i + Z) = \chi(z, Z) = \lambda$. Consequently, f maps $(H_1 + Z) \setminus \{z\}$ onto $(H_2 + Z) \setminus \{z\}$. Therefore, since $Z \setminus \{z\}$ is discrete and $(H + Z) \setminus \{z\} = H + (Z \setminus \{z\})$ for every $H \in \mathcal{H}_\kappa$, we have $H_1 = H_2$ in view of Proposition 2.2.

6. Proof of Theorem 1.2

In order to find a natural way to prove Theorem 1.2 (and also Theorem 1.4) we give a short proof of the following consequence of Theorem 1.2.

(VII) *If* $\mathbf{c} \leq \kappa \leq \lambda \leq 2^\kappa$ *then there exist* 2^λ *mutually non-homeomorphic pathwise connected, paracompact Hausdorff spaces of size* κ *and weight* λ.

From Theorem 1.1 (VII) can easily be derived as follows. Assume $\mathbf{c} \leq \kappa \leq \lambda \leq 2^\kappa$. By Theorem 1.1 there exists a family $\mathcal{P}$ of 2^λ mutually non-homeomorphic, totally disconnected, paracompact Hausdorff spaces X of size κ and weight λ. For every $X \in \mathcal{F}$ let $\mathcal{Q}(X)$ denote the quotient space of $X \times [0, 1]$ by its closed subspace $X \times \{1\}$. The quotient space $\mathcal{Q}(X)$ can be directly defined as follows. Consider the product space $X \times [0,1[$ and fix $p \notin X \times [0,1[$ and put $\mathcal{Q}(X) := \{p\} \cup (X \times [0,1[)$. Declare a subset U of $\mathcal{Q}(X)$ open if and only if $U \setminus \{p\}$ is open in the product space $X \times [0,1[$ and $p \in U$ implies that $(U \setminus \{p\}) \cup (X \times \{1\})$ is open in the space $X \times [0,1]$. One can picture $\mathcal{Q}(X)$ as a *cone* with apex p and all rulings $\{p\} \cup (\{x\} \times [0,1[)(x \in X)$ homeomorphic to the unit interval $[0,1]$. By [2, 5.1.36 and 5.1.28] both $X \times [0,1]$ and $X \times [0,1[$ are paracompact. Consequently, $\mathcal{Q}(X)$ is a regular space and hence $\mathcal{Q}(X)$ is paracompact in view of Lemma 3.2. It is evident that $\mathcal{Q}(X)$ is pathwise connected. Trivially, $|\mathcal{Q}(X)| = \kappa$.

Unfortunately, we can be sure that $w(\mathcal{Q}(X)) = \lambda$ for every $X \in \mathcal{P}$ only if $\lambda = 2^\kappa$. (Since $|\mathcal{Q}(X)| = \kappa$, we have $w(\mathcal{Q}(X)) \leq 2^\kappa$. On the other hand, $w(\mathcal{Q}(X)) \geq w(\mathcal{Q}(X) \setminus \{p\}) = w(X \times [0,1[) = w(X) = \lambda$.) The problem with the weight is that if μ is the character of the apex p then $w(\mathcal{Q}(X)) = \max\{w(X \times [0,1[), \mu\} = \max\{\lambda, \mu\}$. But we cannot rule out $\lambda < \mu$ if $\lambda < 2^\kappa$. Of course, if $X \in \mathcal{P}$ is compact then $\mu = \aleph_0$ and hence $w(\mathcal{Q}(X)) = \lambda$ (but also $\lambda \leq |X| = \kappa$). Fortunately, we can make the character of the apex countable also by harshly reducing the filter of the neighborhoods of p. Let $\mathcal{Q}^*(X)$ be defined as the cone $\mathcal{Q}(X)$ but with the (only) difference that $U \subset \{p\} \cup (X \times [0,1[)$ is an open neighborhood of p if and only if $U \setminus \{p\}$ is open in $X \times [0,1[$ and $U \supset X \times [t,1[$ for some $t \in [0,1[$. Now we have $\chi(p, \mathcal{Q}^*(X)) = \aleph_0$ and hence $w(\mathcal{Q}^*(X)) = w(X)$ for every $X \in \mathcal{P}$. Of course, $\mathcal{Q}^*(X)$ is pathwise connected. By the same arguments as for $\mathcal{Q}(X)$, the space $\mathcal{Q}^*(X)$ is regular and paracompact. Finally, the spaces $\mathcal{Q}(X)(X \in \mathcal{P})$ are mutually non-homeomorphic because every $X \in \mathcal{P}$ can be recovered (up to homeomorphism) from $\mathcal{Q}(X)$. Indeed, since X is totally disconected, if Z is the set of all $z \in \mathcal{Q}(X)$ such that $\mathcal{Q}(X) \setminus \{z\}$ remains pathwise connected then it is evident that $Z = X \times \{0\}$ and hence Z is homeomorphic with X. This concludes the proof of (VII).

In the following proof of Theorem 1.2 we will also work with cones but we cannot use the cones $\mathcal{Q}(X)$ or $\mathcal{Q}^*(X)$ because it is evident that if X is not discrete then neither $\mathcal{Q}(X)$ nor $\mathcal{Q}^*(X)$ is locally connected. Furthermore, by virtue of Corollary 3.4 and since $\{p\}$ is a G_δ-set in the space $\mathcal{Q}^*(X)$, the cone $\mathcal{Q}^*(X)$ is almost metrizable if and only if X is metrizable (then $w(\mathcal{Q}^*(X)) = w(X) = \kappa$). Consequently, $\mathcal{Q}^*(X)$ is locally connected and almost metrizable if and only if X is discrete. Now the clue in the following proof of Theorem 1.2 is to consider $\mathcal{Q}^*(S)$ for one discrete spaces S of size (and weight) κ and to reduce the topology of $\mathcal{Q}^*(S)$ in 2^λ ways such that the weight κ of $\mathcal{Q}^*(S)$ is increased to λ and that 2^λ non-homeomorphic spaces as desired are obtained. First of all we need a lemma.

Lemma 6.1. *If $n \in \mathbb{N}$ and A is a topological space and $a \in A$ and $A_1, \ldots, A_n$ are metrizable, closed subspaces of A and $A = A_1 \cup \cdots \cup A_n$ and $A_i \cap A_j = \{a\}$ whenever $1 \leq i < j \leq n$ then the space A is metrizable.*

Proof. Assume $n \geq 2$. Clearly, if $1 \leq i \leq n$ then $A_i \setminus \{a\} = A \setminus \bigcup_{j \neq i} A_j$ is an open subset of A. Furthermore, if $a \in U_i \subset A_i$ and U_i is open in the subspace A_i for $1 \leq i \leq n$ then $U_1 \cup \cdots \cup U_n$ is an open subset of the space A. (Because if V_i is an open subset of A with $U_i = V_i \cap A_i$ for $1 \leq i \leq n$ then $U_1 \cup \cdots \cup U_n = (V_1 \cap \cdots \cap V_n) \cup \bigcup_{i=1}^n (V_i \cap (A_i \setminus \{a\}))$.) For $1 \leq i \leq n$ consider A_i equipped with a suitable metric d_i. Define a mapping from $A \times A$ into $\mathbb{R}$ in the following way. If $x, y \in A_i$ for some i then put $d(x, y) = d_i(x, y)$. If $x \in A_i$ and $y \in A_j$ for distinct i, j then put $d(x, y) = d_i(x, a) + d_j(y, a)$. Of course, d is a metric on the set A. (One may regard A as a hedgehog with body a and spines $A_1, \ldots, A_n$.) By considering the open neighborhoods of the point a in the space A we conclude that the topology generated by the metric d coincides with the topology of the space A. $\square$

Now we are ready to prove Theorem 1.2. Assume $\mathbf{c} \leq \kappa < \lambda \leq 2^\kappa$. (We ignore the case $\kappa = \lambda$ because this case is covered by Proposition 2.3.) Let S be a discrete space of size κ and $\mathcal{F}$ an ω-free filter on S with $\chi(\mathcal{F}) = \lambda$. Consider the metrizable product space $S \times [0, 1[$ and fix $p \notin S \times [0, 1[$ and define a topological space $\Phi[\mathcal{F}]$ in the following way. The points in the space $\Phi[\mathcal{F}]$ are the elements of $\{p\} \cup (S \times [0, 1[)$ and a subset U of $\{p\} \cup (S \times [0, 1[)$ is open if and only if firstly $U \setminus \{p\}$ is open in the product space $S \times [0, 1[$ and secondly the point p lies in U only if $(S \times [t, 1[) \cup (F \times [0, 1[) \subset U$ for some $t \in [0, 1[$ and some $F \in \mathcal{F}$.

It is plain that this is a correct definition of a topological space such that the subspace $\Phi[\mathcal{F}] \setminus \{p\}$ is identical with the product space $S \times [0, 1[$. Similarly as above we picture $\Phi[\mathcal{F}]$ as a cone with apex p and the rulings $\{p\} \cup (\{x\} \times [0, 1[) (x \in X)$ homeomorphic to the unit interval $[0, 1]$. (Obviously, the topology of $\Phi[\mathcal{F}]$ is strictly coarser than the topology of the cone $\mathcal{Q}^*(S)$.) It is straightforward to verify that $\Phi[\mathcal{F}]$ is a regular space. Hence by Corollary 3.4 the space $\Phi[\mathcal{F}]$ is almost metrizable. (Since $\mathcal{F}$ is ω-free and $[0, 1]$ is second countable, it is clear that $\{p\}$ is a G_δ-set.) Since the subspace $\{p\} \cup (\{s\} \times [t, 1[)$ of $\Phi[\mathcal{F}]$ is a homeomorphic copy of the compact unit interval $[0, 1]$ for every $s \in S$ and every $t \in [0, 1[$ and since S is discrete, it is clear that $\Phi[\mathcal{F}]$ is pathwise connected and locally pathwise connected. Trivially, $|\Phi[\mathcal{F}]| = \kappa$.

Clearly, if $\mathcal{B}$ is a filter base on S generating the filter $\mathcal{F}$ then $\big\{\{p\} \cup ((S \setminus F) \times]1 - 2^{-n}, 1[) \cup (F \times [0, 1[) \big| n \in \mathbb{N}, F \in \mathcal{B}\big\}$ is a local basis at p in the space $\Phi[\mathcal{F}]$. Conversely, if $\mathcal{U}_p$ is a local basis at p and if we choose for every $U \in \mathcal{U}_p$ a real number $t_U \in [0, 1[$ and a set $F_U \in \mathcal{F}$ such that $(S \times [t_U, 1[) \cup (F_U \times [0, 1[) \subset U$ then $\{F_U \mid U \in \mathcal{U}_p\}$ is a filter base on S generating the filter $\mathcal{F}$. Consequently, $\chi(p, \Phi[\mathcal{F}]) = \chi(\mathcal{F})$. Therefore, since $w(S \times [0, 1[) = \kappa$, we have $w(\Phi[\mathcal{F}]) = \chi(\mathcal{F}) = \lambda$.

Now consider the pathwise connected, locally pathwise connected, amost metrizable space $\Phi[\mathcal{F}]$ for each of the 2^λ ω-free filters $\mathcal{F}$ on S with $\chi(\mathcal{F}) = \lambda$. Since the size of each space is κ and the weight of each space is λ, by the same arguments about the size of equivalence classes as in the proof of Theorem 1.1 (for $\mu = \kappa$), the statement in Theorem 1.2 is true in case that $2^\lambda > 2^\kappa$ because it is evident that the topologies of the spaces $\Phi[\mathcal{F}_1]$ and $\Phi[\mathcal{F}_2]$ are distinct topologies on the set $\{p\} \cup (S \times [0, 1[)$ whenever $\mathcal{F}_1$ and $\mathcal{F}_2$ are distinct ω-free filters on S.

Now assume $2^\lambda = 2^\kappa$ and let $\mathcal{P}_\kappa$ be a family as provided by Proposition 2.3. Choose one ω-free filter $\mathcal{F}$ on S with $\chi(\mathcal{F}) = \lambda$ and consider the space $\Phi[\mathcal{F}]$. Note that $x \in \Phi[\mathcal{F}]$ is a noncut point of $\Phi[\mathcal{F}]$ if and only if $x = (s, 0)$ for some $s \in S$. For every $H \in \mathcal{P}_\kappa$ create a space $X(H)$ in the following way. Consider the compact unit square $[0, 1]^2$ and choose a point $a_1 \in [0, 1]^2$. (Clearly, a_1 is a noncut point of $[0, 1]^2$. Note also that no connected open subset of $[0, 1]^2$ has cut points.) Choose a noncut point a_2 in $\Phi[\mathcal{F}]$ and a noncut point a_3 in H. Finally, let $X(H)$ be the quotient of the topological sum of the three spaces $[0,1]^2$ and $\Phi[\mathcal{F}]$ and H by the subspace $\{a_1, a_2, a_3\}$. Roughly speaking, $X(H)$ is created by sticking together the three spaces so that the three points a_1, a_2, a_3 are identified. It is clear that $X(H)$ is pathwise connected and locally pathwise connected and regular and $|X(H)| = \kappa$ and $w(X(H)) = \lambda$.

There is precisely one point $b \in X(H)$ with $\chi(b, X(H)) = \lambda$. This point b corresponds with the point $p \in \Phi[\mathcal{F}]$. By virtue of Lemma 6.1 for $n = 3$ the subspace $X(H) \setminus \{b\}$ of $X(H)$ is metrizable. Consequently, if $H \in \mathcal{P}_\kappa$ then $X(H)$ is almost metrizable. The 2^κ spaces $X(H)(H \in \mathcal{P}_\kappa)$ are mutually non-homeomorphic because each $H \in \mathcal{P}_\kappa$ can be recovered from $X(H)$ as follows.

Since cut points in H resp. in $\Phi[\mathcal{F}]$ lie dense and since $[0,1]^2$ has no cut points, there is precisely one point q in $X(H)$ such that every neighborhood of q contains two nonempty connected open sets U_1, U_2 where U_1 has no cut points and where U_2 has cut points. (This point q must be the point obtained by identifying the three points a_1, a_2, a_3.) The subspace $X(H)\setminus\{q\}$ has precisely three components and every component of $X(H) \setminus \{q\}$ is homeomorphic either with $\Phi[\mathcal{F}] \setminus \{a_2\}$ or with $H \setminus \{a_3\}$ or with $[0,1]^2 \setminus \{a_1\}$. Therefore, precisely one component is not metrizable. (If $s \in S$ then the space $\Phi[\mathcal{F}] \setminus \{(s,0)\}$ is not metrizable since it has no countable local basis at p.) The two metrizable components of $X(H) \setminus \{q\}$ can be distinguished by the observation that one component has infinitely many cut points while the other component has no cut points. If M is a metrizable component of $X(H) \setminus \{q\}$ which has cut points then the subspace $M \cup \{q\}$ of $X(H)$ is homeomorphic with H.

7. Proof of Theorem 1.3

Lemma 7.1. *There exists a second countable, countably infinite Hausdorff space H such that $H \setminus E$ is connected and locally connected for every finite set E.*

Proof. Let H be the set $\mathbb{N}$ equipped with the coarsest topology such that if p is a prime and $a \in \mathbb{N}$ is not divisible by p then $\mathbb{N} \cap \{p + ka \mid k \in \mathbb{Z}\}$ is open. Referring to [11, Nr. 61], H is a locally connected Hausdorff space such that the intersection of the closures of any two nonempty open subsets of H must be an infinite set. Therefore, if E is a finite set then the subspace $H \setminus E$ of H is connected. Since H is locally connected, $H \setminus E$ is locally connected for every finite set E, □

The first step in proving Theorem 1.3 is a proof of the following enumeration theorem about countable connected spaces.

Theorem 7.2. *For every $\lambda \leq \mathbf{c}$ there exist 2^λ mutually non-homeomorphic connected, locally connected Hausdorff spaces of size $\aleph_0$ and weight λ.*

Proof. Let H be a connected, locally connected Hausdorff space with $|H| = w(H) = \aleph_0$ as provided by Lemma 7.1. Fix $e \in H$ and note that e is a noncut point in H. Put $M := H \setminus \{e\}$. So M is connected as well.

Let S be an infinite discrete space and let $\mathcal{F}$ be a free filter on S with $\chi(\mathcal{F}) \geq |S|$. Consider the product space $S \times M$ and fix $p \notin S \times M$ and consider $\Psi[\mathcal{F}] := \{p\} \cup (S \times M)$ equipped with the following topology. A subset U of $\{p\} \cup (S \times M)$ is open if and only if $U \setminus \{p\}$ is open in the product space $S \times M$ and $p \in U$ implies that $(S \times (V \setminus \{e\})) \cup (F \times M) \subset U$ for some neighborhood V of e in H and some $F \in \mathcal{F}$. Similarly as in the proof of Theorem 1.2, $\Psi[\mathcal{F}]$ is a connected and locally connected Hausdorff space and $|\Psi[\mathcal{F}]| = |S|$ and $w(\Psi[\mathcal{F}]) = \chi(\mathcal{F})$.

Now let S be the discrete Euclidean space $\mathbb{N}$. If $2^\lambda > \mathbf{c}$ then with the help of 2^λ free filters on $\mathbb{N}$ with $\chi(\mathcal{F}) = \lambda$ we can track down 2^λ mutually non-homeomorphic spaces $\Psi[\mathcal{F}]$. (Note that there are only $\mathbf{c}$ permutations on $\mathbb{N}$ and use the argument on sizes of equivalence classes.) So it remains to settle the case $2^\lambda = \mathbf{c}$.

Let Z be the space $\Psi[\mathcal{F}]$ for some free filter $\mathcal{F}$ on $\mathbb{N}$ with $\chi(\mathcal{F}) = \lambda$. So the underlying set of Z is $\{p\} \cup (\mathbb{N} \times (H \setminus \{e\}))$ and the countable Hausdorff space Z is connected and locally connected and $w(Z) = \lambda$ due to $\chi(p, Z) = \lambda$. The point p is the only cut point of Z and $Z \setminus \{p\}$ has infinitely many components. Keep in mind that $|H| = w(H) = \aleph_0$ and that if $a \in H$ then the spaces H and $H \setminus \{a\}$ and $H \setminus \{a, e\}$ are connected and locally connected. Fix $b \in H \setminus \{e\}$ and consider the subset $\hat{Z} := \{(s, b) \mid s \in \mathbb{N}\}$ of Z. Clearly, $\hat{Z}$ is closed and discrete and $Z \setminus \{z\}$ is connected and locally connected for every $z \in \hat{Z}$. Choose for every $m \in \mathbb{N}$ and every $i \in \{1, \ldots, m\}$ spaces $H_i^{(m)}$ such that $H_i^{(m)}$ is homeomorphic with H and $H_i^{(m)} \cap H_j^{(n)} = \emptyset$ whenever $m \neq n$ or $i \neq j$. Furthermore assume that $H_i^{(m)} \cap Z = \emptyset$ for every m and every i. Let φ be a choice function on the class of all infinite sets, i.e. $\varphi(A) \in A$ for every infinite set A. Now define for every nonempty set $T \subset \mathbb{N}$ a Hausdorff space $Q[T]$ as follows. Consider the topological sum $\Sigma[T]$ of countably infinite and mutually disjoint spaces where the summands are Z and all spaces $H_i^{(m)}$ with $m \in T$ and $i \in \{1, \ldots, m\}$. Define an equivalence relation on $\Sigma[T]$ such that the non-singleton equivalence classes are precisely the sets $\{(m, b)\} \cup \{\varphi(H_1^{(m)}), \ldots, \varphi(H_m^{(m)})\}$ with $m \in T$. (Note that $(m, b) \in \hat{Z}$ for every $m \in T$.) Finally, let $Q[T]$ denote the quotient space of $\Sigma[T]$ with respect to this equivalence relation. Roughly speaking, $Q[T]$ is the union of Z and all spaces $H_i^{(m)}$ with $m \in T$ and $i \in \{1, \ldots, m\}$ where for every $m \in T$ the $m + 1$ points $(m, b), \varphi(H_1^{(m)}), \ldots, \varphi(H_m^{(m)})$ are identified. We consider Z to be a subset of $Q[T]$. One may picture $Q[T]$ as an expansion of Z created by attaching m copies of H to Z at the point $(m, b) \in \hat{Z}$ for every $m \in T$. It is evident that $Q[T]$ is a connected and locally connected countably infinite Hausdorff space. We have $w(Q[T]) = \lambda$ since Z is a subspace of $Q[T]$ with $w(Z) = \lambda$ and $\chi(x, Q[T]) = \aleph_0$ if $p \neq x \in Q[T]$. Thus the case $2^\lambda = \mathbf{c}$ is settled by verifying that two spaces $Q[T_1]$ and $Q[T_2]$ cannot be homeomorphic if $\emptyset \neq T_1, T_2 \subset \mathbb{N}$ and $T_1 \neq T_2$. This must be true because the set $T \subset \mathbb{N}$ is completely determined by the topology of $Q[T]$ by the following observation.

Let $\emptyset \neq T \subset \mathbb{N}$. For every point $x \in Q[T]$ let $\nu(x)$ denote the total number of all components of the subspace $Q[T] \setminus \{x\}$. The following three statements for $x \in Q[T]$ are evident. Firstly, $\nu(x) \geq \aleph_0$ if and only if $x = p$. Secondly, $1 < \nu(x) < \aleph_0$ if and only if $x = (m, b) \in \hat{Z}$ for some $m \in T$. Thirdly, $\nu(x) = 1$ if and only if x is an element of the set $Q[T] \setminus ((T \times \{b\}) \cup \{p\})$. Concerning the second statement we compute

$\nu((m,b)) = m+1$ for every $m \in T$. Consequently, $\{\nu(x)-1 \mid x \in Q[T] \wedge \nu(x) \in \mathbb{N}\} \setminus \{0\} = T$ whenever T is one of the $\mathbf{c}$ non-empty subsets of $\mathbb{N}$. □

Now in order to prove Theorem 1.3 assume $\aleph_0 \leq \kappa < \mathbf{c}$ and $\kappa \leq \lambda \leq 2^\kappa$. Referring to Theorem 7.2 there is nothing more to show in case that $\kappa = \aleph_0$. So we also assume that $\kappa > \aleph_0$. Let S be a discrete space of size κ. By Proposition 4.2 there are 2^λ free filters $\mathcal{F}$ on S with $\chi(\mathcal{F}) = \lambda$. For each one of these filters $\mathcal{F}$ consider the connected and locally connected Hausdorff space $\Psi[\mathcal{F}]$ of size κ and weight λ as defined in the previous proof. Hence in case that $2^\lambda > 2^\kappa$ we can track down 2^λ filters $\mathcal{F}$ on S such that the corresponding spaces $\Psi[\mathcal{F}]$ are mutually non-homeomorphic.

So it remains to settle the case $2^\lambda = 2^\kappa$. Choose any free filter $\mathcal{F}$ on S with $\chi(\mathcal{F}) = \lambda$ and and consider the space $\Psi := \Psi[\mathcal{F}]$ of size κ and weight λ. Fix a noncut point $z \in \Psi$. Keep in mind that Ψ has precisely one cut point p and that $\Psi[\mathcal{F}] \setminus \{p\}$ has precisely κ and hence *uncountably many* components.

In view of our proof of Theorem 7.2 there is a family $\mathcal{C}$ of mutually non-homeomorphic *countable* Hausdorff spaces of weight κ (and hence not necessarily of weight λ) such that $|\mathcal{C}| = 2^\kappa$ and if $C \in \mathcal{C}$ then C is connected and locally connected and contains precisely one cut point $q(C)$ such that $C \setminus \{q(C)\}$ has infinitely many components. In particular, all these components are countable and $\aleph_0$ is their total number.

For every $C \in \mathcal{C}$ consider the topological sum $\Psi + C$ and define an equivalence relation such that $\{z, q(C)\}$ is an equivalence class and all other equivalence classes are singletons. Let $Q[C]$ denote the quotient space of $\Psi + C$ with respect to this equivalence relation. So $Q[C]$ is obtained by sticking together the spaces Ψ and C at one point and this point is the identification of $z \in \Psi$ and $q(C) \in C$. It is clear that $Q[C]$ is a connected, locally connected Hausdorff space of size κ and weight λ. So we are done by verifying that for distinct $C_1, C_2 \in \mathcal{C}$ the spaces $Q[C_1]$ and $Q[C_2]$ are never homeomorphic. This must be true because each $C \in \mathcal{C}$ can be recovered from $Q[C]$ as follows.

There is a unique point ξ in $Q[C]$ such that $Q[C] \setminus \{\xi\}$ has precisely $\aleph_0$ components. (This point ξ is the one corresponding with the equivalence class $\{z, q(C)\}$.) Among these components there is precisely one of uncountable size. (This component is the one which contains the point $p \in \Psi$.) Let K be the unique uncountable component of $Q[C] \setminus \{\xi\}$. Then $Q[C] \setminus K$ is essentially identical, at least homeomorphic with the space C.

8. Proof of Theorem 1.4

Our goal is to derive Theorem 1.4 from Theorem 1.1 by using appropriate modifications of the cones $\mathcal{Q}^*(X)$ considered in Section 6. In order to accomplish this we need building blocks provided by the following lemma.

Lemma 8.1. *There exists a second countable, connected, totally pathwise disconnected, nowhere locally connected, metrizable space M of size* $\mathbf{c}$ *which contains precisely one noncut point b, where $M \setminus \{x, b\}$ has precisely two components whenever $b \neq x \in M$.*

Proof. Let f be a function from $\mathbb{R}$ into $\mathbb{R}$ such that the graph of f is a dense and connected subset of the Euclidean plane $\mathbb{R}^2$ (see [8] for a construction of such a function f.) Automatically, f is discontinuous everywhere. Let M be the intersection of $[0,\infty[\times\mathbb{R}$ and the graph of f. It is straightforward to check that M fits with $b=(0,f(0))$. □

Now we are ready to prove Theorem 1.4. Assume $\mathbf{c}\leq\kappa\leq\lambda\leq 2^\kappa$ and let $\mathcal{Y}=\mathcal{Y}(\kappa,\lambda)$ be a family of precisely 2^λ mutually non-homeomorphic scattered, normal spaces of size κ and weight λ such that if $Y\in\mathcal{Y}$ then for a certain finite set $\gamma(Y)\subset Y$ the subspace $Y\setminus\gamma(Y)$ is metrizable (and hence of weight κ) and $\gamma(Y)$ is a G_δ-set in Y. Precisely, the set $\gamma(Y)$ is empty when $\kappa=\lambda$ and a singleton $\{y\}$ when $\kappa<\lambda$. (Clearly, if $\gamma(Y)=\{y\}$ then $\chi(y,Y)=\lambda$.) If $2^\lambda>2^\kappa$ then such a family $\mathcal{Y}$ exists by considering the 2^λ almost discrete spaces provided by Theorem 1.1. If $\lambda>\kappa$ and $2^\lambda=2^\kappa$ then such a family $\mathcal{Y}$ exists in view of the construction in Section 5 which proves Theorem 1.1 in case that $2^\lambda=2^\kappa$. If $\lambda=\kappa$ then such a family $\mathcal{Y}$ exists by Proposition 2.2.

Let M be a metrizable space as in Lemma 8.1 and let b denote the noncut point of M and fix a point $a\in M\setminus\{b\}$. For an infinite, scattered, normal space X consider the product space $X\times M$ and fix $p\notin X\times M$ and put $K(X):=\{p\}\cup(X\times(M\setminus\{b\})$. Declare a subset U of $K(X)$ open if and only if $U\setminus\{p\}$ is open in the product space $X\times(M\setminus\{b\})$ and $p\in U$ implies that U contains $X\times(N\setminus\{b\})$ for some neighborhood N of b in the space M. It is plain that $K(X)$ is a well-defined regular space. Since M is metrizable and $\chi(p,K(X))=\aleph_0$, if X is metrizable then $K(X)$ has a σ-locally finite base and hence $K(X)$ is metrizable.

Now for $Y\in\mathcal{Y}$ consider the subspace $L(Y):=K(Y)\setminus(\gamma(Y)\times(M\setminus\{a,b\}))$ of $K(Y)$ and the subspace $S(Y):=L(Y)\setminus(\gamma(Y)\times\{a\})$ of $L(Y)$. Trivially, the spaces $K(Y)$ and $L(Y)$ and $S(Y)$ coincide if $\kappa=\lambda$. Furthermore the space $S(Y)$ coincides with the metrizable space $K(Y\setminus\gamma(Y))$. Therefore and by Corollary 3.4, $L(Y)$ is an almost metrizable space since $\gamma(Y)\times\{a\}$ is a G_δ-set in $K(Y)$ of size 0 or 1. We have $|L(Y)|=\kappa$ and $w(L(Y))=\lambda$ because if $\gamma(Y)=\{y\}$ then $\chi((y,a),L(Y))=\lambda$. It is evident that $S(Y)$ is connected and totally pathwise disconnected and nowhere locally connected. Consequently, $L(Y)$ is totally pathwise disconnected and nowhere locally connected. And $L(Y)$ is connected since the connected set $S(Y)$ is dense in $L(Y)$.

Finally, the spaces $L(Y)(Y\in\mathcal{Y})$ are mutually non-homeomorphic because every $Y\in\mathcal{Y}$ can be recovered from $L(Y)$. Indeed, for $x\in L(Y)$ let $\mathcal{C}(x)$ denote the family of all components of the subspace $L(Y)\setminus\{x\}$ of $L(Y)$. Then $\mathcal{C}(x)$ is an infinite set if and only if $x=p$. Because the scattered space Y has infinitely many isolated points and if $u\in Y$ is isolated then $\{u\}\times(M\setminus\{b\})$ lies in $\mathcal{C}(x)$. If $u\in Y\setminus\gamma(Y)$ and $b\neq v\in M$ then $|\mathcal{C}((u,v))|\leq 2$ (and $|\mathcal{C}((u,v))|=2$ when u is isolated in Y). And if $\gamma(Y)=\{y\}$ then $|\mathcal{C}((y,a))|=1$. Thus $\{p\}=\{x\in L(Y)\mid|\mathcal{C}(x)|\geq\aleph_0\}$, whence the point p can be recovered from the space $L(Y)$. Now let $\mathcal{C}$ be the family of all components of the space $L(Y)\setminus\{p\}$. Since Y is totally disconnected, the members of $\mathcal{C}$ are precisely the sets $\{u\}\times(M\setminus\{b\})$ with $u\in Y\setminus\gamma(Y)$ plus the singleton $\gamma(Y)\times\{a\}$ if and only if $\gamma(Y)\neq\emptyset$. Naturally, the quotient space of $L(Y)\setminus\{p\}$ by the equivalence relation defined via the partition $\mathcal{C}$ is homeomorphic with Y for every $Y \in \mathcal{Y}$. This concludes the proof of Theorem 1.4.

9. Compact spaces of excessive weights

While $w(X) \leq |X|$ for every compact Hausdorff space X (see [2, 3.3.6]), for compact T_1-spaces X one cannot rule out $w(X) > |X|$ and actually we can prove the following enumeration theorem by applying Theorems 1.1–1.3.

THEOREM 9.1. *If $\kappa \leq \lambda \leq 2^\kappa$ then there exist two families $\mathcal{C}_1, \mathcal{C}_2$ of mutually non-homeomorphic compact T_1-spaces of size κ and weight λ such that $|\mathcal{C}_1| = |\mathcal{C}_2| = 2^\lambda$ and all spaces in $\mathcal{C}_1$ are scattered, all spaces in $\mathcal{C}_2$ are connected and locally connected, and if $\kappa \geq \mathbf{c}$ then all spaces in $\mathcal{C}_2$ are arcwise connected and locally arcwise connected.*

In order to prove Theorem 9.1 we consider T_1-compactifications of Hausdorff spaces. If Y is an infinite Hausdorff space with $|Y| \leq w(Y)$ then define a topological space $\Gamma(Y)$ which expands Y in the following way. Put $\Gamma(Y) = Y \cup \{z\}$ where $z \notin Y$ and declare $U \subset \Gamma(Y)$ open either when U is an open subset of Y or when $z \in U$ and $Y \setminus U$ is finite. It is clear that in this way a topology on $\Gamma(Y)$ is well-defined such that Y is a dense subspace of $\Gamma(Y)$. Obviously, $\Gamma(Y) \setminus \{x\}$ is open for every $x \in \Gamma(Y)$ and hence $\Gamma(Y)$ is a T_1-space. Since all neighborhoods of z cover the whole space $\Gamma(Y)$ except finitely many points, $\Gamma(Y)$ is compact. Trivially, $|\Gamma(Y)| = |Y|$. We have $w(\Gamma(Y)) = w(Y)$ since $w(Y) \geq |Y|$ and Y is a subspace of $\Gamma(Y)$ and, by definition, there is a local basis at z of size $|Y|$.

Evidently, if Y is scattered then $\Gamma(Y)$ is scattered. On the other hand it is clear that if Y is dense in itself then $\Gamma(Y)$ is connected and every neighborhood of z is connected. So if Y is connected and locally connected then $\Gamma(Y)$ is connected and locally connected.

We claim that if Y is pathwise connected then $\Gamma(Y)$ is arcwise connected. Assume that the Hausdorff space Y is pathwise connected and hence arcwise connected and let $a \in Y$. Of course it is enough to find an arc which connects the point a with the point $z \notin Y$. Since Y is arcwise connected, we can define a homeomorphism φ from $[0,1]$ onto a subspace of Y such that $\varphi(0) = a$. Define an injective function f from $[0,1]$ into $\Gamma(Y)$ via $f(1) = z$ and $f(t) = \varphi(t)$ for $t < 1$. Let U be an open subset of $\Gamma(Y)$. If $z \in U$ then $U \setminus Y$ is finite and thus $f^{-1}(U)$ is a cofinite and hence open subset of $[0,1]$. If $z \notin U$ then U is an open subset of Y and hence $f^{-1}(U) = \varphi^{-1}(U) \setminus \{1\}$ is an open subset of $[0,1]$. Thus the injective function f is continuous.

Since every neighborhood of z contains all but finitely many points from Y, by exactly the same arguments we conclude that if Y is locally pathwise connected then every neighborhood of z is an arcwise connected subspace of $\Gamma(Y)$. Consequently, if the Hausdorff space Y is locally pathwise connected then the T_1-space $\Gamma(Y)$ is locally arcwise connected.

The space Y can be recovered from $\Gamma(Y)$ (up to homeomorphism) provided that Y has at least two limit points. Because then it is evident that z is the unique point

$x \in \Gamma(Y)$ such that the subspace $\Gamma(Y) \setminus \{x\}$ of $\Gamma(Y)$ is Hausdorff.

By virtue of Theorem 1.1, for $\kappa \leq \lambda \leq 2^\kappa$ let $\mathcal{Y}_1(\kappa, \lambda)$ be a family of mutually non-homeomorphic, scattered Hausdorff spaces of size κ and weight λ such that $|\mathcal{Y}_1(\kappa, \lambda)| = 2^\lambda$. By virtue of Theorem 1.3, for $\kappa < \mathbf{c}$ and $\kappa \leq \lambda \leq 2^\kappa$ let $\mathcal{Y}_2(\kappa, \lambda)$ be a family of mutually non-homeomorphic connected and locally connected Hausdorff spaces of size κ and weight λ such that $|\mathcal{Y}_2(\kappa, \lambda)| = 2^\lambda$. By virtue of Theorem 1.2, for $\mathbf{c} \leq \kappa \leq \lambda \leq 2^\kappa$ let $\mathcal{Y}_3(\kappa, \lambda)$ be a family of mutually non-homeomorphic pathwise connected and locally pathwise connected Hausdorff spaces of size κ and weight λ such that $|\mathcal{Y}_3(\kappa, \lambda)| = 2^\lambda$. Now put $\mathcal{C}_1 := \{\Gamma(Y) \mid Y \in \mathcal{Y}_1(\kappa, \lambda)\}$ and $\mathcal{C}_2 := \{\Gamma(Y) \mid Y \in \mathcal{Y}_i(\kappa, \lambda)\}$ where $i = 2$ when $\kappa < \mathbf{c}$ and $i = 3$ when $\kappa \geq \mathbf{c}$. Then $\mathcal{C}_1, \mathcal{C}_2$ are families which prove Theorem 9.1.

The condition $\kappa \geq \mathbf{c}$ in Theorem 9.1 is inevitable since, trivially, $|X| \geq \mathbf{c}$ for every infinite, *arcwise connected* space. There arises the question whether $|X| \geq \mathbf{c}$ is inevitable for infinite, *pathwise connected* T_1-spaces. (Of course, every finite T_1-space X is discrete and hence not pathwise connected when $|X| \geq 2$.) It is well-known that a pathwise connected T_1-space of size $\aleph_0$ does not exist (see also Proposition 9.2 below). So the essential question is whether there are pathwise connected T_1-spaces X with $\aleph_0 < |X| < \mathbf{c}$ (provided that there are cardinals μ with $\aleph_0 < \mu < \mathbf{c}$). The following proposition shows that there is no chance to track down such spaces X.

PROPOSITION 9.2. *Pathwise connected* T_1*-spaces* X *with* $2 \leq |X| \leq \aleph_0$ *do not exist. It is consistent with* ZFC *that* $|\{\kappa \mid \aleph_0 < \kappa < \mathbf{c}\}| > \aleph_0$ *and pathwise connected* T_1*-spaces* X *with* $\aleph_0 < |X| < \mathbf{c}$ *do not exist.*

If X is a T_1-space and $f : [0,1] \to X$ is continuous then $\{f^{-1}(\{x\}) \mid x \in X\} \setminus \{\emptyset\}$ is a decomposition of $[0,1]$ into precisely $|f([0,1])|$ nonempty closed subsets. Therefore, Proposition 9.2 is an immediate consequence of

PROPOSITION 9.3. *Every partition of* $[0,1]$ *into at least two closed sets is uncountable. It is consistent with* ZFC *that uncountably many cardinals* κ *with* $\aleph_0 < \kappa < \mathbf{c}$ *exist while still a partition* $\mathcal{P}$ *of* $[0,1]$ *into closed sets with* $\aleph_0 < |\mathcal{P}| < \mathbf{c}$ *does not exist.*

Certainly, the first statement in Proposition 9.3 is an immediate consequence of Sierpiński's theorem [2, 6.1.27]. However, in order to prove Proposition 9.3 we need another approach than in the proof of [2, 6.1.27].

Assume that $\mathcal{P}$ is a partition of $[0,1]$ into closed sets with $|\mathcal{P}| \geq 2$. For $S \subset [0,1]$ let ∂S denote the boundary of S in the compact space $[0,1]$. (Notice that then $\partial[0,1] = \emptyset$.) Put $\mathcal{V} := \{\partial A \mid A \in \mathcal{P}\}$ and $W := \bigcup \mathcal{V}$. Then $\emptyset \notin \mathcal{V}$ since $[0,1] \notin \mathcal{P}$ and hence $\mathcal{V}$ is a partition of W with $|\mathcal{V}| = |\mathcal{P}|$. The nonempty set W is a closed subset of $[0,1]$ because $W = [0,1] \setminus \bigcup\{A \setminus \partial A \mid A \in \mathcal{P}\}$ since $\mathcal{P}$ is a partition of $[0,1]$. We claim that the closed sets $V \in \mathcal{V}$ are nowhere dense in the compact metrizable space W.

Let $A \in \mathcal{P}$ and assume indirectly that a is an interior point of ∂A in W. Then there is an interval I open in the compact space $[0,1]$ with $a \in I$ and $I \cap W \subset \partial A$. Since a lies in the boundary of A the interval I intersects $[0,1] \setminus A$ and hence for some $B \neq A$ in the family $\mathcal{P}$ we have $I \cap B \neq \emptyset$. However, $I \cap \partial B = \emptyset$ in view of $(\partial A) \cap (\partial B) = \emptyset$ and $I \cap W \subset \partial A$. Therefore, $I \cap B$ is a nonempty set which is open and closed in the connected space I and hence $I \cap B = I$ contrarily with $A \cap I \neq \emptyset$ and $A \cap B = \emptyset$.

Thus $\mathcal{V}$ is a partition of the compact Hausdorff space W into nowhere dense subsets with $|\mathcal{V}| = |\mathcal{P}|$. Therefore $|\mathcal{P}| \leq \aleph_0$ is impossible since W is a space of second category. This concludes the proof of the first statement. Under the assumption of *Martin's Axiom* (see [4, 16.11]) also the weaker inequality $|\mathcal{V}| = |\mathcal{P}| < \mathbf{c}$ is impossible because it is well-known that Martin's axiom implies that no separable, compact Hausdorff space can be covered by less than $\mathbf{c}$ nowhere dense subsets. (Actually, Martin's axiom is *equivalent* to the statement that in every compact Hausdorff space of countable cellularity any intersection of less than $\mathbf{c}$ dense, open sets is dense.) Therefore, the proof of Proposition 9.3 is concluded by checking that the existence of uncountably many infinite cardinals below $\mathbf{c}$ is consistent with ZFC plus Martin's Axiom. This is certainly true because by applying the Solovay-Tennenbaum theorem [4, 16.13] there is a model of ZFC in which Martin's Axiom holds and the identity $2^{\aleph_0} = \aleph_{\omega_1+1}$ is enforced. (If $\mathbf{c} = \aleph_{\omega_1+1}$ then $|\{\kappa \mid \kappa < \mathbf{c}\}| = \aleph_1 > \aleph_0$.)

Remark 9.4. There is an interesting observation concerning compactness and the Hausdorff separation axiom. By applying Theorem 9.1 and (II), *there exist precisely* $\mathbf{c}$ *compact, countable, second countable* T_1*-spaces up to homeomorphism.* If in this statement T_1 is sharpened to T_2 then we obtain an *unprovable hypothesis.* Indeed, due to Mazurkiewicz and Sierpiński [10], there exist precisely $\aleph_1$ countable (and hence second countable) compact Hausdorff spaces up to homeomorphism. The hypothesis $\aleph_1 < \mathbf{c}$ is irrefutable since it is a trivial consequence of (I). This discrepancy of provability vanishes when *uncountable* compacta are counted up to homeomorphism. Indeed, by virtue of [6, Theorem 1.3] it can be accomplished that in Theorem 9.1 for $\kappa = \lambda > \aleph_0$ all spaces in the family $\mathcal{C}_1$ are Hausdorff spaces. (Note that $w(X) = |X|$ for every scattered, compact Hausdorff space.)

10. Pathwise connected, scattered spaces

Naturally, a scattered T_1-space is totally disconnected and hence far from being pathwise connected. Furthermore it is plain that no scattered space is *arcwise* connected. Therefore and in view of Proposition 9.2 the following enumeration theorem is worth mentioning.

Theorem 10.1. *If $\kappa \leq \lambda \leq 2^\kappa$ then there exist two families $\mathcal{C}, \mathcal{L}$ of mutually non-homeomorphic pathwise connected, scattered T_0-spaces of size κ and weight λ such that $|\mathcal{C}| = |\mathcal{L}| = 2^\lambda$ and all spaces in $\mathcal{C}$ are compact and if $\kappa \leq \mathbf{c}$ or $2^\kappa < 2^\lambda$ then all spaces in $\mathcal{L}$ are locally pathwise connected.*

The existence of the family $\mathcal{C}$ in Theorem 10.1 can be derived from Theorem 1.1 in view of the following considerations. Let X be an infinite Hausdorff space. Fix $b \notin X$ and define a topology on the set $B(X) = X \cup \{b\}$ by declaring $U \subset B(X)$ open when either $U = B(X)$ or U is an open subset of X. Then $\{b\}$ is closed and $B(X)$ is the only neighborhood of b. Obviously, $B(X)$ is a compact T_0-space and b is a limit point of every nonempty subset of $X = B(X) \setminus \{b\}$. It is trivial that $|B(X)| = |X|$ and clear that $w(B(X)) = w(X)$. For any pair x, y of distinct points in $B(X)$ define a function f from $[0, 1]$ into $B(X)$ via $f(t) = x$ when $t < \frac{1}{2}$ and $f(\frac{1}{2}) = b$ and $f(t) = y$ when $t > \frac{1}{2}$. It is plain that f is continuous, whence $B(X)$ is pathwise connected. Obviously, if X is scattered then $B(X)$ is scattered. Finally, the space X can be recovered from $B(X)$ since a singleton $\{a\}$ is closed in $B(X)$ if and only if $a = b$.

Unfortunately, if X is scattered and not discrete then $B(X)$ is not locally connected. Fortunately, finishing the proof of Theorem 10.1 we can track down a family $\mathcal{L}$ as desired by adopting the proofs of Theorem 1.3 and Theorem 7.2 in Section 7 line by line such that, throughout, the building block H in the definition of $\Phi[\mathcal{F}]$ provided by Lemma 10.2 is replaced with the space G provided by the following lemma. In Section 7 the restriction $\kappa < \mathbf{c}$ is only for avoiding an overlap between Theorem 1.2 and Theorem 1.3 and can clearly be expanded to $\kappa \leq \mathbf{c}$. The case $2^\kappa < 2^\lambda$ is settled by the 2^λ spaces $\Psi[\mathcal{F}]$ of *arbitrary* size κ.

Lemma 10.2. *There exists a second countable, scattered, countably infinite* T_0*-space* G *such that* $G \setminus E$ *is pathwise connected and locally pathwise connected for every finite set* E.

Proof. Let G be the set $\{n \in \mathbb{Z} \mid n \geq 2\}$ equipped with *divisor topology* as defined in [11, **57**]. A basis of the divisor topology is the family of all sets $\{m \in \mathbb{Z} \mid m \geq 2 \wedge m \mid n\}$ with $n \in G$. In view of the considerations in [11], it is straightforward to verify that G fits. ☐

Remark 10.3. If $i \in \{0, 1, 2\}$ and $\mathcal{F}_i$ is a family of mutually non-homeomorphic compact T_i-spaces X with $w(X) \leq \kappa$ then $|\mathcal{F}_i| \leq 2^\kappa$ is true for $i = 2$. (Because any compact Hausdorff space of weight at most κ is embeddable into the Hilbert cube $[0, 1]^\kappa$ and, since $w([0, 1]^\kappa) = \kappa$ and $|X| = 2^\kappa$, the compact Hausdorff space $[0, 1]^\kappa$ has precisely 2^κ closed subspaces.) However, the estimate $|\mathcal{F}_i| \leq 2^\kappa$ is false for $i = 0$ because $|\mathcal{F}_0| = 2^{2^\kappa}$ can be achieved for every κ. (In view of (II) and since $\max\{|X|, w(X)\} \leq \min\{2^{|X|}, 2^{w(X)}\}$ for every infinite T_0-space X, 2^{2^κ} is the maximal possible cardinality.) Indeed, consider for $X = [0, 1]^\kappa$ the compact T_0-space $B(X) = X \cup \{b\}$ of size 2^κ and weight κ defined as above. Clearly, for every nonempty $S \subset X$ the subspace $S \cup \{b\}$ of $B(X)$ is compact. Since X is Hausdorff and $w(X) = \kappa$, there are $2^{|X|} = 2^{2^\kappa}$ mutually non-homeomorphic subspaces of X and hence 2^{2^κ} mutually non-homeomorphic compact subspaces of $B(X)$. There arises the interesting question whether the estimate $|\mathcal{F}_i| \leq 2^\kappa$ is generally true for $i = 1$.

11. Counting P-spaces

A natural modification of the proof of Theorem 1.1 leads to a noteworthy enumeration theorem about P-spaces. As usual (see [1]), a Hausdorff space is a *P-space* if and only if any intersection of countably many open sets is open. More generally, a Hausdorff space X is a P_α-*space* if and only if α is an infinite cardinal number and $\bigcap\mathcal{U}$ is an open subset of X whenever $\mathcal{U}$ is a family of open subsets of X with $0 \neq |\mathcal{U}| < \alpha$. So if $\alpha = \aleph_0$ then every Hausdorff space is a P_α-space and if $\alpha = \aleph_1$ then X is a P_α-space if and only if X is a P-space. Clearly, if X is a P_α-space and $|X| < \alpha$ then X is discrete. (It is plain that if X is a P_α-space and $|X| = \alpha$ and α is a singular cardinal then X is discrete.)

For an infinite cardinal α let us call a Hausdorff space α-*normal* when it is completely normal and every closed set is an intersection of at most α open sets. So a Hausdorff space is perfectly normal if and only if it is $\aleph_0$-normal. It is dull to consider *perfectly normal* P-spaces because, trivially, a perfectly normal P-space must be discrete. More generally, if $\mu < \alpha$ then every μ-normal P_α-space is discrete. However, the enumeration problem concerning *completely normal* P-spaces and α-normal P_α-spaces is not trivial and can be solved under certain cardinal restrictions.

As usual, κ^+ denotes the smallest cardinal greater than κ, whence $\kappa^+ \leq 2^\kappa$ and $\aleph_1 = (\aleph_0)^+$. Furthermore, for arbitrary κ, μ the cardinal number $\kappa^{<\mu}$ is defined as usual (see [4]). Note that if $\mu \leq \kappa^+$ then $\kappa^{<\mu} = |\{T \mid T \subset S \wedge |T| < \mu\}|$ whenever S is a set of size κ. In particular, $\kappa^{<\aleph_0} = \kappa$ and $\kappa^{<\aleph_1} = \kappa^{\aleph_0}$ for every κ. Naturally, if $\mu = \kappa^+$ then $\kappa^{<\mu} = 2^\kappa$. Consequently, if $\mu > \kappa$ then $\kappa < \kappa^{<\mu}$. (If κ is a cardinal number of cofinality smaller than μ^{++} then $\kappa < \kappa^{<\mu}$ due to [4, Theorem 5.14].) On the other hand, for every μ the cardinals κ satisfying $\kappa^{<\mu} = \kappa$ form a proper class $\mathcal{K}_\mu$ such that $2^\theta \in \mathcal{K}_\mu$ for every cardinal θ with $\theta^+ \geq \mu$ and if $\kappa \in \mathcal{K}_\mu$ then the cardinal successor κ^+ of κ also lies in $\mathcal{K}_\mu$ due to the Hausdorff formula [4, (5.22)]. In particular, the cardinals $\mathbf{c}, \mathbf{c}^+, \mathbf{c}^{++}, \ldots$ lie in $\mathcal{K}_\mu$ for $\mu = \aleph_1$. Furthermore, if $\kappa^{<\mu} = \kappa \leq \lambda$ and there are only finitely many cardinals θ with $\kappa \leq \theta \leq \lambda$ then $\lambda^{<\mu} = \lambda$. (Note, again, that $\kappa^{<\alpha} = \kappa$ implies $\alpha \leq \kappa$.)

Theorem 11.1. *Let α be an uncountable cardinal. Assume $\kappa = \kappa^{<\alpha}$ and $\kappa \leq \lambda \leq 2^\kappa$ and $\lambda^{<\alpha} = \lambda \leq 2^\mu < 2^\lambda$ for some $\mu \leq \kappa$ with $\mu^{<\alpha} = \mu$. Then there exist 2^λ mutually non-homeomorphic scattered, strongly zero-dimensional, hereditarily paracompact, α-normal P_α-spaces of size κ and weight λ. In particular, for every κ with $\kappa = \kappa^{\aleph_0}$ there exist precisely 2^{2^κ} mutually non-homeomorphic paracompact P-spaces of size κ and weight 2^κ up to homeomorphism.*

As usual (see [1,4]), a filter $\mathcal{F}$ is κ-*complete* if and only if $\bigcap\mathcal{A} \in \mathcal{F}$ for every $\mathcal{A} \subset \mathcal{F}$ with $0 \neq |\mathcal{A}| < \kappa$. Trivially, every filter is $\aleph_0$-complete. Obviously, an ω-free filter is not κ-complete for any $\kappa > \aleph_0$. Let us call a filter $\mathcal{F}$ κ-*free* if and only if $\bigcap\mathcal{A} = \emptyset$ for some $\mathcal{A} \subset \mathcal{F}$ with $|\mathcal{A}| \leq \kappa$. (So a filter is ω-free if and only if it is $\aleph_0$-free.) Clearly, if the topology of an almost discrete space X is the single filter topology defined with a free filter $\mathcal{F}$ then for every infinite cardinal α the (completely normal) space X is α-normal if and only if $\mathcal{F}$ is α-free, and X is a P_α-space if and only if $\mathcal{F}$ is α-complete. Therefore, in view of the following counterpart of Proposition 4.2, Theorem 11.1 can be easily proved by simply adopting the proof of the case $2^\lambda > 2^\mu$ in Theorem 1.1 line by line while replacing the property *ω-free* with *α-complete and α-free* throughout.

Proposition 11.2. *If $|Y| = \kappa = \kappa^{<\mu}$ and $\kappa \leq \lambda = \lambda^{<\mu} \leq 2^\kappa$ then there exist 2^λ μ-complete, μ-free filters $\mathcal{F}$ on Y such that $\chi(\mathcal{F}) = \lambda$.*

For the proof of Proposition 11.2 we need a lemma that also guarantees the existence of the family $\mathcal{A}_\omega$ in the proof of Proposition 4.2 since $\kappa^{<\mu} = \kappa$ for $\mu = \aleph_0$.

Lemma 11.3. *Let Y be an infinite set of size κ and assume $\kappa^{<\mu} = \kappa$. Then there exists a family $\mathcal{A}$ of subsets of Y such that $|\mathcal{A}| = 2^\kappa$ and $\mathcal{A}$ has a subfamily $\mathcal{K}$ of size μ with $\bigcap \mathcal{K} = \emptyset$ and if $\mathcal{D}, \mathcal{E} \neq \emptyset$ are disjoint subfamilies of $\mathcal{A}$ of size smaller than μ then $\bigcap \mathcal{D}$ is not a subset of $\bigcup \mathcal{E}$.*

Proof. For an infinite set S put $\mathcal{P}_\mu(S) := \{T \mid T \subset S \wedge |T| < \mu\}$. Let Y be a set of size κ and assume $\kappa^{<\mu} = \kappa$, whence $\kappa \geq \mu$. Choose any set X of size κ. Then $|\mathcal{P}_\mu(X)| = \kappa^{<\mu} = \kappa$ and hence $|\mathcal{P}_\mu(\mathcal{P}_\mu(X))| = \kappa^{<\mu} = \kappa$. Therefore we may identify Y with the set $\mathcal{P}_\mu(X) \times \mathcal{P}_\mu(\mathcal{P}_\mu(X))$. Now for $Y := \mathcal{P}_\mu(X) \times \mathcal{P}_\mu(\mathcal{P}_\mu(X))$ put $A[S] := \{(H, \mathcal{H}) \in Y \mid \emptyset \neq H \cap S \in \mathcal{H}\}$ whenever $S \subset X$. Clearly, $A[S] = \emptyset$ if and only if $S = \emptyset$. We observe that $A[S_1] \neq A[S_2]$ whenever $S_1, S_2 \subset X$ are distinct. Indeed, if S_1, S_2 are subsets of X and $s \in S_1 \setminus S_2$ then $(\{s\}, \{\{s\}\}) \in A[S_1] \setminus A[S_2]$. Put $\mathcal{A} := \{A[S] \mid \emptyset \neq S \subset X\}$. Then $|\mathcal{A}| = 2^\kappa$ and we claim that $\mathcal{A}$ is a family as desired.

For $0 \neq |I \times J| < \mu$ let $\{S_i \mid i \in I\}$ and $\{T_j \mid j \in J\}$ be disjoint families of nonempty subsets of X. Choose $a_{i,j} \in (S_i \setminus T_j) \cup (T_j \setminus S_i)$ for every $(i, j) \in I \times J$ and $b_i \in S_i$ for every $i \in I$ and put $V := \{a_{i,j} \mid i \in I, j \in J\} \cup \{b_i \mid i \in I\}$. Then $|V| < \mu$ and $\emptyset \neq V \cap S_i \neq V \cap T_j$ whenever $i \in I$ and $j \in J$. Hence the pair $(V, \{V \cap S_i \mid i \in I\})$ lies in $\bigcap_{i \in I} A[S_i]$ but not in $\bigcup_{j \in J} A[T_j]$. Finally, since $|H| < \mu$ whenever $(H, \mathcal{H}) \in A[S]$, if $\mathcal{K}$ is any subfamily of $\{A[\{x\}] \mid x \in X\}$ with $|\mathcal{K}| = \mu$ then $\bigcap \mathcal{K} = \emptyset$. □

Remark 11.4. The previous proof is very similar to Hausdorff's classic construction of independent sets as carried out in the proof of [4, 7.7]. However, by Hausdorff (and in [4, 7.7]) only the special case $\mu = \aleph_0$ is considered and, unfortunately, from Hausdorff's construction one cannot obtain ω-free resp. α-free filters in a natural way. In order to accomplish this we have modified the proof of [4, 7.7] in a subtle but crucial way by including the condition $\emptyset \neq H \cap S$ in our definition of $A[S]$. This condition guarantees that $\mathcal{A}$ has a subfamily $\mathcal{K}$ as desired and hence that the family $\mathcal{A}_\omega$ in the proof of Proposition 4.2 actually exists.

Now in order to prove Proposition 11.2 let $\mathcal{A}$ and $\mathcal{K}$ be families as in Lemma 11.3. For every family $\mathcal{H}$ with $\mathcal{K} \subset \mathcal{H} \subset \mathcal{A}$ and $|\mathcal{H}| = \lambda$ put $\mathcal{B}_\mathcal{H} := \{\bigcap \mathcal{G} \mid \emptyset \neq \mathcal{G} \subset \mathcal{H} \wedge |\mathcal{G}| < \mu\}$. Then $\emptyset \notin \mathcal{B}_\mathcal{H}$ and thus $\mathcal{B}_\mathcal{H}$ is a filter base for a μ-complete filter $\mathcal{F}[\mathcal{H}]$. Since $\mathcal{K} \subset \mathcal{F}[\mathcal{H}]$, the filter $\mathcal{F}[\mathcal{H}]$ is μ-free. Since $\lambda^{<\mu} = \lambda$, we have $|\mathcal{B}_\mathcal{H}| = \chi(\mathcal{F}[\mathcal{H}]) = \lambda$ by exactly the same arguments as in the proof of Proposition 4.2.

Remark 11.5. Since for no cardinal $\kappa > \aleph_0$ the existence of a κ-complete *ultrafilter* is provable in ZFC (see [4]), in Theorem 11.1 we cannot include the property *extremally disconnected.* While Theorem 11.1 modifies Theorem 1.1 for P-spaces, there is no pendant of Theorem 1.2 for P-spaces because an infinite P-space is clearly not pathwise connected and, moreover, every regular P-space X is zero-dimensional. (If $x \in U$ where $U \subset X$ is open then choose open neighborhoods U_n of x such that

$U \supset \overline{U_n} \supset U_n \supset \overline{U_{n+1}} \supset U_{n+1}$ for every $n \in \mathbb{N}$. Then $V := \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} \overline{U_n}$ is an open-closed neighborhood of x and $V \subset U$.)

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