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SEMILATTICE DECOMPOSITION OF LOCALLY ASSOCIATIVE Γ-AG**-GROUPOIDS

Madad Khan, Saima Anis and Kostaq Hila

Abstract. In this paper, we have shown that a locally associative Γ-AG**-groupoid S has associative powers and S/ρ_Γ is a maximal separative homomorphic image of S, where $a\rho_\Gamma b$ implies that $a\Gamma b_\Gamma^n = b_\Gamma^{n+1}, b\Gamma a_\Gamma^n = a_\Gamma^{n+1}, \forall a, b \in S$. The relation η_Γ is the least left zero semilattice congruence on S, where η_Γ is defined on S as $a\eta_\Gamma b$ if and only if there exist some positive integers m, n such that $b_\Gamma^m \subseteq a\Gamma S$ and $a_\Gamma^n \subseteq b\Gamma S$.

1. Introduction

An Abel-Grassmann's groupoid [11] (abbreviated as an AG-groupoid), is a groupoid S whose elements satisfy the invertive law $(ab)c = (cb)a$, for all $a, b, c \in S$. It is also called a left almost semigroup [3, 7, 8]. In [2], the same structure is called a left invertive groupoid. It is a useful non-associative algebraic structure, midway between a groupoid and a commutative semigroup.

An AG-groupoid S is medial [3], that is, $(ab)(cd) = (ac)(bd)$, for all $a, b, c, d \in S$. If an AG-groupoid satisfies the following property:

$$a(bc) = b(ac), \text{ for all } a, b, c \in S, \tag{1}$$

then it is called an AG**-groupoid (cf. [5, 10]). In an AG**-groupoid S the law $(ab)(cd) = (db)(ca)$ holds for all $a, b, c, d \in S$ (cf. [10]).

An AG-groupoid S is called a locally associative AG-groupoid if $(aa)a = a(aa)$ holds for all $a \in S$. If S is a locally associative AG-groupoid, then it is easy to see that $(Sa)S = S(aS)$ or $(SS)S = S(SS)$. If a locally associative AG-groupoid S satisfies the identity (1), then S is known as a locally associative AG**-groupoid.

An element a of S is called left zero if $ax = a$, for all $x \in S$.

Locally associative LA-semigroups have been studied by Mushtaq et al. [6, 7]. Other notions and results on AG-groupoids and AG**-groupoids, one can find in [2–5, 8–11, 13].

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M.K. Sen [12] introduced the concept of Γ-semigroup in 1981. The non-associative Γ-AG-groupoid is the generalization of an associative Γ-semigroup.

Let S and Γ be two non-empty sets. Denote by the letters of English alphabet the elements of S and by the letters of Greek alphabet the elements of Γ. Any map from $S \times \Gamma \times S$ to S will be called a Γ-multiplication in S and denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in S$ and $\alpha \in \Gamma$ is denoted by $a\alpha b$. A Γ-AG-groupoid [1] S is an ordered pair $(S, (\cdot)_\Gamma)$ where S and Γ are non-empty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on S which satisfies the following Γ-left invertive law: $\forall(a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2$,

$$(x\alpha y)\beta z = (z\alpha y)\beta x. \tag{2}$$

A Γ-AG-groupoid also satisfies the Γ-medial law $\forall(w, x, y, z, \alpha, \beta, \gamma) \in S^4 \times \Gamma^3$,

$$(w\alpha x)\beta(y\gamma z) = (w\alpha y)\beta(x\gamma z). \tag{3}$$

Note that if a Γ-AG-groupoid contains a left identity, then it becomes an AG-groupoid with left identity. A Γ-AG-groupoid is called a Γ-AG**-groupoid [1] if it satisfies the following law $\forall(x, y, z, \alpha, \beta) \in S^3 \times \Gamma^2$,

$$x\alpha(y\beta z) = y\alpha(x\beta z). \tag{4}$$

A Γ-AG**-groupoid also satisfies the following Γ-paramedial law

$$\forall(w, x, y, z, \alpha, \beta, \gamma) \in S^4 \times \Gamma^3, (w\alpha x)\beta(y\gamma z) = (z\alpha y)\beta(x\gamma w).$$

Other concepts and results on Γ-AG**-groupoids one can find in [1].

In this paper, we introduce a new non-associative algebraic structure namely locally associative Γ-AG**-groupoids and decompose it using Γ-congruences. An AG-groupoid S is called a locally associative Γ-AG-groupoid if $(a\alpha a)\beta a = a\alpha(a\beta a)$ holds for all $a \in S$ and $\alpha, \beta \in \Gamma$. If S is a locally associative AG-groupoid, then it is easy to see that $(S\Gamma a)\Gamma S = S\Gamma(a\Gamma S)$ or $(S\Gamma S)\Gamma S = S\Gamma(S\Gamma S)$. For particular $\alpha \in \Gamma$, let us denote $a\alpha a = a^2_\alpha$ for some $\alpha \in \Gamma$ and $a\alpha a = a^2_\Gamma$, for all $\alpha \in \Gamma$, i.e., $a\Gamma a = a^2_\Gamma$ and generally $a\Gamma a\Gamma a \ldots a\Gamma a = a^n_\Gamma$ (n times).

2. Main results

Let S be an Γ-AG**-groupoid and a relation ρ_Γ be defined on S as follows : $a\rho_\Gamma b$ if and only if there exists a positive integer n such that $a\Gamma b^n_\Gamma = b^{n+1}_\Gamma$ and $b\Gamma a^n_\Gamma = a^{n+1}_\Gamma$, for all a and b in S.

Proposition 2.1. *If S is a locally associative Γ-AG**-groupoid, then $a\Gamma a^{n+1}_\Gamma = (a^{n+1}_\Gamma)\Gamma a$, for all a in S and positive integer n.*

Proof.
$$\begin{aligned} a\Gamma a^{n+1}_\Gamma &= a\Gamma(a^n_\Gamma\Gamma a) = a^n_\Gamma\Gamma(a\Gamma a) = (a^{n-1}_\Gamma\Gamma a)\Gamma(a\Gamma a) \\ &= (a\Gamma a)\Gamma(a\Gamma a^{n-1}_\Gamma) = (a\Gamma a)\Gamma a^n_\Gamma = (a^n_\Gamma\Gamma a)\Gamma a = (a^{n+1}_\Gamma)\Gamma a. \end{aligned}$$ □

Proposition 2.2. *In a locally associative Γ-AG**-groupoid S, $a^m_\Gamma\Gamma a^n_\Gamma = a^{m+n}_\Gamma$, for all $a \in S$ and positive integers m, n.*

Proof. $a_\Gamma^{m+1}\Gamma a_\Gamma^n = (a_\Gamma^m\Gamma a)\Gamma a_\Gamma^n = (a_\Gamma^n\Gamma a)\Gamma a_\Gamma^m = (a\Gamma a_\Gamma^n)\Gamma a_\Gamma^m = (a_\Gamma^m\Gamma a_\Gamma^n)\Gamma a$
$= a_\Gamma^{m+n}\Gamma a = a_\Gamma^{m+n+1}$. □

Proposition 2.3. *If S is a locally associative Γ-AG^{**}-groupoid, then for all $a, b \in S$, $(a\Gamma b)_\Gamma^n = a_\Gamma^n\Gamma b_\Gamma^n$ for a positive integer $n \geq 1$ and $(a\Gamma b)_\Gamma^n = b_\Gamma^n\Gamma a_\Gamma^n$, for $n \geq 2$.*

Proof. We have

$$(a\Gamma b)_\Gamma^2 = (a\Gamma b)\Gamma(a\Gamma b) = (a\Gamma a)\Gamma(b\Gamma b) = a_\Gamma^2\Gamma b_\Gamma^2$$

$$(a\Gamma b)_\Gamma^{k+1} = (a\Gamma b)_\Gamma^k\Gamma(a\Gamma b) = (a_\Gamma^k\Gamma b_\Gamma^k)\Gamma(a\Gamma b) = (a_\Gamma^k\Gamma a)\Gamma(b_\Gamma^k\Gamma b) = a_\Gamma^{k+1}\Gamma b_\Gamma^{k+1}.$$

Let $n \geq 2$. Then by (4) and (2), we get

$$\begin{aligned}(a\Gamma b)_\Gamma^n &= a_\Gamma^n\Gamma b_\Gamma^n = (a\Gamma a_\Gamma^{n-1})\Gamma(b\Gamma b_\Gamma^{n-1}) = b\Gamma((a\Gamma a_\Gamma^{n-1})\Gamma b_\Gamma^{n-1})) = b\Gamma((b_\Gamma^{n-1}\Gamma a_\Gamma^{n-1})\Gamma a)\\ &= b\Gamma((b\Gamma a)_\Gamma^{n-1}\Gamma a) = (b\Gamma a)_\Gamma^{n-1}\Gamma(b\Gamma a) = (b\Gamma a)_\Gamma^n = b_\Gamma^n\Gamma a_\Gamma^n.\end{aligned}$$ □

Proposition 2.4. *In a locally associative Γ-AG^{**}-groupoid S, $(a_\Gamma^m)_\Gamma^n = a_\Gamma^{mn}$ for all $a \in S$ and positive integers m, n.*

Proof. $(a_\Gamma^{m+1})_\Gamma^n = (a_\Gamma^m\Gamma a)_\Gamma^n = (a_\Gamma^m)_\Gamma^n\Gamma a_\Gamma^n = a_\Gamma^{mn}\Gamma a_\Gamma^n = a_\Gamma^{mn+n} = a_\Gamma^{n(m+1)}$. □

Theorem 2.5. *Let S be a locally associative Γ-AG^{**}-groupoid. If $a\Gamma b_\Gamma^m = b_\Gamma^{m+1}$ and $b\Gamma a_\Gamma^n = a_\Gamma^{n+1}$, for $a, b \in S$ and positive integers m, n, then $a\rho_\Gamma b$.*

Proof. If $n > m$, then $b_\Gamma^{n-m}\Gamma(a\Gamma b_\Gamma^m) = b_\Gamma^{n-m}\Gamma b_\Gamma^{m+1}$, $a\Gamma(b_\Gamma^{n-m}\Gamma b_\Gamma^m) = b_\Gamma^{n-m+m+1}$, $a\Gamma b_\Gamma^{n-m+m} = b_\Gamma^{n+1}$, $a\Gamma b_\Gamma^n = b_\Gamma^{n+1}$. □

Theorem 2.6. *The relation ρ_Γ on a locally associative Γ-AG^{**}-groupoid is a congruence relation.*

Proof. Evidently ρ_Γ is reflexive and symmetric. For transitivity we may proceed as follows.

Let $a\rho_\Gamma b$ and $b\rho_\Gamma c$ so that there exist positive integers n, m such that $a\Gamma b_\Gamma^n = b_\Gamma^{n+1}, b\Gamma a_\Gamma^n = a_\Gamma^{n+1}$, and $b\Gamma c_\Gamma^m = c_\Gamma^{m+1}, c\Gamma b_\Gamma^m = b_\Gamma^{m+1}$.

Let $k = (n+1)(m+1) - 1$, that is, $k = n(m+1) + m$. Using (2), (4) and Proposition 2.2, 2.3 and 2.4, we get

$$\begin{aligned}a\Gamma c_\Gamma^k &= a\Gamma c_\Gamma^{n(m+1)+m} = a\Gamma(c_\Gamma^{n(m+1)}\Gamma c_\Gamma^m) = a\Gamma\{(c_\Gamma^{m+1})_\Gamma^n\Gamma c_\Gamma^m\} = a\Gamma\{(b\Gamma c_\Gamma^m)_\Gamma^n\Gamma c_\Gamma^m\}\\ &= a\Gamma\{(b_\Gamma^n\Gamma c_\Gamma^{mn})\Gamma c_\Gamma^m\} = a\Gamma(c_\Gamma^{m(n+1)}\Gamma b_\Gamma^n) = c_\Gamma^{m(n+1)}\Gamma(a\Gamma b_\Gamma^n)\\ &= c_\Gamma^{m(n+1)}\Gamma b_\Gamma^{n+1} = (c_\Gamma^m\Gamma b)_\Gamma^{n+1} = b_\Gamma^{n+1}\Gamma c_\Gamma^{m(n+1)} = (b\Gamma c_\Gamma^m)_\Gamma^{n+1} = c_\Gamma^{k+1}.\end{aligned}$$

Similarly, $c\Gamma a^k = a_\Gamma^{k+1}$. Thus ρ_Γ is an equivalence relation. To show that ρ_Γ is compatible, assume that $a\rho_\Gamma b$ such that for some positive integer n, $a\Gamma b_\Gamma^n = b_\Gamma^{n+1}$ and $b\Gamma a_\Gamma^n = a_\Gamma^{n+1}$.

Let $c \in S$, then by identity (3) and Propositions 2.4 and 2.1, we get

$$(a\Gamma c)\Gamma(b\Gamma c)_\Gamma^n = (a\Gamma c)\Gamma(b_\Gamma^n\Gamma c_\Gamma^n) = (a\Gamma b_\Gamma^n)\Gamma(c\Gamma c_\Gamma^n) = b_\Gamma^{n+1}\Gamma c_\Gamma^{n+1} = (b\Gamma c)_\Gamma^{n+1}.$$

Similarly, $(b\Gamma c)\Gamma(a\Gamma c)_\Gamma^n = (a\Gamma c)_\Gamma^{n+1}$. Hence ρ_Γ is a congruence relation on S. □

Lemma 2.7. *Let S be a locally associative Γ-AG^{**}-groupoid; then $a\Gamma b\rho_\Gamma b\Gamma a$, for all $a, b \in S$.*

Proof.
$$(a\Gamma b)\Gamma(b\Gamma a)_\Gamma^{n+1} = (a\Gamma b)\Gamma(a_\Gamma^{n+1}\Gamma b_\Gamma^{n+1}) = (a\Gamma a_\Gamma^{n+1})\Gamma(b\Gamma b_\Gamma^{n+1})$$
$$= a_\Gamma^{n+2}\Gamma b_\Gamma^{n+2} = (b\Gamma a)_\Gamma^{n+2}.$$
Similarly, $(b\Gamma a)\Gamma(a\Gamma b)_\Gamma^{n+1} = (a\Gamma b)_\Gamma^{n+2}$. Hence $a\Gamma b\rho_\Gamma b\Gamma a$, for all $a, b \in S$. □

A relation ρ_Γ on an AG-groupoid S is called separative if $a\Gamma b\rho_\Gamma a_\Gamma^2$ and $a\Gamma b\rho_\Gamma b_\Gamma^2$ imply that $a\rho_\Gamma$.

Theorem 2.8. *The relation ρ_Γ is separative.*

Proof. Let $a, b \in S$, $a\Gamma b\rho_\Gamma a_\Gamma^2$ and $a\Gamma b\rho_\Gamma b_\Gamma^2$. Then by the definition of ρ_Γ, there exist positive integers m and n such that,
$$(a\Gamma b)\Gamma(a_\Gamma^2)_\Gamma^m = (a_\Gamma^2)_\Gamma^{m+1}, a_\Gamma^2\Gamma(a\Gamma b)_\Gamma^m = (a\Gamma b)_\Gamma^{m+1}$$
and
$$(a\Gamma b)\Gamma(b_\Gamma^2)_\Gamma^n = (b_\Gamma^2)_\Gamma^{n+1}, b_\Gamma^2\Gamma(a\Gamma b)_\Gamma^n = (a\Gamma b)_\Gamma^{n+1}.$$
Then
$$(a\Gamma b)\Gamma a_\Gamma^{2m} = (a\Gamma b)\Gamma(a_\Gamma^m\Gamma a_\Gamma^m) = (a\Gamma a_\Gamma^m)\Gamma(b\Gamma a_\Gamma^m)$$
$$= (a_\Gamma^{m+1})\Gamma(b\Gamma a_\Gamma^m) = b\Gamma(a_\Gamma^{m+1}\Gamma a_\Gamma^m) = b\Gamma a_\Gamma^{2m+1},$$
but $(a\Gamma b)\Gamma a_\Gamma^{2m} = (a_\Gamma^2)_\Gamma^{m+1} = a_\Gamma^{2m+2}$, which implies that $b\Gamma a_\Gamma^{2m+1} = a_\Gamma^{2m+2}$. Also, $(a\Gamma b)\Gamma(b_\Gamma^2)_\Gamma^n = (b_\Gamma^2)_\Gamma^{n+1}$ implies that $b_\Gamma^{2n+1}\Gamma a = b_\Gamma^{2n+2}$. Also, we get $b_\Gamma^{2n+2}\Gamma b_\Gamma^2 = (b_\Gamma^{2n+1}\Gamma a)\Gamma b_\Gamma^2$, which implies that $b_\Gamma^{2n+4} = b_\Gamma^2\Gamma(a\Gamma b_\Gamma^{2n+1}) = a\Gamma(b_\Gamma^2\Gamma b_\Gamma^{2n+1}) = a\Gamma b_\Gamma^{2n+3}$. Hence by Theorem 2.5, $a\rho_\Gamma b$. □

Theorem 2.9. *Let S be a locally associative Γ-AG^{**}-groupoid. Then S/ρ_Γ is a maximal separative commutative image of S.*

Proof. By Theorem 2.8, ρ_Γ is separative, and hence S/ρ_Γ is separative. We now show that ρ_Γ is contained in every separative congruence relation σ_Γ on S. Let $a\rho_\Gamma b$, so that there exists a positive integer n such that $a\Gamma b_\Gamma^n = b_\Gamma^{n+1}$ and $b\Gamma a_\Gamma^n = a_\Gamma^{n+1}$.

We need to show that $a\sigma_\Gamma b$, where σ_Γ is a separative congruence on S. Let k be any positive integer such that
$$a\Gamma b_\Gamma^k\Gamma\sigma_\Gamma b_\Gamma^{k+1} \text{ and } b\Gamma a_\Gamma^k\sigma_\Gamma a_\Gamma^{k+1} \tag{5}$$
Suppose that $k \geq 3$.
$$(a\Gamma b_\Gamma^{k-1})_\Gamma^2 = (a\Gamma b_\Gamma^{k-1})\Gamma(a\Gamma b_\Gamma^{k-1}) = a_\Gamma^2\Gamma b_\Gamma^{2k-2} = (a\Gamma a)\Gamma(b_\Gamma^{k-2}\Gamma b_\Gamma^k)$$
$$= (a\Gamma b_\Gamma^{k-2})\Gamma(a\Gamma b_\Gamma^k) = (a\Gamma b_\Gamma^{k-2})\Gamma b_\Gamma^{k+1}.$$
Therefore $(a\Gamma b_\Gamma^{k-2})\Gamma(a\Gamma b_\Gamma^k)\sigma_\Gamma(a\Gamma b_\Gamma^{k-2})\Gamma b_\Gamma^{k+1}$.

Using the identity (2) and Proposition 2.2, we get
$$(a\Gamma b_\Gamma^{k-2})\Gamma b_\Gamma^{k+1} = (b_\Gamma^{k+1}\Gamma b_\Gamma^{k-2})\Gamma a = b_\Gamma^{2k-1}\Gamma a = (b_\Gamma^k\Gamma b_\Gamma^{k-1})\Gamma a = (a\Gamma b_\Gamma^{k-1})\Gamma b_\Gamma^k$$
Also
$$(a\Gamma b_\Gamma^{k-1})\Gamma b_\Gamma^k = (b_\Gamma^k\Gamma b_\Gamma^{k-1})\Gamma a = b_\Gamma^{2k-1}\Gamma a = (b_\Gamma^{k-1}\Gamma b_\Gamma^k)\Gamma a = (a\Gamma b_\Gamma^k)\Gamma b_\Gamma^{k-1},$$
implying that $(a\Gamma b_\Gamma^{k-1})_\Gamma^2\sigma_\Gamma(a\Gamma b_\Gamma^k)\Gamma b_\Gamma^{k-1}$.

Since $a\Gamma b_\Gamma^k\sigma_\Gamma b_\Gamma^{k+1}$ and $(a\Gamma b_\Gamma^k)\Gamma b_\Gamma^{k-1}\sigma_\Gamma b_\Gamma^{k+1}\Gamma b_\Gamma^{k-1}$, hence $(a\Gamma b_\Gamma^{k-1})_\Gamma^2\sigma_\Gamma(b_\Gamma^k)_\Gamma^2$. It further implies that $(a\Gamma b_\Gamma^{k-1})_\Gamma^2\sigma_\Gamma(a\Gamma b_\Gamma^{k-1})\Gamma b_\Gamma^k\sigma_\Gamma(b_\Gamma^k)_\Gamma^2$. Thus $a\Gamma b_\Gamma^{k-1}\sigma_\Gamma b_\Gamma^k$. Similarly, $b\Gamma a_\Gamma^{k-1}\sigma_\Gamma a_\Gamma^k$. Thus if (5) holds for k, it holds for $k-1$.

Now obviously (5) yields $a\Gamma b^3_\Gamma \sigma'_\Gamma b^4_\Gamma$ and $b\Gamma a^3_\Gamma \sigma'_\Gamma a^4_\Gamma$. Also, we get

$$\begin{array}{rcl}
(a\Gamma b^3_\Gamma)\Gamma a^2_\Gamma \sigma'_\Gamma b^4_\Gamma \Gamma a^2_\Gamma & \text{and} & (b\Gamma a^3_\Gamma)\Gamma b^2_\Gamma \sigma'_\Gamma a^4_\Gamma \Gamma b^2_\Gamma, \\
(a^2_\Gamma \Gamma b^3_\Gamma)\Gamma a \sigma'_\Gamma \Gamma b^4_\Gamma \Gamma a^2_\Gamma & \text{and} & (b^2_\Gamma \Gamma a^3_\Gamma)\Gamma b \sigma'_\Gamma a^4_\Gamma \Gamma b^2_\Gamma, \\
(b^3_\Gamma \Gamma a^2_\Gamma)\Gamma a \sigma'_\Gamma a^2_\Gamma \Gamma b^4_\Gamma & \text{and} & (a^3_\Gamma \Gamma b^2_\Gamma)\Gamma b \sigma'_\Gamma b^2_\Gamma \Gamma a^4_\Gamma, \\
a^3_\Gamma \Gamma b^3_\Gamma \sigma'_\Gamma a^2_\Gamma \Gamma b^4_\Gamma & \text{and} & b^3_\Gamma \Gamma a^3_\Gamma \sigma'_\Gamma b^2_\Gamma \Gamma a^4_\Gamma, \\
a^3_\Gamma \Gamma b^3_\Gamma \sigma'_\Gamma a^2_\Gamma \Gamma b^4_\Gamma & \text{and} & a^3_\Gamma \Gamma b^3_\Gamma \sigma'_\Gamma b^2_\Gamma \Gamma a^4_\Gamma,
\end{array}$$

which imply that $(b^2_\Gamma \Gamma a)^2_\Gamma \sigma'_\Gamma a^3_\Gamma \Gamma b^3_\Gamma \sigma'_\Gamma (a^2_\Gamma \Gamma b)^2_\Gamma$, and as σ'_Γ is separative and $(b^2_\Gamma \Gamma a)\Gamma(a^2_\Gamma \Gamma b) = (b^2_\Gamma \Gamma a^2_\Gamma)\Gamma(a\Gamma b) = (a^2_\Gamma \Gamma b^2_\Gamma)\Gamma(a\Gamma b) = a^3_\Gamma \Gamma b^3_\Gamma$, so $a^2_\Gamma \Gamma b \sigma'_\Gamma b^2_\Gamma \Gamma a$. Now we get: $(a^2_\Gamma \Gamma b)\Gamma a \sigma'_\Gamma \Gamma (b^2_\Gamma \Gamma a)\Gamma a$, $(a\Gamma b)\Gamma a^2_\Gamma \sigma'_\Gamma a^2_\Gamma \Gamma b^2_\Gamma$, $a^2_\Gamma \Gamma (b\Gamma a)\sigma'_\Gamma a^2_\Gamma \Gamma b^2_\Gamma$, $b\Gamma a^3_\Gamma \sigma'_\Gamma a^2_\Gamma \Gamma b^2_\Gamma$, but $b\Gamma a^3_\Gamma \sigma'_\Gamma a^4_\Gamma$.

Thus $(b\Gamma a)^2_\Gamma \sigma'_\Gamma b\Gamma a^3_\Gamma \sigma'_\Gamma (a^2_\Gamma)^2_\Gamma$. Now since σ'_Γ is separative and $a^2_\Gamma \Gamma (b\Gamma a) = b\Gamma a^3_\Gamma$, so we get $b\Gamma a \sigma'_\Gamma a^2_\Gamma$.

Similarly we can obtain $a\Gamma b \sigma'_\Gamma b^2_\Gamma$.

Also it is easy to show that (5) holds for $k = 2$. Thus if (5) holds for k, it holds for $k = 1$. By induction down from k, it follows that (5) holds for $k = 1$, $a\Gamma b \sigma_\Gamma b^2_\Gamma$ and $b\Gamma a \sigma_\Gamma a^2_\Gamma$. Now using (2) and Proposition 2.4 on $a\Gamma b \sigma_\Gamma b^2_\Gamma$, we get $(b\Gamma a)^2_\Gamma \sigma_\Gamma b^3_\Gamma \Gamma a$, and again using (4) and (2) on $a\Gamma b \sigma_\Gamma b^2_\Gamma$ we get $b^3_\Gamma \Gamma a \sigma_\Gamma b^4_\Gamma$. So $(b\Gamma a)^2_\Gamma \sigma_\Gamma b^3_\Gamma \Gamma a \sigma_\Gamma b^4_\Gamma$ implies that $b\Gamma a \sigma_\Gamma b^2_\Gamma$ which further implies that $a\Gamma b \sigma_\Gamma b\Gamma a$. Thus we obtain $a\sigma_\Gamma b$. Hence $\rho_\Gamma \subseteq \sigma_\Gamma$ and so S/ρ_Γ is the maximal separative commutative image of S. □

Lemma 2.10. *If $x\Gamma a = x$ $(a = a^2_\Gamma)$ for some x in a locally associative Γ-AG^{**}-groupoid S, then $x^n_\Gamma \Gamma a = x^n_\Gamma$ for some positive integer n.*

Proof. Let $n = 2$. By using (3), we get

$$x^2_\Gamma \Gamma a = (x\Gamma x)\Gamma(a\Gamma a) = (x\Gamma a)\Gamma(x\Gamma a) = x\Gamma x = x^2_\Gamma.$$

Let the result be true for k, that is, $x^k_\Gamma \Gamma a = x^k_\Gamma$. Then by (3) and Proposition 2.1, we get $x^{k+1}_\Gamma \Gamma a = (x\Gamma x^k_\Gamma)\Gamma(a\Gamma a) = (x\Gamma a)\Gamma(x^k_\Gamma \Gamma a) = x\Gamma x^k_\Gamma = x^{k+1}_\Gamma$. Hence $x^n_\Gamma \Gamma a = x^n_\Gamma$ for all positive integers n. □

Lemma 2.11. *If S is a Γ-AG-groupoid, then $Q_\Gamma = \{x \in S \mid x\Gamma a = x$ and $a = a^2_\Gamma\}$ is a commutative subsemigroup.*

Proof. As $a\Gamma a = a$, we have $a \in Q_\Gamma$. Now if $x, y \in Q_\Gamma$, then by identity (3), $x\Gamma y = (x\Gamma a)\Gamma(y\Gamma a) = (x\Gamma y)\Gamma(a\Gamma a) = (x\Gamma y)\Gamma a$.

To prove that Q_Γ is commutative and associative, assume that x, y and z belong to Q_Γ. Then by using (2), we get $x\Gamma y = (x\Gamma a)\Gamma y = (y\Gamma a)\Gamma x = y\Gamma x$. Also, $(x\Gamma y)\Gamma z = (z\Gamma y)\Gamma x = x\Gamma(y\Gamma z)$. Hence Q_Γ is a commutative subsemigroup of S. □

Theorem 2.12. *Let ρ_Γ and σ_Γ be separative congruences on locally associative Γ-AG^{**}-groupoid S and $x^2_\Gamma \Gamma a = x^2_\Gamma$ $(a = a^2_\Gamma)$ for all $x \in S$. If $\rho_\Gamma \cap (Q_\Gamma \times Q_\Gamma) \subseteq \sigma_\Gamma \cap (Q_\Gamma \times Q_\Gamma)$, then $\rho_\Gamma \subseteq \sigma_\Gamma$.*

Proof. If $x\rho_\Gamma y$, then $(x^2_\Gamma \Gamma(x\Gamma y))^2_\Gamma \rho_\Gamma (x^2_\Gamma \Gamma(x\Gamma y)\Gamma(x^2_\Gamma \Gamma y^2_\Gamma) \rho_\Gamma (x^2_\Gamma \Gamma y^2_\Gamma)^2_\Gamma$. It follows that $(x^2_\Gamma \Gamma(x\Gamma y))^2_\Gamma, (x^2_\Gamma \Gamma y^2_\Gamma)^2_\Gamma \subseteq Q_\Gamma$. Now by (3), (2), (4), respectively, we get

$$(x^2_\Gamma \Gamma(x\Gamma y))\Gamma(x^2_\Gamma \Gamma y^2_\Gamma) = (x^2_\Gamma \Gamma x^2_\Gamma)\Gamma(x\Gamma y)\Gamma y^2_\Gamma) = (x^2_\Gamma \Gamma x^2_\Gamma)\Gamma(y^3_\Gamma \Gamma x)$$

$$= x_\Gamma^4\Gamma(y_\Gamma^3\Gamma x) = y_\Gamma^3\Gamma(x_\Gamma^4\Gamma x) = y_\Gamma^3\Gamma x_\Gamma^5$$

and $$(y_\Gamma^3\Gamma x_\Gamma^5)\Gamma a = (y_\Gamma^3\Gamma x_\Gamma^5)\Gamma(a\Gamma a) = (y_\Gamma^3\Gamma a)\Gamma(x_\Gamma^5\Gamma a) = y_\Gamma^3\Gamma x_\Gamma^5 \quad .$$

So $x_\Gamma^2\Gamma(x\Gamma y)\Gamma(x_\Gamma^2\Gamma y_\Gamma^2) \subseteq Q_\Gamma$. Hence $x_\Gamma^2\Gamma(x\Gamma y))_\Gamma^2\sigma_\Gamma(x_\Gamma^2\Gamma(x\Gamma y)\Gamma(x_\Gamma^2\Gamma y_\Gamma^2)\sigma_\Gamma(x_\Gamma^2\Gamma y_\Gamma^2)_\Gamma^2$ implies that $x_\Gamma^2\Gamma(x\Gamma y)\sigma_\Gamma x_\Gamma^2\Gamma y_\Gamma^2$.

Since $x_\Gamma^2\Gamma y_\Gamma^2\rho_\Gamma x_\Gamma^4$ and $(x_\Gamma^2\Gamma y_\Gamma^2), x_\Gamma^4 \subseteq Q_\Gamma$, thus $x_\Gamma^2\Gamma y_\Gamma^2\sigma_\Gamma x_\Gamma^4$. From Proposition 2.4, we get $(x_\Gamma^2)_\Gamma^2\sigma_\Gamma x_\Gamma^2\Gamma(x\Gamma y)\sigma_\Gamma(x\Gamma y)_\Gamma^2$, which implies that $x_\Gamma^2\sigma_\Gamma x\Gamma y$. Finally, $x_\Gamma^2\rho_\Gamma y_\Gamma^2$ and $x_\Gamma^2, y_\Gamma^2 \subseteq Q_\Gamma$, implying that $x_\Gamma^2\sigma_\Gamma y_\Gamma^2, x_\Gamma^2\sigma_\Gamma x\Gamma y\sigma_\Gamma y_\Gamma^2$. Thus $x\sigma_\Gamma y$ because σ_Γ is separative. □

LEMMA 2.13. *Every left zero congruence is commutative.*

Proof. Let $a\sigma_\Gamma a$ and $b\sigma_\Gamma b$ which imply that $a\Gamma b\sigma_\Gamma a\Gamma b, (a\Gamma b)\Gamma(a\Gamma b)\sigma(a\Gamma b)_\Gamma^2 = (b_\Gamma\Gamma a)_\Gamma^2$ and so we obtain $a\Gamma b\sigma_\Gamma b\Gamma a$. □

The relation η_Γ is defined on S by $a\eta_\Gamma b$ if and only if there exist some positive integers m, n such that $b_\Gamma^m \subseteq a\Gamma S$ and $a_\Gamma^n \subseteq b\Gamma S$.

THEOREM 2.14. *Let S be a locally associative Γ-AG**-groupoid. Then the relation η_Γ is the least semilattice congruence on S.*

Proof. The relation η_Γ is obviously reflexive and symmetric. To show the transitivity, let $a\eta_\Gamma b$ and $b\eta_\Gamma c$, where $a, b, c \in S$. Then $a\Gamma x = b_\Gamma^m$ and $b\Gamma y = c_\Gamma^n$, for some $x, y \in S$. Using (4), we get $c_\Gamma^{mn} = (c_\Gamma^n)_\Gamma^m = (b\Gamma y)_\Gamma^m = y_\Gamma^m\Gamma b_\Gamma^m = y_\Gamma^m\Gamma(a\Gamma x) = a\Gamma(y_\Gamma^m\Gamma x)$, implying that $c_\Gamma^k = a\Gamma z$, where $k = mn$ and $z = (y_\Gamma^m\Gamma x)$. Similarly, $b\Gamma x' = a_\Gamma^{m'}$ and $c\Gamma y' = b_\Gamma^{n'}$ implying that $a_\Gamma^{k'} = c\Gamma z'$.

Let $a, b, c \in S$ and $a\eta_\Gamma b \Leftrightarrow (\exists m, n \in Z^+)(\exists x, y \in S), b_\Gamma^m = a\Gamma x, a_\Gamma^n = b\Gamma y$. If $m = 1, n > 1$, that is, $b = a\Gamma x$, $a_\Gamma^n = b\Gamma y$ for some $x, y \in S$, then $b_\Gamma^3 = (b\Gamma b)\Gamma(a\Gamma x) = a\Gamma(b_\Gamma^2\Gamma x) \subseteq a\Gamma S$.

Similarly we can consider the case $m = n = 1$. Suppose that $m, n > 1$. Then using (2) and Proposition 2.4, we obtain

$$(b\Gamma c)_\Gamma^m = b_\Gamma^m\Gamma c_\Gamma^m = (a\Gamma x)\Gamma c_\Gamma^m = (a\Gamma x)\Gamma(c\Gamma c_\Gamma^{m-1}) = (a\Gamma c)\Gamma(x\Gamma c_\Gamma^{m-1}) = (a\Gamma c)\Gamma y,$$

where $y = x\Gamma c_\Gamma^{m-1}$. Thus $a\Gamma c\eta_\Gamma b\Gamma c$ and $c\Gamma a\eta_\Gamma c\Gamma b$.

Now to show that η_Γ is a semilattice congruence on S, first we need to show that $a\eta_\Gamma b$ implies $a\Gamma b\eta_\Gamma a$.

Let $a\eta_\Gamma b$; then $b_\Gamma^m = a\Gamma x$ and $a_\Gamma^n = b\Gamma y$ for some $x, y \in S$. So by Proposition 2.4 and (3), we get $(a\Gamma b)_\Gamma^m = a_\Gamma^m\Gamma b_\Gamma^m = a_\Gamma^m\Gamma(a\Gamma x) = a\Gamma(a_\Gamma^m\Gamma x)$. Also, $a_\Gamma^n = b\Gamma y$ implies that $a_\Gamma^{n+2} = a_\Gamma^2\Gamma a_\Gamma^n = (a\Gamma a)\Gamma(b\Gamma y) = (a\Gamma b)\Gamma(a\Gamma y)$. Hence $a\Gamma b\eta_\Gamma a$ which implies that $a_\Gamma^2\eta_\Gamma a, (a_\Gamma^2)_{\eta_\Gamma} = (a)_{\eta_\Gamma}$ and so S/η_Γ is idempotent.

Next we show that η_Γ is commutative. By Proposition 2.4, $(a\Gamma b)_\Gamma^2 = (b\Gamma a)_\Gamma^2$, which shows that $a\Gamma b\eta_\Gamma b\Gamma a$ that is, $(a)_{\eta_\Gamma}\Gamma(b)_{\eta_\Gamma} = (b)_{\eta_\Gamma}\Gamma(a)_{\eta_\Gamma}$, that is, S/η_Γ is a commutative AG-groupoid and so it is left zero commutative semigroup of idempotents. Therefore, η_Γ is a semilattice congruence on S. Next we will show that η_Γ is contained in any other left zero semilattice congruence ρ_Γ on S. Let $a\eta_\Gamma b$; then $b_\Gamma^m = a\Gamma x$ and $a_\Gamma^n = b\Gamma y$. Now since $a\rho_\Gamma a_\Gamma^2$ and $b\rho_\Gamma b_\Gamma^2$, it implies that $a\Gamma x\rho_\Gamma a_\Gamma^2\Gamma x, a\rho_\Gamma a_\Gamma^n$ and $b\rho_\Gamma b_\Gamma^m$ which further implies that $a\rho_\Gamma b\Gamma y$ and $b\rho_\Gamma a\Gamma x$. It can be easily seen that $a\Gamma b\rho_\Gamma b\Gamma a$.

Also since $b\rho_\Gamma b_\Gamma^2$ and ρ_Γ is compatible, so we get $b\Gamma y\rho_\Gamma b_\Gamma^2\Gamma y$. We can easily see that $b\Gamma a\rho_\Gamma a\Gamma b\rho_\Gamma a\rho_\Gamma b\Gamma y\rho_\Gamma b_\Gamma^2\Gamma y$ which implies that $b\Gamma a\rho_\Gamma b_\Gamma^2\Gamma y$. Similarly, we can show that $a\Gamma b\rho_\Gamma a_\Gamma^2\Gamma x$. So $a\rho_\Gamma b\Gamma y\rho_\Gamma b_\Gamma^2\Gamma y\rho_\Gamma b\Gamma a\rho_\Gamma a\Gamma b\rho_\Gamma a_\Gamma^2\Gamma x\rho_\Gamma a\Gamma x\rho_\Gamma b$ implies that $a\rho_\Gamma b$. Thus η_Γ is a least semilattice congruence on S. □

THEOREM 2.15. *η_Γ is separative.*

Proof. Let $a_\Gamma^2\eta_\Gamma a\Gamma b$ and $a\Gamma b\eta_\Gamma b_\Gamma^2$, then there exist positive integers m, m', n, n' such that $(a_\Gamma^2)_\Gamma^m = (a\Gamma b)_\Gamma^2\Gamma x$, $(a\Gamma b)_\Gamma^{m'} = (a_\Gamma^2)_\Gamma^2\Gamma x'$ and $(a\Gamma b)_\Gamma^{n'} = (b_\Gamma^2)_\Gamma^2\Gamma y'$, $(b_\Gamma^2)_\Gamma^n = (a\Gamma b)_\Gamma^2\Gamma y$.

Now we get,
$$\begin{aligned} a_\Gamma^{2m+2} &= a_\Gamma^{2m}\Gamma a_\Gamma^2 = (a_\Gamma^2)_\Gamma^m\Gamma a_\Gamma^2 = ((a\gamma b)_\Gamma^2\Gamma x)\Gamma a_\Gamma^2 \\ &= (a_\Gamma^2\Gamma x)\Gamma(a\Gamma b)_\Gamma^2 = (a_\Gamma^2\Gamma x)\Gamma(a_\Gamma^2\Gamma b_\Gamma^2) = (a_\Gamma^2\Gamma x)\Gamma(b_\Gamma^2\Gamma a_\Gamma^2) \\ &= b_\Gamma^2\Gamma((a_\Gamma^2\Gamma x)\Gamma a_\Gamma^2 = b_\Gamma^2\Gamma t_6, \quad \text{where} \quad t_6 = ((a_\Gamma^2\Gamma x)\Gamma a_\Gamma^2) \end{aligned}$$

Similarly,
$$\begin{aligned} b_\Gamma^{2n+2} &= b_\Gamma^{2n}\Gamma b_\Gamma^2 = ((a\Gamma b)_\Gamma^2\Gamma y)\Gamma b_\Gamma^2 = (b_\Gamma^2\Gamma y)\Gamma(a_\Gamma^2\Gamma b_\Gamma^2) \\ &= a_\Gamma^2\Gamma((b_\Gamma^2\Gamma y)\Gamma b_\Gamma^2) = a_\Gamma^2\Gamma t_7, \quad \text{where} \quad t_7 = ((b_\Gamma^2\Gamma y)\Gamma b_\Gamma^2). \end{aligned}$$

Hence η_Γ is separative. □

THEOREM 2.16. *Let S be a locally associative Γ-AG^{**}-groupoid. Then S/η_Γ is a maximal semilattice separative image of S.*

Proof. By Theorem 2.14, η_Γ is the least semilattice congruence on S and S/η_Γ is a semilattice. Hence S/η_Γ is a maximal semilattice separative image of S. □

THEOREM 2.17. *Every locally associative Γ-AG^{**}-groupoid S is uniquely expressible as a semilattice Y of Archimedean locally associative Γ-AG^{**}-groupoids $(S_\pi)_\Gamma(\pi \in Y)$. The semilattice Y is isomorphic with the maximal semilattice separative image S/η_Γ of S and $(S_\pi)_\Gamma(\pi \in Y)$ are the equivalence classes of S modη_Γ.*

Proof. By Theorem 2.14, η_Γ is the least semilattice congruence on S. Next we will prove that the equivalence classes modη_Γ are Archimedean locally associative Γ-AG**-groupoids and the semilattice Y is isomorphic to S/η_Γ. Let $a, b \in (S_\pi)_\Gamma$, where $\pi \in Y$; then $a\eta_\Gamma b$ implies that $a_\Gamma^m \subseteq b\Gamma S, b_\Gamma^n \subseteq a\Gamma S$, so $a_\Gamma^m = b\Gamma x$ and $b_\Gamma^n = a\Gamma y$, where $x, y \in S$. If $x \in S_\vartheta, \vartheta \neq \pi$, then $\pi = \pi\vartheta$, using (4), and we get $a_\Gamma^{m+1} = a\Gamma a_\Gamma^m = a\Gamma(b\Gamma x) = b\Gamma(a\Gamma x) \subseteq b\Gamma(S_{\pi\vartheta})_\Gamma = b\Gamma(S_\pi)_\Gamma$. Similarly, one can show that $b_\Gamma^{n+1} \subseteq a\Gamma(S_\pi)_\Gamma$. This shows that $(S_\pi)_\Gamma$ is right Archimedean and so it is locally associative Archimedean Γ-AG**-groupoid S.

Next we show the uniqueness. Let S be a semilattice Y of Archimedean AG**-groupoid $(S_\pi)_\Gamma, \pi \in Y$. We need to show that $(S_\pi)_\Gamma$ are equivalent classes of S modη_Γ. Let $a, b \in S$. Then we show that $a\eta_\Gamma b$ if and only if a and b belong to the same $(S_\pi)_\Gamma$. If a and b both belong to the same $(S_\pi)_\Gamma$, then each divides the power of the other. Since $(S_\pi)_\Gamma$ is Archimedean, $a\eta_\Gamma b$ by the definition. Conversely, if $a\eta_\Gamma b$, then $a\Gamma x = b_\Gamma^m$ and $b\Gamma y = a_\Gamma^n$ for some $x, y \in S$ and some $m, n \in Z^+$. If $x \in (S_\vartheta)_\Gamma$, then $a\Gamma x \subseteq (S_{\pi\vartheta})_\Gamma$ and $b_\Gamma^m \subseteq (S_\vartheta)_\Gamma$, so that $\pi\vartheta = \vartheta$. Hence $\vartheta \leq \pi$, in the semilattice Y. By symmetry, it follows that $\pi \leq \vartheta$, that is, $\pi = \vartheta$. □

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Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus, Pakistan

E-mail: madadmath@yahoo.com

Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus, Pakistan

E-mail: saimaanis_pk@yahoo.com

Department of Mathematical Engineering, Polytechnic University of Tirana, Albania

E-mail: kostaq_hila@yahoo.com