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ON PRIME STRONG IDEALS OF A SEMINEARRING

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Abstract. The concept prime ideals and corresponding radicals play an important role in the study of nearrings. In this paper, we define different prime strong ideals of a seminearring S and study corresponding prime radicals. In particular, we prove that $P_e = \{S | P_e(S) = S\}$ is a Kurosh-Amitsur radical class where $P_e(S)$ denotes the intersection of equiprime strong ideals of S.

1. Introduction

A non-empty set S is said to be a right (left) seminearring, if S is a semigroup with respect to the addition $(+)$, multiplication $(\cdot)$ and satisfies right (left) distributive law. A natural example of a seminearring is the set of all mappings from a semigroup to itself with pointwise addition and composition of mappings. Hoorn and Rootsellar [20] defined an ideal of a seminearring as the kernel of a seminearring homomorphism. By using this concept, Javed [1, 2] discussed different types of prime ideals in seminearrings. Booth, Gronewald and Veldsman [6] defined the concept of equiprime ideal of a nearring and proved that the equiprime radical leads to a Kurosh-Amitsur radical class in nearings. Later, Veldsman [22] investigated equiprimeness of well-known examples of nearrings and established its relation with different types of primeness in nearrings. Groenewald [11] defined completely prime radicals in nearrings and gave an element-wise characterization of these radicals. Birkenmeier, Heatherly and Lee [5] showed the interrelationship between different types of prime radicals and prime ideals in nearrings. Anderson, Kaarli and Weigandt [3] discussed radicals and semisimple classes in Ω-groups and in nearrings. Rao and Prasad [19] introduced the concept of an R-group of type 5/2 in nearrings. Further, they defined the Jacobson radical of type 5/2 ($J_{5/2}$) and proved that $J_{5/2}$ is an ideal-heriditary Kurosh-Amitsur radical in the class of zero-symmetric nearrings.

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The present paper aims to extend the concept of radicals to seminearrings. Koppula, Kedukodi and Kuncham [13] defined the concept of strong ideal of a seminearring and proved the classical isomorphism theorems in seminearrings. In this paper, we define various prime ideals of seminearrings and obtain the interrelations among them. Then we define various prime radicals in seminearings and prove that if $P_e(S)$ is an equiprime radical, then P_e is a Kurosh-Amitsur radical class. Section 2 of this paper contains basic definitions and results related to rings, nearrings, seminearrings and radicals. Section 3 contains definitions and examples of various prime ideals in seminearrings. Section 4 establishes a Kurosh-Amitsur radical class for seminearrings.

2. Preliminaries

Definition 2.1 ([20]). A right seminearring is a system $(S,+,\cdot)$ such that
(i) $(S,+)$ is a semigroup. (ii) $(S,\cdot)$ is a semigroup.

(iii) $(s_1+s_2)s_3 = s_1s_3 + s_2s_3$ for all $s_1,s_2,s_3 \in S$.

(iv) There exists $0 \in S$ such that $0+s = s+0 = s$ for all $s \in S$. (v) $0s = 0$ for all $s \in S$.
In this paper, all seminearrings are right seminearrings.

Definition 2.2 ([13]). A non-empty subset I of a seminearring S is said to be a strong ideal of S, if I satisfies the following conditions:
(i) For $x,y \in I$, $x+y \in I$ $(I+I \subseteq I)$. (ii) For $s \in S$, $s+I \subseteq I+s$.

(iii) If $x \equiv_I 0$ then $x \in I$. (iv) $Is \subseteq I$ for all $s \in S$ (right strong ideal).

(v) $s(I+s') \subseteq I+ss'$ for all $s,s' \in S$ (left strong ideal).

Proposition 2.3 ([13]). *If I is a strong ideal of a seminearring S then the canonical mapping $\pi : S \to S/I$ is a seminearring strong onto homomorphism. Conversely, if $h : S \to R$ is a seminearring strong onto homomorphism then $\ker h$ is a strong ideal of S.*

Theorem 2.4 ([13]). *(i) If I and J are strong ideals of a seminearring S then $I \cap J$ is a strong ideal of J and $(I+J)/I \cong J/(I \cap J)$.*

(ii) If I and J are strong ideals of a seminearring S and $I \subseteq J$ then $S/J \cong (S/I)/(J/I)$.

The results related to prime ideals in semirings can be found in Bataineh, Malas [4] and Dubey, Sarohe [8].

Definition 2.5 ([14]). $S_c = \{s \in S \mid ss' = s, \text{ for all } s' \in S\}$ is called a constant part of a seminearring S. $S_0 = \{s \in S \mid s0 = 0\}$ is called a zero-symmetric part of a seminearring S

Definition 2.6 ([15]). A non-empty subset M of a semigroup $(S,+)$ is said to be a subsemigroup, if $x,y \in M$ implies $x+y \in M$.

Definition 2.7 ([14]). An additive subsemigroup M of a seminearing S is said to be a subseminearring, if $0 \in M$ and $MM \subseteq M$.

Definition 2.8 ([18]). A subseminearring M of a seminearing S is called invariant if $MS \subseteq M$ and $SM \subseteq M$.

Definition 2.9 ([10]). Let S be a hemiring and I be a non-empty subset of S. Then I is said to be subtractive, if $x \in I$ and $x + y \in I$ imply $y \in I$.

Definition 2.10 ([9]). A class of rings ρ is said to be hereditary, if $R \in \rho$ and A is an ideal of R imply $A \in \rho$. A class of rings ρ is said to be regular, if $0 \neq A$ is an ideal of S and $S \in \rho$ imply that A has a non zero homomorphic image in ρ.

Note that hereditary implies regularity.

Definition 2.11 ([9]). An ideal I of a nearring N is said to be essential if $I \cap E \neq 0$ for all ideals $0 \neq E$ of N; it is denoted by $I \triangleleft \cdot N$.

Definition 2.12 ([6]). A class μ of nearrings is called closed under essential extensions (resp. essential left invariant extensions) if $I \in \mu$, $I \triangleleft \cdot N$ (resp. I is an essential ideal of N which is left invariant) imply that $N \in \mu$.

An ideal I of a nearring N is called equiprime if $a \in N/I$, $x, y \in N$ and $arx - ary \in I$ for all $r \in N$ imply $x - y \in I$.

For the results related to lattices, we refer to [12] and for boolean nearrings we refer to [17].

Definition 2.13 ([6]). A class σ of seminearrings is said to satisfy condition (F_1) if for each strong ideal B of A, where A is a left invariant strong ideal of S, such that $A/B \in \sigma$, B is a strong ideal of S.

Definition 2.14 ([9]). A seminearring S is a subdirect sum $S = \sum\limits_{Subdirect} (S_\lambda \mid \lambda \in \Lambda)$, if

(i) S is a subseminearring of the direct product $A = \prod(S_\lambda \mid \lambda \in \Lambda)$;

(ii) $\pi_\lambda(S) = S_\lambda$ for every projection $\pi_\lambda : A \to S_\lambda,\ \lambda \in \Lambda$.

Definition 2.15 ([21]). Let σ be a class of seminearrings and μ be a subclass of σ. Then the subdirect closure of a class μ is $\overline{\mu} = \{S \in \sigma \mid S$ is a subdirect sum of the seminearrings from $\mu\}$.

The class of seminearrings σ is said to be C-hereditary, if $S \in \sigma$ and T is a left invariant ideal of S imply $T \in \sigma$.

We refer to Kedukodi, Kuncham, Jagadeesha [16] for the related results in nearrings.

3. Prime ideals in seminearrings

DEFINITION 3.1. A strong ideal I of a seminearring S is said to be equiprime strong if $a, x, y \in S$ with $asx \equiv_I asy$, $\forall s \in S$ implies that either $a \in I$ or $x \equiv_I y$.

A strong ideal I of a seminearring S is said to be 3-prime strong if $a, b \in S$ with $asb \in I$ $\forall s \in S$ implies that either $a \in I$ or $b \in I$.

EXAMPLE 3.2. Let $S = \{0, a, b, c, d\}$, and $+$ and $\cdot$ be defined as follows.

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	b	b
c	c	d	d	c	d
d	d	d	d	d	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	d	d
c	0	a	b	d	d
d	0	a	b	d	d

Then S is a seminearring with respect to $+$ and $\cdot$, and $I = \{0, a\}$ is an equiprime and 3-prime strong ideal of S. The equivalence classes with respect to I are $a/I = 0/I = \{0, a\}$, $b/I = \{b\}$ and $c/I = \{c, d\}$.

EXAMPLE 3.3. Let $S = \{0, a, b, c\}$, and $+$ and $\cdot$ be defined as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

Then S is a seminearring with respect to $+$ and $\cdot$, and $I = \{0\}$ is a strong ideal of S. The equivalence classes with respect to I are $0/I = \{0\}$, $a/I = \{a\}, b/I = \{b\}$ and $c/I = \{c\}$. Note that I is a 3-prime strong ideal. However, I is not an equiprime strong ideal, because $ar0 \equiv_I ara$, $\forall r \in S$ implies $a \notin I$ and $0 \not\equiv_I a$.

PROPOSITION 3.4. *Let S be a seminearring and I be a strong ideal of S. Then I is subtractive if and only if $x \equiv_I 0$ implies $x \in I$, $\forall x \in S$.*

Proof. Let I be subtractive. Take $x \in S$ such that $x \equiv_I 0$. Then there exist $i_1, i_2 \in I$ such that $i_1 + x = i_2 + 0$. This implies $i_1 + x \in I$. As I is subtractive, we get $x \in I$. Conversely, let $i \in I$ and $i + x \in I$. Then there exists $i_3 \in I$ such that $i + x = i_3 + 0$. This implies $x \equiv_I 0$. Hence by hypothesis, we get $x \in I$. Thus I is subtractive. □

PROPOSITION 3.5. *If I is an equiprime strong ideal of S then $S_c \subseteq I$.*

Proof. Let $s \in S_c$. Then $ss' = s$, $\forall s' \in S$. Now, take $x \in S$. Then $0 + ss'x = 0 + ss'0$, $\forall s' \in S$. This implies $ss'x \equiv_I ss'0$, $\forall s' \in S$. As I is equiprime, we get $s \in I$ or $x \equiv_I 0$. This implies $s \in I$ or $x \in I$. If I is a proper strong ideal then $x \in I$, a contradiction. Hence $s \in I$. Thus $S_c \subseteq I$. □

Theorem 3.6. *Every equiprime strong ideal of a seminearring S is a 3-prime strong ideal.*

Proof. Let I be an equiprime strong ideal of a seminearring S and let $a, b \in S$, with $asb \in I$, $\forall s \in S$. If $a \in I$ then I is 3-prime strong. Suppose $a \notin I$. As $S_c \subseteq I$, we have $as0 \in I$, $\forall s \in S$. Now, fix $s \in S$. Then $asb \in I + as0$. This implies that $asb \equiv_I as0$. As $s \in S$ is arbitrary, we have $asb \equiv_I as0\ \forall s \in S$. Since I is an equiprime strong ideal of S and $a \notin I$, we get $b \equiv_I 0$. This implies $b \in I$. Hence, I is a 3-prime strong. □

If $\{0\}$ is an equiprime (3-prime) strong ideal of S, then S is said to be an equiprime (3-prime) seminearring.

Example 3.7 ([22]). Let $(S, +)$ be a cyclic group of prime order p (≥ 3). Define multiplication on S as follows. For $x, y \in S$, $xy = \begin{cases} x & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$. Then $(S, +, \cdot)$ is a 3-prime seminearring. We give an example to show that, in general, S need not be an equiprime seminearring.

Let $S = \mathbb{Z}_3$ with $+$ and $\cdot$ defined as follows.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

Then S is a seminearring and $I = \{0\}$ is a strong ideal of S. However I is not an equiprime strong ideal, because $1s1 = 1s2$, $\forall s \in S$, but $1 \neq 0$ and $1 \neq 2$.

Definition 3.8. A seminearring S is said to be equiprime if
(i) For all $0 \neq x, y \in S$, $xSy \neq (0)$.

(ii) If $x, y \in S$ and $(0) \neq T$ is any invariant subsemigroup of S then $tx = ty$ for all $t \in T$ implies $x = y$.
If S/I is an equiprime seminearring then the strong ideal I is called an equiprime strong ideal of S.

Lemma 3.9. *Let S be a seminearring. Then the following are equivalent.*
(i) S is equiprime.

(ii) If $x, y,\ 0 \neq a \in S$ with $asx = asy$, $\forall s \in S$ then $x = y$.

Proof. (i)⇒(ii) Let $x, y, 0 \neq a \in S$ with $asx = asy$, $\forall s \in S$; we will show that $x = y$. As $a \neq 0$ and S is equiprime, we get $aSa \neq (0)$. This implies $aS \neq (0)$. Now, define B_K inductively as follows. Let $B_0 = aS$ and assume that B_{k-1} has been defined. Let $B_K = \{\Sigma_i u_i : u_i \in B_{K-1}\} \cup \{s'u : s' \in S, u \in B_{K-1}\}$. Then $B = \bigcup B_K$ is an invariant subsemigroup of $(S, +)$. Now, we will prove that $ux = uy$, $\forall u \in B = \cup B_K$ by taking induction on k. Take $k = 0$ and $u \in B_0$. Then there exists $s \in S$ such that $u = as$. Hence $ux = asx = asy = uy$, $\forall u \in B_0$. Now, take $k = 1$ and $u \in B_1$. Then $ux = \sum_i u_i x\ [u_i \in B_0] = \sum_i u_i y = uy$. Let $u_0 \in B_0$. Then $ux = (su_0)x$

$= s(u_0x) = s(u_0y) = (su_0)y = uy$. Hence $ux = uy$, $\forall u \in B_1$. Similarly, for any k, $ux = uy$, $\forall\ u \in B_K$. Since S is equiprime and $(0) \neq B$ is an invariant subsemigroup of S, $ux = uy$, $\forall u \in B$ then by (i), we get $x = y$.

(ii)⇒(i) First, we will show that (ii) implies that S is zero-symmetric. That is, we have to prove that $s'0 = 0$, $\forall s' \in S$. Assume that $s'0 \neq 0$. We have $(s'0)S(s'0) = (s'0)S0$. Then by (ii) we get $s'0 = 0$, which is a contradiction. Therefore S is zero-symmetric.

Now, consider $xSy = (0)$ with $x \neq 0$. We will prove that $y = 0$. As S is zero-symmetric, we have $xs'y = 0 = xs'0$, $\forall s' \in S$. Then, by (ii), we get $y = 0$.

Let $(0) \neq B$ be an invariant subsemigroup of S and suppose that $x, y \in S$ with $ax = ay\ \forall\ a \in B$. Assume that $0 \neq a \in B$. As B is an invariant subsemigroup of S, we have $as \in B$, $\forall s \in S$ and $asx = asy$, $\forall s \in S$. Hence, by (ii), we get $x = y$. □

Definition 3.10. A seminearring S is said to satisfy the condition (V_1), if I and J are strong ideals of S implies that $(I \cap J) + x = (I + x) \cap (J + x)$, $\forall\ x \in S$.

Proposition 3.11 ([13]). *Let S be a seminearring and I, J be strong ideals of S. If S satisfies the condition V_1 then $I \cap J$ is a strong ideal of S.*

In the sequel, we assume that seminearrings satisfy the condition V_1.

4. Kurosh-Amitsur prime radical for seminearrings

Definition 4.1 ([7]). A class ρ of seminearrings is said to be a Kurosh-Amitsur radical class if

(i) ρ is homomorphically closed.

(ii) For every seminearring S, the sum $\rho(S) = \Sigma(I \triangleleft S \mid I \in \rho)$ is in ρ.

(iii) $\rho(S/\rho(S)) = 0$ for every seminearring S.

$\rho(S)$ is called a ρ-radical of S.

Proposition 4.2 ([9]). *If the conditions (i) and (ii) of Definition 4.1 are satisfied on a class ρ of seminearrings, then the condition (iii) is equivalent to the following one. (iii') If I is a strong ideal of the seminearring S and $I, S/I \in \rho$ then $S \in \rho$.*

If ρ satisfies (iii') then ρ is said to be closed under extensions.

Proposition 4.3 ([9]). *If the conditions (i) and (iii') are satisfied on a class ρ of seminearrings, then the condition (ii) is equivalent to the following one. (ii') If $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_\alpha \subseteq \cdots$ is an ascending chain of strong ideals of a seminearring S and if each I_α is in ρ then $\bigcup I_\alpha$ is in ρ.*

If ρ satisfies (ii') then we say that ρ has the inductive property.

Theorem 4.4 ([9]). *A class ρ of seminearrings is a radical class if and only if* *(i) ρ is homomorphically closed,* *(ii) ρ has the inductive property,*

(iii) ρ is closed under extensions.

THEOREM 4.5 ([9]). *Let ρ be a class of seminearrings. Then the following conditions are equivalent:*
(A) ρ is a radical class.

(B1) If $S \in \rho$ then for every onto homomorphism $S \to T \neq 0$ there exists an ideal U of T such that $0 \neq U \in \rho$.

(B2) If S is a seminearring and for every onto homomorphism $S \to T \neq 0$ there is an ideal U of T such that $0 \neq U \in \rho$ then $S \in \rho$.

(C) ρ satisfies (B1), is closed under extensions and has the inductive property.

THEOREM 4.6. *If ρ is a regular class of seminearrings then the class $U\rho = \{S \mid S$ has no nonzero homomorphic image in $\rho\}$ is a radical class.*

Proof. We will prove that $U\rho$ satisfies the conditions (B1) and (B2) of Theorem 4.5. Suppose S has a nonzero homomorphic image T and T has no nonzero ideal in $U\rho$. Then T has a nonzero homomorphic image R in ρ. This implies that S has a nonzero homomorphic image in ρ. That is, $S \notin U\rho$. Hence $U\rho$ satisfies (B1).

Now, assume that $S \notin U\rho$. Then S has a nonzero homomorphic image R in ρ. Let $0 \neq K$ be an ideal of R. As ρ is regular, K has a nonzero homomorphic image in ρ. This implies $K \notin U\rho$. Hence $U\rho$ satisfies (B2). Then by Theorem 4.5, we get $U\rho$ is a radical class. □

DEFINITION 4.7 ([9]). A Hoehnke radical ρ is a mapping which assigns to each seminearring S an ideal $\rho(S)$ subject to
(i) $h(\rho(S)) \subseteq \rho(h(S))$, for every homomorphism $h : S \to h(S)$,

(ii) $\rho(S/\rho(S)) = 0$ for every seminearring S.

A Hoehnke radical ρ is:

(iii) complete, if T is an ideal of S and $\rho(T) = T$ imply $T \subseteq \rho(S)$;

(iv) idempotent, if $\rho(\rho(S)) = \rho(S)$ for every seminearring S.

THEOREM 4.8. *If ρ is a Kurosh-Amitsur radical class then ρ is a complete, idempotent Hoehnke radical. Conversely, if ρ is a complete, idempotent Hoehnke radical, then there exists a Kurosh-Amitsur radical class μ such that $\mu = \{S \mid \rho(S) = S\}$ with $\rho(S) = \mu(S)$ for every seminearring S.*

Proof. Let ρ be a Kurosh-Amitsur radical class and $h : S \to h(S)$ be a homomorphism.
(i) As ρ is a KA-radical class, we have, from the Definition 4.1, taht $\rho(S) \in \rho$ and hence $h(\rho(S)) \in \rho$. Since h is an onto homomorphism and $\rho(S)$ is an ideal of S, we have that $h(\rho(S))$ is an ideal of $h(S)$ and hence $h(\rho(S)) \subseteq \rho(h(S))$.

(ii) Clearly holds from the Definition 4.1.

(iii) Let T be an ideal of S and $\rho(T) = T$. Then we have $T = \rho(T) \in \rho$ and hence $T \subseteq \rho(S)$.

(iv) We know that $\rho(S)$ is an ideal of S and $\rho(S) \in \rho$, hence $\rho(\rho(S)) = \rho(S)$.
Now, we assume that ρ is a complete, idempotent Hoehnke radical. Define a class μ as $\mu = \{S \mid \rho(S) = S\}$. Let $S \in \mu$ and $h : S \to S'$ be an onto homomorphism. Then $S' = h(S) = h(\rho(S)) \subseteq \rho(h(S)) = \rho(S')$. Hence $S' \in \mu$. Now, $\mu(S) = \sum(T \triangleleft S \mid T \in \mu) = \sum(T \triangleleft S \mid \rho(T) = T)$. As $T \subseteq \mu(S)$ and $\rho(T) = T$, by completeness we have $T \subseteq \rho(\mu(S))$. This implies $\mu(S) \subseteq \rho(\mu(S))$. Hence we get $\mu(S) = \rho(\mu(S))$. This gives $\mu(S) \in \mu$.

As $\mu(S) = \rho(\mu(S))$, by completeness we get $\mu(S) \subseteq \rho(S)$ and by idempotence we have $\rho(S) = \rho(\rho(S))$. This implies $\rho(S) \in \mu$. That is, $\rho(S) \subseteq \mu(S)$, hence $\rho(S) = \mu(S)$. Now, from Definition 4.7 (ii), we get $\mu(S/\mu(S)) = \rho(S/\rho(S)) = 0$. Thus, μ is a KA-radical class with $\mu(S) = \rho(S)$. □

Theorem 4.9 ([9]). *A seminearring S is a subdirect sum of the seminearrings S_λ, $\lambda \in \Lambda$ if and only if there exist strong ideals $T_\lambda, \lambda \in \Lambda$, in S such that $\bigcap T_\lambda = 0$ and $S_\lambda = S/T_\lambda$.*

Definition 4.10. Let μ be a class of seminearrings. Then μ is said to be a semisimple class
(i) If $S \in \mu$, then every nonzero strong ideal T of S has a nonzero homomorphic image in μ (regular).

(ii) If for any seminearring S and for every nonzero strong ideal T of S, there exists an onto homomorphism $T \to U \neq 0$ such that $U \in \mu$, then $S \in \mu$.

In the following, μ denotes a class of seminearrings.

Proposition 4.11. *If μ is a radical class, then $S\mu = \{B \mid \mu(B) = 0\}$ is a semisimple class.*

Proof. Let $B \in S\mu$. Then $\mu(B) = 0$. Now assume that T is a nonzero strong ideal of B and T has no nonzero homomorphic image in $S\mu$. As μ is a radical class, we have $\mu(T/\mu(T)) = 0$. This gives $T/\mu(T) \in S\mu$. This implies $T/\mu(T) = 0$. Then $T = \mu(T) \in \mu$. As $T \in \mu$ and T is a strong ideal of B, we get $0 \neq T \subseteq \mu(B)$, which is a contradiction. Hence T has a nonzero homomorphic image in $S\mu$.

Now, assume that $B \notin S\mu$. Then $\mu(B) \neq 0$. As μ is homomorphically closed, there is no nonzero homomorphic image of $\mu(B)$ in $S\mu$. Thus, $S\mu$ is a semisimple class. □

Theorem 4.12. *If μ is a semisimple class, then $SU\mu = \mu$.*

Proof. Let $B \in \mu$. Then every nonzero ideal of B has a nonzero homomorphic image in μ. This implies B has no nonzero strong ideals in $U\mu$. That is, $U\mu(B) = 0$. Then $B \in SU\mu$ and hence $\mu \subseteq SU\mu$.

Now, take $B \in SU\mu$. Then $U\mu(B) = 0$. This implies no nonzero strong ideal of B is in $U\mu$. That is, no nonzero strong ideal of B has a nonzero homomorphic image in μ. Hence $B \in \mu$ and, thus, $SU\mu = \mu$. □

For the sake of completeness, we give proof of the following proposition.

Proposition 4.13. *If μ is a semisimple class, then μ is closed under subdirect sums.*

Proof. Let $S_\lambda \in \mu,\ \lambda \in \Lambda$. Then we will prove that $\sum_{Subdirect} S_\lambda \in \mu$. Take T is a nonzero ideal of $\sum_{Subdirect} S_\lambda$. Then there exists an index λ such that the restriction of the homomorphism $\phi_\lambda : \sum_{Subdirect} S_\lambda \to S_\lambda$ to T is nonzero. That is, $\phi_\lambda(T) \neq 0$ and it is an ideal of S_λ. As μ is regular, $\phi_\lambda(T)$ has a nonzero homomorphic image U in μ. This implies U is a nonzero homomorphic image of T and $U \in \mu$, hence we get $\sum_{Subdirect} S_\lambda \in \mu$. □

Definition 4.14. The class of seminearrings μ is said to satisfy the coinductive property, if for any descending chain of strong ideals $T_1 \supseteq T_2 \supseteq \cdots T_\lambda \supseteq \cdots$ of a seminearring S with $S/T_\lambda \in \mu,\ \forall\lambda,\ S/\cap T_\lambda \in \mu$ holds.

Proposition 4.15 ([9]). *If μ is a semisimple class, then μ has the coinductive property.*

Proof. Since $S/\cap T_\lambda = \sum_{Subdirect} S_\lambda \in \mu$, thus μ has the coinductive property. □

Proposition 4.16. *If μ is a semisimple class, then μ is closed under extensions.*

Proof. Let T and B/T be in μ. Then we will show that B is in μ. As μ is regular, we have $U\mu$ is a radical class. Then $U\mu(B) \in U\mu$. As $U\mu(B) \to U\mu(B)/(U\mu(B) \cap T)$ is an onto homomorphism, we get $U\mu(B)/(U\mu(B) \cap T) \in U\mu$. This implies $U\mu(B)/(U\mu(B) \cap T) \cong (U\mu(B) + T)/T \in U\mu$. Since $(U\mu(B) + T)/T$ is a strong ideal of $B/T \in \mu = SU\mu$, then $(U\mu(B) + T)/T \in SU\mu$ and we have $(U\mu(B) + T)/T \in U\mu$. Hence $U\mu((U\mu(B) + T)/T) = (U\mu(B) + T)/T = 0 \Rightarrow U\mu(B) \subseteq T$. As $U\mu(B)$ is a strong ideal of T and $U\mu(B) \in U\mu$, we get $U\mu(B) \subseteq U\mu(T)$. As $T \in \mu = SU\mu$, we get $U\mu(T) = 0$. This implies $U\mu(B) = 0$. Hence $B \in SU\mu = \mu$. □

Definition 4.17. Let S be a seminearing and μ be a class of seminearrings. Then $(S)\mu = \bigcap\{T_\lambda \triangleleft S \mid S/T_\lambda \in \mu\}$.

Proposition 4.18 ([9]). *If μ is a regular class, closed under extensions and subdirect sums then the following conditions are equivalent.*
(i) $U\mu(S) = (S)\mu$ for every seminearring S,

(ii) $((S)\mu)\mu = (S)\mu$ for every seminearring S,

(iii) $((S)\mu)\mu$ is a strong ideal of S for every seminearring S.

Proposition 4.19. *If μ is any semisimple class, then $U\mu(S) = (S)\mu$ for any seminearring S.*

Proof. As μ is regular, we have $U\mu$ is a radical class. Then $U\mu(S/U\mu(S)) = 0$. This implies $S/U\mu(S) \in SU\mu = \mu$. Hence $(S)\mu \subseteq U\mu(S)$. As each $S/T_\lambda \in \mu$ and μ is closed under subdirect sums, we get $S/(S)\mu = S/\cap T_\lambda \cong \sum_{Subdirect} S/T_\lambda \in \mu$. Suppose we assume that $U\mu(S) \nsubseteq (S)\mu$. Then $U\mu(S)/(S)\mu$ is a nonzero strong ideal of the seminearring $S/(S)\mu \in \mu$. As $U\mu(S) \to U\mu(S)/(S)\mu$ is an onto homomorphism, we get $U\mu(S)/(S)\mu \in U\mu$ and $U\mu(S)/(S)\mu \subseteq U\mu(S/(S)\mu) = 0$, which is a contradiction. Hence $U\mu(S) = (S)\mu$. □

Theorem 4.20 ([9]). *The following conditions are equivalent.*
(i) μ is a semisimple class.

(ii) μ is regular, closed under subdirect sums, closed under extensions and $((S)\mu)\mu \lhd S$ for every seminearring S.

(iii) μ is regular, closed under subdirect sums and $((S)\mu)\mu = (S)\mu$ for every seminearring S.

(iv) μ is regular, closed under extensions, has the coinductive property and satisfies:

$\star$ *If $T \lhd S$ and T is minimal with respect to $S/T \in \mu$ and if U is a strong ideal of T and U is minimal with respect to $T/U \in \mu$ then U is a strong ideal of S.*

Theorem 4.21 ([21]). *Let μ_0 be a class of zerosymmetric seminearrings. If μ_0 is regular, satisfies condition (F_1) and is closed under essential left invariant extensions, then $U\mu_0$ is a C-hereditary radical class in the variety of all seminearrings, $SU\mu_0 = \overline{\mu_0}$ and $SU\mu_0$ is hereditary.*

Denote the class of all equiprime seminearrings by μ_e and 3-prime seminearrings by μ_3.

Definition 4.22. Let S be a seminearring. Then $P_e(S) = \bigcap\{I \lhd S \mid S/I \in \mu_e\}$ is an equiprime radical and $P_3(S) = \bigcap\{I \lhd S \mid S/I \in \mu_3\}$ is a 3-prime radical.

Proposition 4.23. *The class μ_e is hereditary on invariant subsemigroups. In particular, the class μ_e is hereditary.*

Proof. Let $S \in \mu_e$ and I be an invariant subsemigroup of S. If $I = 0$ then $I \in \mu_e$. Let $I \neq 0$. Then take $0 \neq p, q \in I$. Now, we will prove that $pIq \neq 0$. Suppose $pIq = 0$. Then $pSIq \subseteq pIq = 0$. As $p \neq 0$ and S is an equiprime seminearring, we get $Iq = 0$. Now, $ISq \subseteq Iq = 0$. As $q \neq 0$, we get $I = 0$, a contradiction. Hence $0 \neq p, q$ implies $pIq \neq 0$. Now, take $a, b \in I$ and $0 \neq T$ be any invariant subsemigroup of I with $tx = ty$ for all $t \in T$. Then we will prove that $x = y$. Let $K = \{k \in S \mid ksx = ksy,\ \forall s \in S\}$. We will prove that $K \neq 0$. Now take $0 \neq t \in T$. As S is an equiprime seminearring and $0 \neq I$, we get $0 \neq tSI \subseteq tI$. Now, take $i \in I$ and $s \in S$. Then $tis \in tI \subseteq T$. Hence $tisx = tisy$. This implies $ti \in K$. Then $0 \neq tI \subseteq K$and hence $K \neq 0$.

Now, we will prove that K is an invariant subsemigroup of S. Let $z \in KS$. Then there exist $k \in K, s \in S$ such that $z = ks$. As $k \in K$, we have $ksx = ksy$, $\forall s \in S$. Let $s_1 \in S$. Then $(ks)s_1x = k(ss_1)x = ks_2x = ks_2y = k(ss_1)y = (ks)s_1y$. This implies $z = ks \in K$. Hence $KS \subseteq K$. Similarly, we can prove that $SK \subseteq K$. Hence K is an invariant subsemigroup of S. For each $s \in S$, $k \in K$ we have $ksx = ksy$. As S is an equiprime seminearring and K is an invariant subsemigroup of S, $ksx = ksy$, $\forall k \in K$ implies $sx = sy$, $\forall s \in S$. Again by considering that S is equiprime, $sx = sy$, $\forall s \in S$ implies $x = y$. Hence $0 \neq I$ is equiprime. As every equiprime seminearring is zerosymmetric, we have that all the strong ideals of an equiprime seminearring are invariant. Thus, the class μ_e is hereditary. $\square$

Definition 4.24. Let A be a strong ideal of B and B be a left invariant strong ideal of a seminearring S such that $B/A \in \mu$ and $x, y \in S$. Then the class μ is said to satisfy the condition (U_1)
(i) If $(xm)/A = (ym)/A,\ \forall\ m \in B$ then $x \in A + y$ and $y \in A + x$,

(ii) If $(mx)/A = (my)/A,\ \forall m \in B$ then $x \in A + y$ and $y \in A + x$.

Proposition 4.25. *If a class of seminearrings μ satisfies the condition U_1 with B a left invariant strong ideal of the seminearring S such that $B/A \in \mu$, then A is a strong ideal of S.*

Proof. Let $x \in S$, $m \in B$ and $a_1, a_2 \in A$. Then $a_1 + (x + a_2)m = a_1 + xm + a_2m$. As A is an ideal of S, there exists $a_3 \in A$ such that $a_1 + xm + a_2m = a_1 + (xm + a_3) = a_1 + (a_4 + xm) = a_5 + xm$, for some $a_4, a_5 \in A$. That is, $a_1 + (x + a_2)m = a_5 + xm$, implying $(x + a_2)m/A = xm/A$. Then by Definition 4.24 (i), we get $x + a_2 \in A + x$, implying $x + A \subseteq A + x$. Let $x \in S$ and $x \equiv_A 0$. This implies there exist $a_1, a_2 \in A$ such that $a_1 + x = a_2 + 0$. Now, take $m \in B$. Then

$$\begin{aligned}(a_1 + x)m = a_2m &\Rightarrow a_1m + xm = a_2m\\ &\Rightarrow a_3 + xm = a_2m\ [a_1m = a_3 \in A] \Rightarrow xm/A = a_2m/A.\end{aligned}$$

Again by Definition 4.24 (i), we get $x \in A + a_2$. This implies $x \in A$.

Now, we will prove that $AS \subseteq A$. Let $y \in AS$. Then there exists $a \in A$ such that $y = as$. Now, take $m \in B$. Then $(as)m = a(sm) \in aB \subseteq A$. This implies there exists $a_2 \in A$ such that $(as)m = a_2 + 0m$. This gives $(as)m/A = om/A$. Then by Definition 4.24 (i), we get $as \in A$.

Now, we will show that $s(A + s') \subseteq A + ss',\ \forall s, s' \in S$. Suppose that $(s(a + s'))m/A \neq ss'm/A$. Now, take $b \in B$. Then we get $b(s(a+s')m)/A = b(ss'm/A)$ $[b(s(a+s')m) = bs(am+s'm) = a_2+bss'm]$, which is a contradiction. Hence $[s(a+s')]m/A = (ss'm)/A,\ \forall m \in B$. Then by Definition 4.24 (i), we get $s(a + s') \in A + ss'$. Thus A is a strong ideal of S. □

Proposition 4.26. *If B is a left invariant strong ideal of a seminearring S with $B/A \in \mu$ and μ satisfies the condition U_1, then $(A : B)_S$ is a strong ideal of S.*

Proof. By Proposition 4.25, A is a strong ideal of S. Now, take $x, y \in (A : B)_S$. Then $xB \subseteq A$ and $yB \subseteq A$. This gives $xB + yB \subseteq A + A = A$. That is, $(x + y)B \subseteq A$. Hence $x + y \in (A : B)_S$. Let $x \in S$ and $i \in (A : B)_S$. Then we will prove that $x + i \in (A : B)_S + x$. Take $m \in B$ and $a_1 \in A$. Then $a_1 + (x + i)m = a_1 + (xm + im)$. As A is an ideal of S, there exists $a_2 \in A$ such that $a_1 + (xm + im) = a_1 + (a_2 + xm) = (a_1 + a_2) + xm = a_3 + xm[(a_1 + a_2) = a_3 \in A]$. This implies $(x + i)m/A = xm/A$. Then by Definition 4.24 (i), we get $x + i \in A + x \subseteq (A : B)_S + x$.

Now, take $x, x' \in S,\ i \in (A : B)_S$. Then we will prove that $x(i + x') \subseteq (A : B)_S + xx'$. Let $a_1 \in A$ and $m \in B$. Then $a_1 + [x(i + x')m] = a_1 + [x(im + x'm)] = a_1 + [x(im + x'm)]$. As A is an ideal of S, there exists $a_2 \in A$ such that $a_1+[x(im+x'm)] = a_1+a_2+xx'm = a_3+xx'm$. This implies $x(i+x')m/A = xx'm/A$. Then by the Definition 4.24 (i), we get $x(i + x') \subseteq A + xx' \subseteq (A : B)_S + xx'$. Let

$x \equiv_{(A:B)_S} 0$. Then there exist $y_1, y_2 \in (A : B)_S$ such that $y_1 + x = y_2 + 0$. Now, take $m \in B$. Then

$$\begin{aligned}(y_1 + x)m = y_2m + 0 \Rightarrow y_1m + xm = y_2m + 0 \Rightarrow xm/A = 0/A \\ \Rightarrow xm \in A \Rightarrow x \in (A : B)_S.\end{aligned}$$

Now, we will prove that $(A : B)_S S \subseteq (A : B)_S$. Let $z \in (A : B)_S S$. Then there exist $y \in (A : B)_S$ and $x \in S$ such that $z = yx$. Then $zB = (yx)B = y(xB) \subseteq yB \subseteq A$. Hence $z = yx \in (A : B)_S$. Thus, $(A : B)_S$ is a strong ideal of S. □

Proposition 4.27. *If A is a left invariant strong ideal of a seminearring S with A/B being an equiprime seminearring and μ_e satisfies the condition U_1, then $(B : A)_S$ is an equiprime strong ideal of S.*

Proof. By Proposition 4.26, $(B : A)_S$ is a strong ideal of S. Now, take $x, y \in S$ such that $x, y \notin (B : A)_S$. There exist $a, b \in A$ such that $xa \notin B$ and $ya \notin B$. As B is an equiprime strong ideal of A, we have $xaAyb \nsubseteq B$. Hence $xAy \nsubseteq (B : A)_S$. Let M be an invariant subsemigroup of S such that $(B : A)_S \subset M$. Now, take $x, y \in S$ such that $ax/(B : A)_S = ay/(B : A)_S\ \forall a \in M$. Then there exist $y_1, y_2 \in (B : A)_S$ such that

$$\begin{aligned}y_1 + ax = y_2 + ay \Rightarrow (y_1 + ax)d = (y_2 + ay)d,\ \forall d \in A \\ \Rightarrow y_1d + axd = y_2d + ayd \Rightarrow axd/B = ayd/B.\end{aligned}$$

As $B \subseteq (B : A)_S \subset M$, B is an equiprime strong ideal of A and $a(xd)/B = a(yd)/B$, and we get $xd/B = yd/B, \forall d \in A$. Then by Definition 4.24 (i), we get $x \in B + y \subseteq (B : A)_S + y$. Hence $x/(B : A)_S = y/(B : A)_S$. Thus $(B : A)_S$ is an equiprime strong ideal of S. □

Lemma 4.28. *Let S be a seminearring and T be an essential strong ideal of S such that $ST \subseteq T$ and $T \in \mu_e$. If the class μ_e satisfies the condition U_1 and $0 \neq y \in S$ then $yT \neq 0$ and $Ty \neq 0$.*

Proof. By Proposition 4.26, we have $(0 : T)_S$ is a strong ideal of S. Suppose $(0 : T)_S \neq 0$. Since T is an essential strong ideal of S, then $(0 : T)_S \cap T \neq 0$. Let $0 \neq x \in (0 : T)_S \cap T$. Then $xT = 0$. Now, $xTx \subseteq xT = 0$. Since T is an equiprime seminearring, $xTx = 0$ is a contradiction. Hence $(0 : T)_S = 0$ and $yT \neq 0,\ \forall 0 \neq y \in S$. Suppose $Ty = 0$. As T is zero-symmetric, we have $(yT)(Ty)T = 0$, implying $(yT)T(yT) = 0$. Since T is equiprime, we get $yT = 0$, a contradiction. Hence $Ty \neq 0$. □

Proposition 4.29. *If the class μ_e satisfies condition U_1, then μ_e is closed under essential left invariant extensions.*

Proof. Let S be a seminearring and T be an essential strong ideal of S with $ST \subseteq T$ and $T \in \mu_e$. Let $0 \neq x, y \in S$. Then by Lemma 4.28, $xT \neq 0$ and $Ty \neq 0$. Since T is an equiprime seminearring, we get $xTTTy \neq 0$. This implies $xSy \neq 0$. Now, suppose $0 \neq A$ is an invariant subsemigroup of S and $x, y \in S$ with $x \neq y$. Then we will show that there exists an $a \in A$ with $ax \neq ay$. Suppose that $ax = ay,\ \forall a \in A$. As $x \neq y$, by Definition 4.24, there exists $t \in T$ such that $xt \neq yt$. As $0 \neq A$, by Lemma 4.28 we get $0 \neq AT \subseteq ST \subseteq T$ and $0 \neq AT \subseteq AS \subseteq A$. Hence, $A \cap T \neq 0$. Note that $A \cap T$ is an invariant subsemigroup of T. Let $e \in A \cap T$. Then $e(xt) = (ex)t = (ey)t = e(yt)$. Since T is an equiprime seminearring, we get $xt = yt$, which is a contradiction. Hence $ax \neq ay$. □

By Proposition 4.23, we have that the class μ_e is hereditary and hence by Proposition 4.6, we get $U\mu_e$ is a Kurosh-Amitsur radical class.

THEOREM 4.30. *If S is a seminearring then $U\mu_e = P_e = \{S \mid P_e(S) = S\}$.*

Proof. Let $S \in U\mu_e$. Then S has no nonzero homomorphic image in μ_e. This implies S has no nonzero equiprime ideals. That is, $S = P_e(S)$. Hence $U\mu_e \subseteq P_e$. Now, take $S \in P_e$, that is $S = P_e(S)$. This implies S has no nonzero equiprime strong ideals. That is, S has no nonzero homomorphic image in μ_e. Hence $S \in U\mu_e$. Thus we get $U\mu_e = P_e$. □

Using Theorem 4.21 and Propositions 4.27, 4.23 and 4.29 we have the following.

PROPOSITION 4.31. *P_e is a C-hereditary Kurosh-Amitsur radical class in the variety of all the seminearrings, $SP_e = \overline{\mu_e}$ and SP_e is hereditary.*

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