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GENERALIZED CONTRACTIONS AND FIXED POINT THEOREMS OVER BIPOLAR CONE$_{tvs}$ b-METRIC SPACES WITH AN APPLICATION TO HOMOTOPY THEORY

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Abstract. In this paper, we introduce the concept of bipolar cone$_{tvs}$ b-metric space and prove some generalized fixed point theorems on it. These theorems extend and generalize some recent results obtained by other authors for mappings on a bipolar metric space. Also, a brief study on topological properties of this newly introduced space has been made and in support of our theorems, we give some examples. Moreover, our fixed point result is applied to homotopy theory on such spaces.

1. Introduction

In 2007, Huang and Zhang [5] introduced the concept of normal cone metric spaces and proved some fixed point theorems on it. Consequently many mathematicians have proved several fixed point theorems on such spaces. In continuation of this thread, Azam et al. [2] had been able to prove some fixed point theorems in a topological vector space-valued cone metric space equipped with a non-normal cone. Following this concept Kadelburg et al. [7] proved some common fixed point theorems on such spaces and had shown some basic differences between normal cone metric spaces and non-normal topological vector space-valued cone metric spaces.

In the year 2003, Akram et al. [1] introduced a new class of generalized contractions commonly known as $A-$contractions, which contains a class of contractive mappings namely Kannan's contractive mappings [8], Reich's contractive mappings [14] etc. By considering such type of mappings a good number of mathematicians have proved several fixed point theorems on a variety of topological spaces (see [15]).

In 2016, Mutlu and Gürdal [10, 11] have initiated the concept of bipolar metric spaces and proved some contractive fixed point theorems and coupled fixed point

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theorems. Recently Kishore et al. [9] and Rao et al. [12, 13] have proved several fixed point and common fixed point theorems in this setting.

The aim of this paper is to establish some generalized fixed point theorems in the setting of bipolar cone$_{tvs}$ b-metric spaces with supporting examples. Also here we obtain a homotopy result as an application of our established theorem.

2. Preliminaries

Let E be a Hausdorff topological vector space (in short tvs) with null vector θ. A proper nonempty and closed subset P of E is called a cone if $P+P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \theta$. The cone P is called solid if P has a nonempty interior.

Each cone P induces a partial order $\preceq$ on E by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$. The pair (E, P) is an ordered topological vector space.

The definition of normal cones in ordered topological vector spaces can be found, e.g. in [7]. For a pair of elements $x, y \in E$, such that $x \preceq y$, the order interval is given by $[x, y] = \{z \in E : x \preceq z \preceq y\}$. A subset F of E is said to be order-convex if $[x, y] \subset F$, whenever $x, y \in F$ and $x \preceq y$. An ordered topological vector space (E, P) is order-convex if it has a base of neighborhoods of θ consisting of order-convex subsets. In this case the underlying cone P is called normal. For a normed space, this condition means that the unit ball is order-convex, which is actually equivalent to the condition that there is a number $K > 0$ such that $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying above is said to be the normal constant. Another equivalent condition is that $\inf\{\|x+y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0$. For more information one can see [4].

Example 2.1 ([2, 4]). Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and let $P = \{x \in E : x(t) \geq 0 \text{ on } [0,1]\}$. This cone is solid but is not normal. Consider for example, $x_n(t) = \frac{(1-\sin nt)}{(n+2)}$ and $y_n(t) = \frac{(1+\sin nt)}{(n+2)}$. Since $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = \frac{2}{(n+2)} \to 0$, it follows that P is a nonnormal cone.

Now consider the space $E = C^1_{\mathbb{R}}[0,1]$ endowed with the strongest locally convex topology t^*. Then P is also t^*-solid, but not t^*-normal. Indeed, if it were normal then the space (E, t^*) would be normed, which is impossible since an infinite-dimensional space with the strongest locally convex topology cannot be metrizable.

Definition 2.2 ([7]). Let X be a nonempty set and (E, P) be an ordered tvs. A function $d : X^2 \to E$ is called a tvs-cone metric and (X, d) is called a tvs-cone metric space if the following conditions hold:

(C1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(C3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition 2.3 ([7]). Let $\{x_n\} \subset X$ and $x \in X$. Then

(i) $\{x_n\}$ is said to be tvs-cone convergent to x if for every $c \in E$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$; we denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is said to be a *tvs*-cone Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n > n_0$.

(iii) (X, d) is called *tvs*-cone complete if every *tvs*-cone Cauchy sequence is *tvs*-cone convergent in X.

Now we recall some basic properties of a real *tvs* E with a solid cone P and a *tvs*-cone metric space (X, d) over it.

LEMMA 2.4 ([7]). *(a) Let $\theta \preceq x_n \to \theta$ in (E, P), and let $\theta \ll c$. Then there exists n_0 such that $x_n \ll c$ for each $n > n_0$.*

(b) It can happen that $\theta \preceq x_n \ll c$ for each $n > n_0$, but $x_n \nrightarrow \theta$ in (E, P).

(c) It can happen that $x_n \to x$, $y_n \to y$ as $n \to \infty$ in the tvs-cone metric d, but that $d(x_n, y_n) \nrightarrow d(x, y)$ in (E, P). In particular, it can happen that $x_n \to x$ in d but $d(x_n, x) \nrightarrow \theta$ (which is impossible if the cone is normal).

(d) $\theta \preceq u \ll c$ for each $c \in int P$ implies that $u = \theta$.

(e) $x_n \to x \wedge y_n \to y$ (in the tvs-cone metric spaces) implies that $x = y$.

(f) Each tvs-cone metric space is Hausdorff in the sense that for arbitrary distinct points x and y there exist their disjoint neighbourhoods in the topology τ_c having the local base formed by the sets of the form $K_c(x) = \{z \in X : d(x, z) \ll c\}$, $c \in \operatorname{int} P$.

LEMMA 2.5 ([7]). *(a) If $u \preceq v$ and $v \ll w$, then $u \ll w$.*

(b) If $u \ll v$ and $v \preceq w$, then $u \ll w$.

(c) If $u \ll v$ and $v \ll w$, then $u \ll w$.

(d) Let $x \in X$, $\{x_n\}$ and $\{b_n\}$ be two sequences in X and E, respectively, $\theta \ll c$, and $\theta \preceq d(x_n, x) \preceq b_n$ for all $n \in \mathbb{N}$. If $b_n \to \theta$, then there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$.

A generalization of metric spaces, namely b-metric spaces was developed by I.A. Bakhtin [3] following which, Hussain et al. have defined cone b-metric spaces in the following way.

DEFINITION 2.6 ([6]). Let A be a nonempty set and E be a real Banach space with cone P. A vector-valued function $d : A \times A \to P$ is said to be a cone b-metric on A with the constant $k \geq 1$ if the following conditions are satisfied:
(M_1) $d(x, y) = \theta$ if and only if $x = y$; (M_2) $d(x, y) = d(y, x)$ for all $x, y \in A$;

(M_3) $d(x, z) \preceq k(d(x, y) + d(y, z))$ for all $x, y, z \in A$.
The pair (A, d) is called the cone b-metric space.

DEFINITION 2.7 ([10]). Let C and D be two nonempty sets. Suppose that a function $d : C \times D \to \mathbb{R}^+$ satisfies the following conditions:
(B_1) $d(x, y) = 0$ if and only if $x = y$; (B_2) $d(x, y) = d(y, x)$ for all $x, y \in C \cap D$;

(B_3) $d(x_1, y_2) \le d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$ for all $(x_1, y_1), (x_2, y_2) \in C \times D$.
Then the function d is said to be a bipolar metric on (C, D) and the triplet (C, D, d) is called a bipolar-metric space.

Example 2.8 ([10]). Let X be the class of all singleton subsets of $\mathbb{R}$ and Y be the class of all nonempty compact subsets of $\mathbb{R}$. We define $d : X \times Y \to \mathbb{R}^+$ as $d(x, A) = |x - \inf(A)| + |x - \sup(A)|$. Then (X, Y, d) is a bipolar metric space.

3. Introduction to bipolar cone$_{tvs}$ b-metric space

In what follows we always assume that E is a real Hausdorff topological vector space with a solid cone P and $\preceq$ is the partial ordering on E induced by P.

Definition 3.1. Let X and Y be two nonempty sets and $d : X \times Y \to P$ be a function, satisfying the following properties:
(i) $d(x, y) = \theta$ if and only if $x = y$; (ii) $d(x, y) = d(y, x)$ for all $x, y \in X \cap Y$;

(iii) $d(x_1, y_2) \preceq s[d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)]$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, where the coefficient $s \ge 1$.
Then d is called a bipolar cone$_{tvs}$ b-metric on (X, Y) and the triplet (X, Y, d) is called a bipolar cone$_{tvs}$ b-metric space. In particular, if $X \cap Y \neq \emptyset$ then the space is called joint, otherwise it is called disjoint. The sets X and Y are respectively called the left pole and the right pole of (X, Y, d).

Example 3.2. Let $X = \{-1, 0\}$, $Y = \{0, 1\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \ge 0\}$. Let $d : X \times Y \to E$ be defined by $d(0, 0) = (0, 0)$, $d(-1, 0) = (3, 3)$, $d(-1, 1) = d(0, 1) = (1, 1)$. Then (X, Y, d) is a bipolar cone$_{tvs}$ b-metric space with the coefficient $s = \frac{3}{2}$.

Example 3.3. Let L be the set of all Lebesgue measurable functions on $[0, 1]$, such that $\int_0^1 |f(x)|^2 \, dx < \infty$. Let $X = \{f \in L : f(x) \ge 0$ for all $x \in [0, \frac{1}{2}]$ and $f(x) \le 0$ for all $x \in (\frac{1}{2}, 1]\}$ and $Y = \{g \in L : g(x) \le 0$ for all $x \in [0, \frac{1}{2}]$ and $g(x) \ge 0$ for all $x \in (\frac{1}{2}, 1]\}$. Also, $d : X \times Y \to E$, where $E = C^1_{\mathbb{R}}[0, 1]$ and $P = \{\varphi \in E : \varphi \ge 0\}$, is defined by

$$d(f, g)(t) = \left(\int_0^1 |f(x) - g(x)|^2 \, dx \right) \varphi(t)$$

for all $t \in [0, 1]$, $(f, g) \in X \times Y$ and for some $\varphi \in P$. We can choose $\varphi(t) = \frac{1}{1+t^2}$ or e^t for all $t \in \mathbb{R}$ for instance. Then (X, Y, d) is a bipolar cone$_{tvs}$ b-metric space with $s = 3$.

Example 3.4. Let $U_n(\mathbb{R})$ and $L_n(\mathbb{R})$ be the sets of all upper and lower triangular matrices of order n respectively. Also let $E = \mathbb{R}$ and $P = \{x \in E : x \ge 0\}$. Suppose $d : U_n(\mathbb{R}) \times L_n(\mathbb{R}) \to P$ is defined as $d(A, B) = \sum_{i,j=1}^n |a_{ij} - b_{ij}|^2$ for all $A = (a_{ij})_{n \times n} \in U_n(\mathbb{R})$ and $B = (b_{ij})_{n \times n} \in L_n(\mathbb{R})$. Then $(U_n(\mathbb{R}), L_n(\mathbb{R}), d)$ is a bipolar cone$_{tvs}$ b-metric space with the coefficient $s = 3$.

Example 3.5. Let A and B be two nonempty sets, (C, h) be a cone b-metric space with the coefficient s and $g : A \cup B \to C$ be an injective mapping. Then $d : A \times B \to [0, \infty)$, defined by $d(a, b) = h(g(a), g(b))$ for all $(a, b) \in A \times B$, is a bipolar cone$_{tvs}$ b-metric with the coefficient s^2.

Definition 3.6. (i) The opposite of a bipolar cone$_{tvs}$ b-metric space (X, Y, d) is defined as the bipolar cone$_{tvs}$ b-metric space $(Y, X, \bar{d})$, where the function $\bar{d} : Y \times X \to E$ is defined as $\bar{d}(y, x) = d(x, y)$.

(ii) Let (X_1, Y_1) and (X_2, Y_2) be two pairs of sets.

A function $F : X_1 \cup Y_1 \to X_2 \cup Y_2$ is said to be a covariant map if $F(X_1) \subset X_2$ and $F(Y_1) \subset Y_2$ and we denote this as $F : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$.

A function $F : X_1 \cup Y_1 \to X_2 \cup Y_2$ is said to be a contravariant map if $F(X_1) \subset Y_2$ and $F(Y_1) \subset X_2$ and we denote this as $F : (X_1, Y_1) \rightleftharpoons (X_2, Y_2)$.

If (X_1, Y_1, d_1) and (X_2, Y_2, d_2) are two bipolar cone$_{tvs}$ b-metric spaces then we use the notations $F : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ and $F : (X_1, Y_1, d_1) \rightleftharpoons (X_2, Y_2, d_2)$.

Definition 3.7. Let (X, Y, d) be a bipolar cone$_{tvs}$ b-metric space. A point $z \in X \cup Y$ is said to be a left point if $z \in X$, a right point if $z \in Y$ and a central point if both hold.

A sequence $\{x_n\} \subset X$ is called a left sequence and a sequence $\{y_n\} \subset Y$ is called a right sequence.

A sequence $\{u_n\} \subset X \cup Y$ is said to converge to a point u if and only if $\{u_n\}$ is a left sequence, u is a right point and for any $c \gg \theta$ there exists $N_1 \in \mathbb{N}$ such that $d(u_n, u) \ll c$ for all $n \geq N_1$ or $\{u_n\}$ is a right sequence, u is a left point and for any $c \gg \theta$ there exists $N_2 \in \mathbb{N}$ such that $d(u, u_n) \ll c$ for all $n \geq N_2$.

Definition 3.8. A sequence $\{(x_n, y_n)\} \subset X \times Y$ is called a bisequence. If the sequences $\{x_n\}$ and $\{y_n\}$ both converge then the bisequence $\{(x_n, y_n)\}$ is said to be convergent.

If $\{x_n\}$ and $\{y_n\}$ both converge to a same point $z \in X \cap Y$ then the bisequence $\{(x_n, y_n)\}$ is said to be biconvergent.

A sequence $\{(x_n, y_n)\}$ is a Cauchy bisequence if for any arbitrary $c \gg \theta$ there exists $N \in \mathbb{N}$ such that $d(x_n, y_m) \ll c$ whenever $n, m \geq N$.

Definition 3.9. A bipolar cone$_{tvs}$ b-metric space is said to be complete if every Cauchy bisequence is convergent.

Remark 3.10. In Examples 3.2 and 3.4 we have that (X, Y, d) and $(U_n(\mathbb{R}), L_n(\mathbb{R}), d)$, respectively, are complete bipolar cone$_{tvs}$ b-metric spaces.

Definition 3.11. Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar cone$_{tvs}$ b-metric spaces:

(i) A mapping $F : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left-continuous at a point $x_0 \in X_1$ if for every sequence $\{y_n\} \subset Y_1$ with $y_n \to x_0$ we have $F(y_n) \to F(x_0)$ in (X_2, Y_2, d_2).

(ii) A mapping $F : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right-continuous at a point $y_0 \in Y_1$ if for every sequence $\{x_n\} \subset X_1$ with $x_n \to y_0$ we have $F(x_n) \to F(y_0)$ in (X_2, Y_2, d_2).

(iii) A mapping $F : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is said to be continuous, if it is left-continuous at each point $x \in X_1$ and right-continuous at each $y \in Y_1$.

(iv) A contravariant map $F : (X_1, Y_1, d_1) \rightleftharpoons (X_2, Y_2, d_2)$ is continuous if it is continuous as a covariant map $F : (X_1, Y_1, d_1) \rightrightarrows (Y_2, X_2, \bar{d_2})$.

Lemma 3.12. *Let (X, Y, d) be a bipolar cone$_{tvs}$ b-metric space. If a central point is a limit of a sequence, then it is the unique limit of this sequence.*

Proof. Let $\{q_n\}$ be a left-sequence in (X, Y, d) which converges to some $q \in X \cap Y$. If possible, let $q_1 \in Y$ be another limit of this sequence, then

$$d(q, q_1) \preceq s[d(q_n, q) + d(q_n, q_1) + d(q, q)] = s[d(q_n, q) + d(q_n, q_1)].$$

Since $\{q_n\}$ converges to both q and q_1 then for any arbitrary $c \gg \theta$ there exists $N_1, N_2 \in \mathbb{N}$ such that $d(q_n, q) \ll \frac{c}{2s}$ if $n \geq N_1$ and $d(q_n, q_1) \ll \frac{c}{2s}$ whenever $n \geq N_2$. So for $n \geq N = \max\{N_1, N_2\}$ we have $d(q, q_1) \ll c$ and this implies $q_1 = q$. □

Proposition 3.13. *In a bipolar cone$_{tvs}$ b-metric space every biconvergent bisequence is a Cauchy bisequence.*

Proof. Let $\{(x_n, y_n)\}$ be a biconvergent bisequence in a bipolar cone$_{tvs}$ b-metric space (X, Y, d), which biconverges to some $z \in X \cap Y$. Then for $n, m \in \mathbb{N}$

$$d(x_n, y_m) \preceq s[d(x_n, z) + d(z, z) + d(z, y_m)] = s[d(x_n, z) + d(z, y_m)] \tag{1}$$

As $x_n \to z$ and $y_m \to z$ as $n, m \to \infty$ then for any $c \gg \theta$ there exists $N \in \mathbb{N}$ such that $d(x_n, z) \ll \frac{c}{2s}$ and $d(z, y_m) \ll \frac{c}{2s}$ whenever $n \geq N$. Then for $n, m \geq N$ from (1) we get $d(x_n, y_m) \ll c$. Therefore $\{(x_n, y_n)\}$ is a Cauchy bisequence. □

Proposition 3.14. *In a bipolar cone$_{tvs}$ b-metric space every convergent Cauchy bisequence is biconvergent.*

Proof. Let (X, Y, d) be a bipolar cone$_{tvs}$ b-metric space and $\{(p_n, q_n)\}$ be a Cauchy bisequence, such that $p_n \to q \in Y$ and $q_n \to p \in X$. Then $d(p, q) \preceq s[d(p, q_n) + d(p_n, q_n) + d(p_n, q)]$. Now for any arbitrary $c \gg \theta$ there exists $N \geq 1$ such that whenever $n \geq N$, $d(p, q_n) \ll \frac{c}{3s}$, $d(p_n, q) \ll \frac{c}{3s}$ and $d(p_n, q_n) \ll \frac{c}{3s}$ and so we have $d(p, q) = \theta$ implies $p = q$. □

4. Topological construction

In this section we are now in a position to study some topological properties in a bipolar cone$_{tvs}$ b-metric space.

Definition 4.1. Let (X,Y,d) be a bipolar cone$_{tvs}$ b-metric space. For any $(x_0,y_0)\in X\times Y$ and $c\gg\theta$ we define

$$\begin{aligned}
B_r^{(x_0,y_0)}(x_0,c) &= \{y\in Y : d(x_0,y)\ll c+sd(x_0,y_0)\},\\
B_r^{(x_0,y_0)}[x_0,c] &= \{y\in Y : d(x_0,y)\preceq c+sd(x_0,y_0)\},\\
B_l^{(x_0,y_0)}(y_0,c) &= \{x\in X : d(x,y_0)\ll c+sd(x_0,y_0)\},\\
B_l^{(x_0,y_0)}[y_0,c] &= \{x\in X : d(x,y_0)\preceq c+sd(x_0,y_0)\}.
\end{aligned}$$

For $x_0\in X\cap Y$ we denote $B_r^{(x_0,x_0)}(x_0,c)$ and $B_l^{(x_0,x_0)}(x_0,c)$ simply by $B_r(x_0,c)$ and $B_l(x_0,c)$, respectively.

Definition 4.2. Let (X,Y,d) be a bipolar cone$_{tvs}$ b-metric space. Then a pair of sets (U_1,U_2) is called an open pair if $U_1\subset X, U_2\subset Y$ and for any $(x_0,y_0)\in(U_1,U_2)$ there exist $c_1,c_2\gg\theta$ such that $B_r^{(x_0,y_0)}(x_0,c_1)\subset U_2$ and $B_l^{(x_0,y_0)}(y_0,c_2)\subset U_1$. If $x_0\in U_1\cap U_2$ then we set $c_1=c_2$.

Proposition 4.3. *Let (X,Y,d) be a bipolar cone$_{tvs}$ b-metric space, where (E,P) is an ordered topological vector space with the property that for each $c_1,c_2\gg\theta$ there exists $c\gg\theta$ such that $c\preceq c_1,c_2$. Then the collection $\mathfrak{B}$ of all open pairs together with $(\emptyset,\emptyset)$ and (X,Y) forms a base for some topology on $X\times Y$.*

Proof. Let (U_1,U_2) and (U_1',U_2') be two open pairs in $\mathfrak{B}$ such that $(x,y)\in(U_1,U_2)\cap(U_1',U_2')$. Now $(x,y)\in(U_1\cap U_1',U_2\cap U_2')\subset(U_1,U_2)\cap(U_1',U_2')$, we show that $(U_1\cap U_1',U_2\cap U_2')$ is an open pair.

Let $(a,b)\in(U_1\cap U_1',U_2\cap U_2')$. Then $(a,b)\in(U_1,U_2)$ and $(a,b)\in(U_1',U_2')$. So there exist $c_1,c_2,c_1',c_2'\gg\theta$ such that $B_r^{(a,b)}(a,c_1)\subset U_2$, $B_l^{(a,b)}(b,c_2)\subset U_1$, $B_r^{(a,b)}(a,c_1')\subset U_2'$, $B_l^{(a,b)}(b,c_2')\subset U_1'$. By the assumed property of (E,P) there exist $\bar{c_1},\bar{c_2}\gg\theta$ such that $\bar{c_1}\preceq c_1,c_1'$ and $\bar{c_2}\preceq c_2,c_2'$. So $B_r^{(a,b)}(a,\bar{c_1})\subset U_2\cap U_2'$ and $B_l^{(a,b)}(b,\bar{c_2})\subset U_1\cap U_1'$. Therefore $(U_1\cap U_1',U_2\cap U_2')$ is an open pair. So $\mathfrak{B}$ forms a base for some topology on $X\times Y$.

This topology is called the topology induced by d. □

Definition 4.4. Let (X,Y,d) be a bipolar cone$_{tvs}$ b-metric space such that d induces a topology on $X\times Y$. Then a pair of sets (F_1,F_2) is said to be a closed pair if there exists an open set $\mathcal{U}$ such that $(F_1,F_2)=\mathcal{U}^c$, where $\mathcal{U}^c=(X\times Y)\setminus\mathcal{U}$.

Definition 4.5. Let (A,B) be a pair of sets in (X,Y,d), a bipolar cone$_{tvs}$ b-metric space such that d induces a topology on $X\times Y$. Then the closure of (A,B) is the smallest closed pair (F_1,F_2) such that $(A,B)\subset(F_1,F_2)$. We denote the closure of (A,B) by $(\bar{A},\bar{B})$.

Proposition 4.6. *Let $\{(x_n,y_n)\}$ be a convergent bisequence which converges to (x,y) in (X,Y,d), where d induces a topology on $X\times Y$. Then for any open set $\mathcal{U}$ in $X\times Y$ containing (x,y), there exists $n_0\in\mathbb{N}$ such that $(x_{k_1},y_{k_2})\in\mathcal{U}$ for all $k_1,k_2\geq n_0$.*

Proof. Since $\{(x_n, y_n)\}$ converges to (x, y), then $x_n \to y$ and $y_n \to x$ as $n \to \infty$. As $(x, y) \in \mathcal{U}$, so there exists a basic open set (U_1, U_2) such that $(x, y) \in (U_1, U_2) \subset \mathcal{U}$. Since (U_1, U_2) is an open pair, there exist $c_1, c_2 \gg \theta$ such that $B_l^{(x,y)}(y, c_2) \subset U_1$ and $B_r^{(x,y)}(x, c_1) \subset U_2$. Thus, there exist $n_1, n_2 \in \mathbb{N}$ such that $y_n \in B_r^{(x,y)}(x, c_1)$ for all $n \geq n_1$ and for every $n \geq n_2$, $x_n \in B_l^{(x,y)}(y, c_2)$. If we take $n_0 = \max\{n_1, n_2\}$, then clearly $(x_{k_1}, y_{k_2}) \in \mathcal{U}$ for all $k_1, k_2 \geq n_0$. □

Proposition 4.7. *Let (X, Y, d) be a bipolar cone$_{tvs}$ b-metric space, where d induces a topology on $X \times Y$. Suppose that $\{(x_n, y_n)\} \subset (F_1, F_2)$, a closed pair, is a convergent bisequence. If $\{(x_n, y_n)\} \to (x, y)$ as $n \to \infty$ then $(x, y) \in (F_1, F_2)$.*

Proof. Since (F_1, F_2) is a closed pair, there exists an open set $\mathcal{U}$ such that $(F_1, F_2) = \mathcal{U}^c$. If possible let $(x, y) \notin (F_1, F_2)$. As $\mathcal{U}$ is open, so by Proposition 4.6 there exists $n_0 \geq 1$ such that $(x_{k_1}, y_{k_2}) \in \mathcal{U}$ for all $k_1, k_2 \in n_0$, a contradiction. So $(x, y) \in (F_1, F_2)$. □

Proposition 4.8. *For any pair of sets (A, B) in a bipolar cone$_{tvs}$ b-metric space, where d induces a topology on $X \times Y$, we have*
$\{(a, b) \in X \times Y$: there exists a bisequence $\{(a_n, b_n)\}$ in (A, B) such that $\{(a_n, b_n)\}$ converges to $(a, b)\} \subset (\bar{A}, \bar{B})$.

Proof. Let $\{(a_n, b_n)\}$ be a convergent bisequence in (A, B) which converges to (a, b). Since $(\bar{A}, \bar{B})$ is a closed pair containing (A, B), so by Proposition 4.7 we have $(a, b) \in (\bar{A}, \bar{B})$. This completes the proof. □

Proposition 4.9. *Let (X, Y, d) be a complete bipolar cone$_{tvs}$ b-metric space, where d induces a topology on $X \times Y$, and (F_1, F_2) be a closed pair. Then $(F_1, F_2, d_{F_1 \times F_2})$ is also complete.*

Proof. Let $\{(x_n, y_n)\}$ be a Cauchy bisequence in (F_1, F_2). Then clearly it is also a Cauchy bisequence in (X, Y, d). Since (X, Y, d) is complete, the bisequence $\{(x_n, y_n)\}$ is biconvergent to some $u \in X \cap Y$. By proposition 4.7 it follows that $u \in F_1 \cap F_2$. So $\{(x_n, y_n)\}$ also converges in $(F_1, F_2, d_{F_1 \times F_2})$ and therefore $(F_1, F_2, d_{F_1 \times F_2})$ is also complete. □

Example 4.10. In Example 3.2, $B_l(0, (3, 3)) = \{0\}$, $B_r(0, (1, 1)) = \{0\}$, $B_l[0, (3, 3)] = \{-1, 0\}$, $B_r[0, (1, 1)] = \{0, 1\}$, $B_l^{(-1,1)}(1, (\frac{3}{2}, \frac{3}{2})) = \{-1, 0\}$, $B_r^{(-1,1)}(-1, (\frac{3}{2}, \frac{3}{2})) = \{1\}$, $B_l^{(-1,1)}[1, (\frac{3}{2}, \frac{3}{2})] = \{-1, 0\}$, $B_r^{(-1,1)}[-1, (\frac{3}{2}, \frac{3}{2})] = \{0, 1\}$.

Example 4.11. In Example 3.2, $(\{0\}, \{0\})$ is an open pair.

Remark 4.12. Complement of an open set may not be a closed pair. Example 4.11 supports our contention.

4.1 A version of Cantor's intersection theorem in a bipolar b-metric space (i.e. a bipolar cone$_{tvs}$ b-metric space with $E = \mathbb{R}$ together with the usual cone $P = \{x \in E : x \geq 0\}$)

Definition 4.13. Let (X, Y, d) be a bipolar b-metric space and G be any subset of $X \times Y$. Then $\operatorname{diam}(G) = \sup\{d(a, b) : (a, b) \in G\}$.

Definition 4.14. In a bipolar b-metric space (X, Y, d), a sequence $\{(F_n, F_n')\}$ of pairs of subsets is said to be decreasing if $F_1 \supset F_2 \supset F_3 \supset \cdots$ and $F_1' \supset F_2' \supset F_3' \supset \cdots$.

Theorem 4.15. *Let (X, Y, d) be a complete bipolar b-metric space and $\{(F_n, F_n')\}$ be a decreasing sequence of nonempty closed pairs such that $\operatorname{diam}((F_n, F_n')) \to 0$ as $n \to \infty$. Then the intersection $\bigcap_{n=1}^{\infty}(F_n, F_n')$ contains exactly one point.*

Proof. Let $x_n \in F_n$ and $y_n \in F_n'$ be arbitrary for all $n \in \mathbb{N}$. Since $\{(F_n, F_n')\}$ is decreasing, we have $\{x_n, x_{n+1}, \ldots\} \subset F_n$ and $\{y_n, y_{n+1}, \ldots\} \subset F_n'$. Now for any $n, m \in \mathbb{N}$ with $n, m \geq k$ we get $d(x_n, y_m) \leq \operatorname{diam}((F_k, F_k'))$, $k \geq 1$. Let $\epsilon > 0$ be given. Then there exists some $p \in \mathbb{N}$ such that $\operatorname{diam}((F_p, F_p')) < \epsilon$ since $\operatorname{diam}((F_n, F_n')) \to 0$ as $n \to \infty$. From this it follows that $d(x_n, y_m) \leq \operatorname{diam}((F_p, F_p')) < \epsilon$ whenever $n, m \geq p$. So $\{(x_n, y_n)\}$ is a Cauchy bisequence in (X, Y, d), therefore it is biconvergent to some $z \in X \cap Y$. So by Proposition 4.7 it follows that $(z, z) \in (F_n, F_n')$ for all $n \in \mathbb{N}$. Now for the uniqueness of (z, z), let $(a, b) \in \bigcap_{n=1}^{\infty}(F_n, F_n')$ be any other point. Then $(a, z) \in (F_n, F_n')$ for all $n \in \mathbb{N}$ and we have $d(a, z) \leq \operatorname{diam}((F_n, F_n')) \to 0$ as $n \to \infty$ implying that $d(a, z) = 0$ that is $a = z$. Similarly we have $b = z$ and thus $\bigcap_{n=1}^{\infty}(F_n, F_n') = \{(z, z)\}$ and this completes our proof. □

5. Fixed point theorems

In this section we prove some fixed point theorems in the setting of bipolar cone$_{tvs}$ b-metric spaces.

Theorem 5.1. *Let (X, Y, d) be a complete bipolar cone$_{tvs}$ b-metric space and $T : (X, Y, d) \rightrightarrows (X, Y, d)$ be a mapping satisfying $d(Tx, Ty) \preceq \alpha d(x, y)$ for all $(x, y) \in X \times Y$ and for some $\alpha \in (0, \frac{1}{s})$. Then the function $T : X \cup Y \to X \cup Y$ has a unique fixed point.*

Proof. Let $(x_0, y_0) \in X \times Y$. We construct two iterative sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ by $x_n = Tx_{n-1} = T^n x_0$ and $y_n = Ty_{n-1} = T^n y_0$ for all $n \in \mathbb{N}$. Now

$$d(x_n, y_n) = d(Tx_{n-1}, Ty_{n-1}) \preceq \alpha d(x_{n-1}, y_{n-1}) \cdots \preceq \alpha^n d(x_0, y_0) \tag{2}$$

and
$$d(x_{n+1}, y_n) = d(Tx_n, Ty_{n-1}) \preceq \alpha d(x_n, y_{n-1}) \cdots \preceq \alpha^n d(x_1, y_0) \tag{3}$$

for all positive integers $n \in \mathbb{N}$. Now for all $m, n \in \mathbb{N}$ with $m > n$ we have

$$\begin{aligned} d(x_n, y_m) &\preceq s[d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_m)] \\ &\preceq s[\alpha^n d(x_0, y_0) + \alpha^n d(x_1, y_0)] + sd(x_{n+1}, y_m) \qquad \text{[using (2) and (3)]} \\ &= s\alpha^n [d(x_0, y_0) + d(x_1, y_0)] + sd(x_{n+1}, y_m) \end{aligned}$$

$$
\begin{aligned}
&\preceq s\alpha^n[d(x_0,y_0)+d(x_1,y_0)]+s^2[d(x_{n+1},y_{n+1})+d(x_{n+2},y_{n+1})+d(x_{n+2},y_m)]\\
&\preceq(s\alpha^n+s^2\alpha^{n+1})[d(x_0,y_0)+d(x_1,y_0)]+s^2d(x_{n+2},y_m)\preceq\cdots\\
&\preceq(s\alpha^n+s^2\alpha^{n+1}+\cdots+s^{m-n}\alpha^{m-1})[d(x_0,y_0)+d(x_1,y_0)]+s^{m-n}d(x_m,y_m)\\
&\preceq\frac{(s\alpha)^n}{1-s\alpha}M,\quad M=d(x_0,y_0)+d(x_1,y_0) \qquad (4)
\end{aligned}
$$

In a similar fashion for $m<n$ we obtain

$$d(x_n,y_m)\preceq\frac{(s\alpha)^m}{1-s\alpha}M',\quad M'=d(x_0,y_0)+d(x_0,y_1). \qquad (5)$$

Since $0<s\alpha<1$, from (4) we have that, for arbitrary $c\gg\theta$ there exists $N_1\geq 1$ such that $d(x_n,y_m)\preceq\frac{(s\alpha)^n}{1-s\alpha}M\ll c$ for all $m>n\geq N_1$. Similarly from (5) we see that there exists some $N_2\in\mathbb{N}$ such that $d(x_n,y_m)\preceq\frac{(s\alpha)^m}{1-s\alpha}M'\ll c$ whenever $n>m\geq N_2$. Hence $\{(x_n,y_n)\}$ is a Cauchy bisequence in (X,Y,d). By the completeness of (X,Y,d) $\{(x_n,y_n)\}$ biconverges to some $u\in X\cap Y$. Therefore $x_n\to u$ as $n\to\infty$. Since T is continuous then we get $Tx_n=x_{n+1}\to Tu$ as $n\to\infty$ and therefore $Tu=u$. Hence u is a fixed point of T.

Now let $v\in X$ be another fixed point of T. Hence, $Tv=v$ and we have $d(v,u)=d(Tv,Tu)\preceq\alpha d(v,u)$, where $0<\alpha<1$, showing that $u=v$. If $v\in Y$ then we can also we see that $u=v$, implying that T has a unique fixed point in (X,Y,d). □

Remark 5.2. If we take $E=\mathbb{R}$ with the usual cone $P=\{a\in E: a\geq 0\}$ and $s=1$ in Theorem 5.1 then we get [10, Theorem 5.1].

Theorem 5.3. *Let $T:(X,Y,d)\rightleftharpoons(X,Y,d)$, where (X,Y,d) is a complete bipolar cone$_{tvs}$ b-metric space, satisfies*

$$d(Ty,Tx)\preceq ad(x,y)+bd(x,Tx)+cd(Ty,y) \qquad (6)$$

for all $x\in X, y\in Y$, where $0\leq a,b<1$, $0\leq c<\frac{1}{s+1}$ and $0<sa+sb+c<1$. Then the function $T:X\cup Y\to X\cup Y$ has a unique fixed point.

Proof. Let $x_0\in X$ be arbitrary. For any non-negative integer n, we define $y_n=Tx_n$ and $x_{n+1}=Ty_n$. Then we have

$$
\begin{aligned}
d(x_n,y_n)&=d(Ty_{n-1},Tx_n)\preceq ad(x_n,y_{n-1})+b(x_n,Tx_n)+cd(Ty_{n-1},y_{n-1})\\
&=ad(x_n,y_{n-1})+b(x_n,y_n)+cd(x_n,y_{n-1})
\end{aligned}
$$

which implies that $d(x_n,y_n)\preceq\frac{a+c}{1-b}d(x_n,y_{n-1})$ for all $n\in\mathbb{N}$. Again,

$$
\begin{aligned}
d(x_n,y_{n-1})&=d(Ty_{n-1},Tx_{n-1})\preceq ad(x_{n-1},y_{n-1})+bd(x_{n-1},Tx_{n-1})+cd(Ty_{n-1},y_{n-1})\\
&=ad(x_{n-1},y_{n-1})+bd(x_{n-1},y_{n-1})+cd(x_n,y_{n-1})
\end{aligned}
$$

Hence for all $n\geq 1$, $d(x_n,y_{n-1})\preceq\frac{a+b}{1-c}d(x_{n-1},y_{n-1})$. Therefore from the above we get, for all $n\in\mathbb{N}$

$$d(x_n,y_n)\preceq\frac{(a+b)(a+c)}{(1-c)(1-b)}d(x_{n-1},y_{n-1})=\lambda d(x_{n-1},y_{n-1}),\quad\lambda=\frac{(a+b)(a+c)}{(1-c)(1-b)}$$

Thus we have $d(x_n,y_n)\preceq\lambda^n d(x_0,y_0)$ and $d(x_{n+1},y_n)\preceq\lambda^n\frac{a+b}{1-c}d(x_0,y_0)$ for all $n\geq 1$.

So for all natural numbers m, n with $m > n$ we get

$$\begin{aligned}
&d(x_n, y_m) \preceq s[d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_m)] \\
&\preceq s\left[\lambda^n + \lambda^n \frac{a+b}{1-c}\right] d(x_0, y_0) + sd(x_{n+1}, y_m) \\
&\preceq s\lambda^n \left[1 + \frac{a+b}{1-c}\right] d(x_0, y_0) + s^2[d(x_{n+1}, y_{n+1}) + d(x_{n+2}, y_{n+1}) + d(x_{n+2}, y_m)] \\
&\preceq s\lambda^n \left[1 + \frac{a+b}{1-c}\right] d(x_0, y_0) + s^2 \left[\lambda^{n+1} + \lambda^{n+1} \frac{a+b}{1-c}\right] d(x_0, y_0) + s^2 d(x_{n+2}, y_m) \\
&= [s\lambda^n + s^2\lambda^{n+1}] \left(1 + \frac{a+b}{1-c}\right) d(x_0, y_0) + s^2 d(x_{n+2}, y_m) \preceq \cdots \\
&\preceq [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n}\lambda^{m-1}] \left(1 + \frac{a+b}{1-c}\right) d(x_0, y_0) + s^{m-n} d(x_m, y_m) \\
&\preceq \frac{(s\lambda)^n}{1 - s\lambda} \left(1 + \frac{a+b}{1-c}\right) d(x_0, y_0).
\end{aligned}$$

Also, if $m < n$ then

$$\begin{aligned}
&d(x_n, y_m) \preceq s[d(x_{m+1}, y_m) + d(x_{m+1}, y_{m+1}) + d(x_n, y_{m+1})] \\
&\preceq s\left[\lambda^m (\frac{a+b}{1-c}) + \lambda^{m+1}\right] d(x_0, y_0) + sd(x_n, y_{m+1}) \\
&\preceq s\lambda^m \left(\frac{a+b}{1-c} + \lambda\right) d(x_0, y_0) + s^2[d(x_{m+2}, y_{m+1}) + d(x_{m+2}, y_{m+2}) + d(x_n, y_{m+2})] \\
&\preceq \left(\frac{a+b}{1-c} + \lambda\right) (s\lambda^m + s^2\lambda^{m+1} + \cdots + s^{n-m}\lambda^{n-1}) d(x_0, y_0) + s^{n-m} d(x_n, y_n) \\
&\preceq \frac{(s\lambda)^m}{1 - s\lambda} \left(\lambda + \frac{a+b}{1-c}\right) d(x_0, y_0)
\end{aligned}$$

Since $0 < s\lambda < 1$, by routine verification we see that $\{(x_n, y_n)\}$ is a Cauchy bisequence in (X, Y, d). Since (X, Y, d) is complete, so $\{(x_n, y_n)\}$ converges and thus biconverges to some $z \in X \cap Y$. Now,

$$d(Tz, Tx_n) \preceq ad(x_n, z) + bd(x_n, Tx_n) + cd(Tz, z) = ad(x_n, z) + bd(x_n, y_n) + cd(Tz, z)$$

and also

$$\begin{aligned}
d(Tz, z) &\preceq s[d(Tz, y_n) + d(x_n, y_n) + d(x_n, z)] = s[d(Tz, Tx_n) + d(x_n, y_n) + d(x_n, z)] \\
&\preceq s[ad(x_n, z) + bd(x_n, y_n) + cd(Tz, z) + d(x_n, y_n) + d(x_n, z)]
\end{aligned}$$

This implies $(1 - sc)d(Tz, z) \preceq s[(1 + a)d(x_n, z) + (1 + b)d(x_n, y_n)]$, and therefore

$$d(Tz, z) \preceq \frac{s(1+a)}{1-sc} d(x_n, z) + \frac{s(1+b)}{1-sc} d(x_n, y_n) \preceq \frac{s(1+a)}{1-sc} d(x_n, z) + \frac{s(1+b)}{1-sc} \lambda^n d(x_0, y_0)$$

for any $n \in \mathbb{N}$. Since $x_n \to z$ and $\lambda^n \to 0$ as $n \to \infty$, so we get $Tz = z$.

Now if u and v are two fixed points of T then $u, v \in X \cap Y$ and we have $d(u, v) = d(Tu, Tv) \preceq ad(v, u) + bd(v, Tv) + cd(u, Tu) = ad(u, v)$. This shows that $d(u, v) = \theta$ that is $u = v$. Hence T has a unique fixed point in (X, Y, d). □

Corollary 5.4. *Let (X, Y, d) be a complete bipolar cone$_{tvs}$ b-metric space and $T : (X, Y, d) \rightleftharpoons (X, Y, d)$ be a mapping satisfying $d(Ty, Tx) \preceq ad(x, y)$ for all $(x, y) \in X \times Y$ and for some $a \in (0, \frac{1}{s})$. Then the function $T : X \cup Y \to X \cup Y$ has a unique fixed point.*

Proof. If we put $b = c = 0$ in (6) of Theorem 5.3 then we get our required result. □

Remark 5.5. If we put $E = \mathbb{R}$ with the usual cone $P = \{a \in E : a \geq 0\}$ and $s = 1$, [10, Theorem 5.2] will be implied by our Corollary 5.4.

Corollary 5.6. *Let $T : (X, Y, d) \rightleftharpoons (X, Y, d)$, where (X, Y, d) is a complete bipolar cone$_{tvs}$ b-metric space, satisfies $d(Ty, Tx) \preceq b[d(x, Tx) + d(Ty, y)]$ for all $x \in X$ and $y \in Y$, where $0 \leq b < \frac{1}{s+1}$. Then the function $T : X \cup Y \to X \cup Y$ has a unique fixed point.*

Proof. If we put $a = 0$ and $b = c$ in (6) of Theorem 5.3 then the corollary follows immediately. □

Remark 5.7. If we take $E = \mathbb{R}$ with the usual cone $P = \{a \in E : a \geq 0\}$ and $s = 1$ in Corollary 5.6 then we get [10, Theorem 5.6].

Definition 5.8. Let $\mathbb{A}$ be the set consisting of all functions $\alpha : P^3 \to P$, where P is a solid cone in a real topological vector space (E, P), satisfying
(A$_1$) α is continuous on the set P^3 of all triplets of P with respect to the topology of (E, P).

(A$_2$) $a \preceq kb$ for some $k \in [0, 1)$ whenever $a \preceq \alpha(a, b, b)$ or $a \preceq \alpha(b, a, b)$ or $a \preceq \alpha(b, b, a)$, for all $a, b \in P$.
A mapping $T : (X, Y, d) \rightleftharpoons (X, Y, d)$, where (X, Y, d) is a bipolar cone$_{tvs}$ b-metric space, is said to be a contravariant A-contraction if it satisfies $d(Ty, Tx) \preceq \alpha(d(x, y), d(x, Tx), d(Ty, y))$ for all $(x, y) \in X \times Y$ and for some $\alpha \in \mathbb{A}$.

Theorem 5.9. *Let (X, Y, d) be a complete bipolar cone$_{tvs}$ b-metric space and $T : (X, Y, d) \rightleftharpoons (X, Y, d)$ be a continuous contravariant A-contraction mapping. Then the function $T : X \cup Y \to X \cup Y$ has a unique fixed point.*

Proof. Since T is a contravariant A-contraction then there exists some $\alpha \in \mathbb{A}$ such that $d(Ty, Tx) \preceq \alpha(d(x, y), d(x, Tx), d(Ty, y))$ for all $(x, y) \in X \times Y$. Let $x_0 \in X$ and for each $n \in \mathbb{N} \cup \{0\}$ let us define $y_n = Tx_n$ and $x_{n+1} = Ty_n$. Then $\{(x_n, y_n)\}$ is a bisequence on (X, Y, d) and we have

$$\begin{aligned} d(x_n, y_n) &= d(Ty_{n-1}, Tx_n) \preceq \alpha(d(x_n, y_{n-1}), d(x_n, Tx_n), d(Ty_{n-1}, y_{n-1})) \\ &= \alpha(d(x_n, y_{n-1}), d(x_n, y_n), d(x_n, y_{n-1})). \end{aligned}$$

By a property of α we get $d(x_n, y_n) \preceq kd(x_n, y_{n-1})$ for some $0 < k < 1$. Also we see that

$$\begin{aligned} d(x_n, y_{n-1}) &= d(Ty_{n-1}, Tx_{n-1}) \preceq \alpha(d(x_{n-1}, y_{n-1}), d(x_{n-1}, Tx_{n-1}), d(Ty_{n-1}, y_{n-1})) \\ &= \alpha(d(x_{n-1}, y_{n-1}), d(x_{n-1}, y_{n-1}), d(x_n, y_{n-1})), \end{aligned}$$

and hence by a property of α we get $d(x_n, y_{n-1}) \preceq kd(x_{n-1}, y_{n-1})$ for all $n \geq 1$. Thus $d(x_n, y_n) \preceq k^{2n} d(x_0, y_0)$ and $d(x_n, y_{n-1}) \preceq k^{2n-1} d(x_0, y_0)$ for all $n \in \mathbb{N}$.

Proceeding in a similar manner as prescribed in Theorem 5.3, we see that $\{(x_n, y_n)\}$ is a Cauchy bisequence. As (X, Y, d) is complete, $\{(x_n, y_n)\}$ is biconvergent to some $z \in X \cap Y$. Since $x_n \to z$, so by the continuity of T we have $y_n = Tx_n \to Tz$ as $n \to \infty$. Therefore $Tz = z$ and z is a fixed point of T.

Let v be another fixed point of T in (X, Y, d). Then $v \in X \cap Y$ and we get $d(u, v) = d(Tu, Tv) \preceq \alpha(d(v, u), d(v, Tv), d(Tu, u)) = \alpha(d(u, v), \theta, \theta)$. Hence $d(u, v) \preceq k\theta$, and it follows that $u = v$. Therefore T has a unique fixed point in (X, Y, d). □

Example 5.10. Let us consider the complete bipolar cone$_{tvs}$ b-metric space $(U_n(\mathbb{R}), L_n(\mathbb{R}), d)$ (see Example 3.4) and define $T : U_n(\mathbb{R}) \cup L_n(\mathbb{R}) \to U_n(\mathbb{R}) \cup L_n(\mathbb{R})$ by $T((a_{ij})_{n\times n}) = (\frac{a_{ij}}{4})_{n\times n}$ for all $a_{ij} \in U_n(\mathbb{R}) \cup L_n(\mathbb{R})$. Then it can be easily seen that $T : (U_n(\mathbb{R}), L_n(\mathbb{R}), d) \rightrightarrows (U_n(\mathbb{R}), L_n(\mathbb{R}), d)$ satisfies $d(T(A), T(B)) \preceq \frac{1}{8}\, d(A, B)$ for all $(A, B) \in (U_n(\mathbb{R}) \times L_n(\mathbb{R}))$. So it satisfies all the conditions of Theorem 5.1. Here $O_{n\times n}$ is the unique fixed point of T, where $O_{n\times n}$ is the null matrix of order n.

6. Application to a homotopy result

In this section, we obtain a homotopy result as an application of Theorem 5.1. For this purpose, first we recall the definition of homotopy between two functions.

Let X, Y be two topological spaces, and let $G, S : X \to Y$ be two continuous mappings. Then, a homotopy from G to S is a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = Gx$ and $H(x, 1) = Sx$, for all $x \in X$. Also, G and S are called homotopic mappings. For more details one can refer to [16].

Theorem 6.1. *Let (X, Y, d) be a complete bipolar cone$_{tvs}$ b-metric space, where d induces a topology on $X \times Y$ such that for any $x_0 \in X \cap Y$ and $c \gg \theta$, the closure of $(B_l(x_0, c), B_r(x_0, c))$ is $(B_l[x_0, c'], B_r[x_0, c'])$ for some $c' \succeq c$ in $X \times Y$. Also let (U_1, U_2) be an open pair and (V_1, V_2) be a closed pair such that $(U_1, U_2) \subset (V_1, V_2)$. Suppose $J : (V_1 \cup V_2) \times [0, 1] \to X \cup Y$ satisfies the following conditions:*
(i) $x \neq J(x, t)$ for all $x \in (V_1 \setminus U_1) \cup (V_2 \setminus U_2)$ and for any $t \in [0, 1]$;

(ii) $J(., t) : (V_1, V_2) \rightrightarrows (X, Y)$ for every $t \in [0, 1]$ such that $d(J(x, t), J(y, t)) \preceq \alpha d(x, y)$ for all $(x, y) \in X \times Y$ and for some $\alpha \in (0, \frac{1}{s})$;

(iii) there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that $d(J(x, t_1), J(x, t_2)) \preceq |f(t_1) - f(t_2)|M$ for some fixed $M \succeq \theta$ and for all $x \in V_1 \cap V_2$, for every $t_1, t_2 \in [0, 1]$. Then $J(., 0)$ has a fixed point if and only if $J(., 1)$ has a fixed point.

Proof. Consider the sets $G_1 = \{t \in [0, 1] : J(x, t) = x$ for some $x \in U_1\}$ and $G_2 = \{t \in [0, 1] : J(y, t) = y$ for some $y \in U_2\}$. Let us denote $G_1 \cap G_2$ by G.

First let us suppose that $J(., 0)$ has a fixed point ξ in $V_1 \cup V_2$, so by assumed condition (1) and (ii) $\xi \in U_1 \cap U_2$. Thus $0 \in G$ and thus $G \neq \emptyset$. We now show that G is a clopen subset of $[0, 1]$ and therefore by the connectedness of $[0, 1]$ we have $G = [0, 1]$, which implies $G_1 = G_2 = [0, 1]$.

Now we show that G is closed. Let a sequence $\{t_n\} \subset G$ converges to t. Since $t_n \in G_1 \cap G_2$ so there exists $x_n \in U_1$ and $y_n \in U_2$ such that $J(x_n, t_n) = x_n$ and $J(y_n, t_n) = y_n$ for all $n \in \mathbb{N}$. Clearly $x_n = y_n \in U_1 \cap U_2$ by assumed condition (ii). Now

$$\begin{aligned}
&d(x_n, x_m) = d(J(x_n, t_n), J(x_m, t_m)) \\
\preceq& s[d(J(x_n, t_n), J(x_n, t_m)) + d(J(x_n, t_m), J(x_n, t_m)) \\
&+ d(J(x_n, t_m), J(x_m, t_m))]s[d(J(x_n, t_n), J(x_n, t_m)) + d(J(x_n, t_m), J(x_m, t_m))] \\
\preceq& s[|f(t_n) - f(t_m)|M + \alpha d(x_n, x_m)],
\end{aligned}$$

implying that $d(x_n, x_m) \preceq \frac{s}{1-s\alpha}|f(t_n) - f(t_m)|M$ for all $n, m \geq 1$. Since $\{t_n\}$ is Cauchy and f is continuous on $[0, 1]$, we have $\{(x_n, x_n)\}$ is a Cauchy bisequence in (V_1, V_2, d). Hence it biconverges to some $x \in V_1 \cap V_2$. We now have

$$\begin{aligned}
&d(J(x, t), x_n) = d(J(x, t), J(x_n, t_n)) \\
\preceq& s[d(J(x, t), J(x, t_n)) + d(J(x, t_n), J(x_n, t_n))] \preceq s[|f(t) - f(t_n)|M + \alpha d(x, x_n)].
\end{aligned}$$

Since $x_n \to x$, $t_n \to t$ and f is continuous on $[0, 1]$, it follows that $x_n \to J(x, t)$. Since $\{(x_n, x_n)\}$ biconverges, so $J(x, t) = x$. Hence $t \in G$ and G is closed.

Next we show that G is open. For this let $t_0 \in G$. Then there exits $x_0 \in U_1$ and $y_0 \in U_2$ such that $J(x_0, t_0) = x_0$ and $J(y_0, t_0) = y_0$. So by assumption (ii) we have $x_0 = y_0 \in U_1 \cap U_2$. Since (U_1, U_2) is an open pair so there exists some $c \gg \theta$ such that $B_l(x_0, c) \subset U_1$ and $B_r(x_0, c) \subset U_2$. So there exists $c' \succeq c$ such that $\left(\overline{B_l(x_0, c)}, \overline{B_r(x_0, c)}\right)$ is $(B_l[x_0, c'], B_r[x_0, c'])$. Let us choose some $\epsilon > 0$ such that $\epsilon M \preceq (\frac{1}{s} - \alpha)c'$. Since f is continuous on $[0, 1]$, so for $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $|f(t) - f(t_0)| < \epsilon$ whenever $|t - t_0| < \delta(\epsilon), t \in [0, 1]$. Now let $t \in (t_0 - \delta(\epsilon), t_0 + \delta(\epsilon)) \subset [0, 1]$ and $(x, y) \in \left(\overline{B_l(x_0, c)}, \overline{B_r(x_0, c)}\right)$. Then

$$\begin{aligned}
&d(J(x, t), x_0) = d(J(x, t), J(x_0, t_0)) \preceq s[d(J(x, t), J(x_0, t)) + d(J(x_0, t), J(x_0, t_0))] \\
\preceq& s[\alpha d(x, x_0) + |f(t) - f(t_0)|M] \preceq s[\alpha d(x, x_0) + \epsilon M] \preceq s\left[\alpha c' + \left(\frac{1}{s} - \alpha\right)c'\right] = c'
\end{aligned}$$

and also

$$\begin{aligned}
&d(x_0, J(y, t)) = d(J(x_0, t_0), J(y, t)) \preceq s[d(J(x_0, t_0), J(x_0, t)) + d(J(x_0, t), J(y, t))] \\
\preceq& s[|f(t_0) - f(t)|M + \alpha d(x_0, y)] \preceq s\left[\left(\frac{1}{s} - \alpha\right)c' + \alpha c'\right] = c'
\end{aligned}$$

Therefore $J(., t) : \left(\overline{B_l(x_0, c)}, \overline{B_r(x_0, c)}\right) \rightrightarrows \left(\overline{B_l(x_0, c)}, \overline{B_r(x_0, c)}\right)$. Since (X, Y, d) is complete so is $\left(\overline{B_l(x_0, c)}, \overline{B_r(x_0, c)}\right)$ (see Proposition 4.9) and thus $J(., t)$ has a fixed point in $\overline{B_l(x_0, c)} \cap \overline{B_r(x_0, c)} \subset V_1 \cap V_2$. So by the assumed condition (i) this fixed point must lie in $U_1 \cap U_2$ and we have $t \in G$. Therefore $(t_0 - \delta(\epsilon), t_0 + \delta(\epsilon)) \subset G$ and it follows that G is open.

Therefore $J(., 1)$ has a fixed point in $X \cup Y$. The reverse part can also be proved in a similar way. $\square$

Remark 6.2. Theorem 6.1 holds good in a bipolar metric space, i.e. a bipolar cone$_{tvs}$ b-metric space with $s = 1$ and $E = \mathbb{R}$ with the usual cone $P = \{x \in E : x \geq 0\}$.

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