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SOME CHEBYSHEV TYPE INEQUALITIES INVOLVING THE HADAMARD PRODUCT OF HILBERT SPACE OPERATORS

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Abstract. In this paper, we prove that if $\mathcal{A}$ is a Banach $*$-subalgebra of $B(H)$, T is a compact Hausdorff space equipped with a Radon measure μ and $\alpha : T \to [0,\infty)$ is a integrable function and $(A_t), (B_t)$ are appropriate integrable fields of operators in $\mathcal{A}$ having the almost synchronous property for the Hadamard product, then

$$\int_T \alpha(s)d\mu(s)\int_T \alpha(t)\big(A_t \circ B_t\big)d\mu(t) \geq \int_T \alpha(t)A_t d\mu(t) \circ \int_T \alpha(t)B_t d\mu(t).$$

We also introduce a semi-inner product for square integrable fields of operators in a Hilbert space and using it, we prove the Schwarz and Chebyshev type inequalities dealing with the Hadamard product and the trace of operators.

1. Introduction

Let H be a complex Hilbert space with an orthonormal basis $\{e_j\}$ and $B(H)$ denote the C^*-algebra of all bounded linear operators on H. The Hadamard product $A \circ B$ of two operators A and B in $B(H)$ is defined by $\langle A \circ Be_i, e_j\rangle = \langle Ae_i, e_j\rangle\langle Be_i, e_j\rangle$. Clearly, $A \circ B = B \circ A$. It is known that the Hadamard product can be presented by filtering the tensor product $A \otimes B$ through a positive linear map. In fact,

$$A \circ B = U^*(A \otimes B)U, \tag{1}$$

where $U : H \to H \otimes H$ is the isometry defined by $Ue_j = e_j \otimes e_j$; see [9]. It follows from (1) that $\|A \circ B\| \leq \|A\|\|B\|$. If $A \geq 0, B \geq 0$, then $A \circ B \geq 0$, because there are two operators $C, D \in B(H)$ such that $A = C^*C$ and $B = D^*D$ so

$$A \circ B = U^*(C^*C \otimes D^*D)U = U^*(C \otimes D)^*(C \otimes D)U \geq 0.$$

For matrices, one easily observes [18] that the Hadamard product of $A = (a_{ij})$ and $B = (b_{ij})$ is $A \circ B = (a_{ij}b_{ij})$, a principal submatrix of the tensor product $A \otimes B =$

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$(a_{ij}B)_{1\le i,j\le n}$. From now on when we deal with the Hadamard product of operators, we explicitly assume that an orthonormal basis is fixed.

Let us consider real sequences $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$ and a non-negative sequence $w=(w_1,\dots,w_n)$. Then the weighed Chebyshev function is defined by

$$T(w;a,b)=\sum_{j=1}^{n}w_j\sum_{j=1}^{n}w_ja_jb_j-\sum_{j=1}^{n}w_ja_j\sum_{j=1}^{n}w_jb_j. \tag{2}$$

In 1882, Chebyshev [4] proved that if a and b are monotone in the same sense, then $T(w;a,b)\ge 0$.

The integral version of (2) states the following: For two Lebesgue integrable functions $f,g:[a,b]\to\mathbb{R}$, consider the following Chebyshev functional

$$T(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{b-a}\int_a^b f(t)dt\frac{1}{b-a}\int_a^b g(t)dt.$$

In 1934, G. Grüss [11] showed that

$$|T(f,g)|\le\frac{1}{4}(M-m)(N-n), \tag{3}$$

provided m,M,n,N are real numbers with the property $-\infty<m\le f\le M<\infty$ and $-\infty<n\le g\le N<\infty$, a.e. on $[a,b]$. The constant $\frac{1}{4}$ in (3) is best possible in the sense that it cannot be replaced by a smaller quantity and is achieved for $f(x)=g(x)=\operatorname{sgn}\big(x-\frac{a+b}{2}\big)$.

In recent years several extensions and generalizations have been considered for Chebyshev inequality. For some fundamental results and more information, see [6] and the references therein. This inequality is a complement of the Grüss inequality (see [16]). Some integral inequalities of Chebyshev type were given by Barza, Persson and Soria [3]. In [1], generalizations of Chebyshev type inequalities for continuous functions of self adjoint linear operators in Hilbert spaces are proved.

A number of papers have been written on this inequality providing some inequalities analogous to Chebyshev's inequality given in (3) involving the Hadamard product of linear operators, see [7, 15]. Moslehian and Bakherad in [15] considered the Hadamard product of continuous fields of operators in $\mathcal{C}(T,\mathcal{A})$ with the norm $\|(A_t)\|=\sup_{t\in T}\|A_t\|$, where $\mathcal{A}$ is a C^*-algebra of operators acting on a Hilbert space. They proved that if (A_t) and (B_t) are fields in $\mathcal{C}(T,\mathcal{A})$ with the synchronous Hadamard property, then

$$\int_T\alpha(s)d\mu(s)\int_T\alpha(t)\big(A_t\circ B_t\big)d\mu(t)\ge\int_T\alpha(t)A_td\mu(t)\circ\int_T\alpha(t)B_td\mu(t). \tag{4}$$

Motivated by the above results, in this paper we prove that inequality (4) holds for some integrable field of operators (not necessarily continuous) on a Hilbert space. We also introduce a semi-inner product involving the Hadamard product of linear operators in a suitable field of Hilbert space operators. Using it, we provide some new operator extensions of the Schwarz and Chebyshev type inequalities dealing with the Hadamard product of operators.

Before we state our main results we remind that the Bochner integrals are generalized Lebesgue integrals for functions whose values lie in a Banach space or, more generally, in a topological vector space. It is a straightforward abstraction of the Lebesgue integral. Most of the basic properties of Bochner integration are forced on it by the classical Lebesgue integration (see for example [5,13]). For Bochner integrals in C^*-algebras (see [2,12]). Hereafter, all integrals considered are Bochner integrals.

2. Main results

Let H be a complex Hilbert space with an orthonormal basis $\{e_j\}$, A be a linear operator on H, A^* be the adjoint operator of A and $|A| = (A^*A)^{\frac{1}{2}}$. Recall that the operator norm, trace class norm and Hilbert-Schmidt norm of A, are respectively:

$$\|A\| = \sup\{\|A(x)\| : \|x\| \leq 1\}, \ \|A\|_1 = \mathrm{tr}|A| = \sum_j \langle |A|e_j, e_j\rangle, \ \|A\|_2 = \left(\sum_j \|A(e_j)\|^2\right)^{\frac{1}{2}}.$$

An operator A is bounded if $\|A\| < \infty$. The set of all bounded linear operators on H is denoted by $B(H)$. If $\|A\|_1 < \infty$, we call A a trace-class operator and we denote the set of trace-class operators on H by $L^1(H)$. If $\|A\|_2 < \infty$, we call A a Hilbert-Schmidt operator and we denote the set of these operators on H by $L^2(H)$. Also the set of finite-rank operators on H is denoted by $F(H)$ and the set of compact operators on H is denoted by $K(H)$.

It is easy to check that $\|\cdot\| \leq \|\cdot\|_2 \leq \|\cdot\|_1$ and $F(H) \subseteq L^1(H) \subseteq L^2(H) \subseteq K(H) \subseteq B(H)$, and for $i = 1, 2$, $L^i(H)$ is an ideal in $K(H)$ and $F(H)$ is dense in $L^i(H)$ in the norm $\|.\cdot\|_i$, see [17].

Let $\mathcal{A}$ be a Banach $*$-subalgebra of $B(H)$, T a compact Hausdorff space equipped with a Radon measure μ and $\alpha : T \to [0, \infty)$ be an integrable function. If $1 \leq p < \infty$ is a real number and if the function $t \mapsto A_t$ is strongly measurable, define

$$|||(A_t)|||_p := \left(\int_T \alpha(t)\|A_t\|^p \, d\mu(t)\right)^{\frac{1}{p}}$$

and let $\mathcal{L}^p_\alpha(T, \mathcal{A})$ consist of all fields (A_t) of operators in $\mathcal{A}$ for which $|||(A_t)|||_p < \infty$. A field $(A_t)_{t\in T}$ of operators in $\mathcal{A}$ is called essentially bounded measurable field if the function $t \mapsto A_t$ is strongly measurable and $|||(A_t)|||_\infty := \operatorname{ess\,sup} \|A_t\| < \infty$.

We denote by $\mathcal{L}^\infty(T, \mathcal{A})$ the set of all essentially bounded measurable fields of operators in $\mathcal{A}$. It is trivial that $\mathcal{C}(T, \mathcal{A}) \subset \mathcal{L}^p_\alpha(T, \mathcal{A})$. If $\mathcal{A}$ is an ideal in $B(H)$ and its norm is unitarily invariant, then the norm on $\mathcal{L}^p_\alpha(T, \mathcal{A})$ is unitarily invariant:

$$|||U(A_t)V|||_p = \left(\int_T \alpha(t)\|UA_tV\|^p \, d\mu(t)\right)^{\frac{1}{p}} = \left(\int_T \alpha(t)\|A_t\|^p \, d\mu(t)\right)^{\frac{1}{p}} = |||(A_t)|||_p.$$

Definition 2.1. (i) Let T be a compact Hausdorff space equipped with a Radon measure μ. Two fields (A_t) and (B_t) of operators in $B(H)$ are said to have the almost synchronous property for Hadamard product if $(A_t - A_s) \circ (B_t - B_s) \geq 0$ almost everywhere with respect to $\mu \times \mu$.

(ii) Two fields (A_t) and (B_t) of operators in $B(H)$ are said to have the weakly synchronous property for Hadamard product if $\Lambda\big((A_t - A_s) \circ (B_t - B_s)\big) \geq 0$ almost everywhere with respect to $\mu \times \mu$, for all positive linear functionals Λ on $B(H)$.

THEOREM 2.2. *Let p and q be conjugate exponents, $1 \leq p \leq \infty$. Let $\mathcal{A}$ be a Banach $*$-algebra and T be a compact Hausdorff space equipped with a Radon measure μ. Let $(A_t)_{t\in T}$ be a field in $\mathcal{L}^p_\alpha(T,\mathcal{A})$ and $(B_t)_{t\in T}$ be a field in $\mathcal{L}^q_\alpha(T,\mathcal{A})$ with the almost synchronous Hadamard property and let $\alpha : T \to [0,+\infty)$ be an integrable function. Then*

$$\int_T \alpha(s)\,d\mu(s) \int_T \alpha(t)\big(A_t \circ B_t\big)\,d\mu(t) \geq \left(\int_T \alpha(t)A_t\,d\mu(t)\right) \circ \left(\int_T \alpha(s)B_s\,d\mu(s)\right). \quad (5)$$

Proof. Let $1 < p < \infty$ and $(A_t) \in \mathcal{L}^p_\alpha(T,\mathcal{A})$; we must state that integrands (5) are Bochner integrable on Ω. These functions are strongly measurable and from inequality $\|A \circ B\| \leq \|A\|\|B\|$, we have

$$\int_T \alpha(t)\|A_t \circ B_t\|\,d\mu(t) \leq \int_T \alpha(t)\|A_t\|\ \|B_t\|\,d\mu(t)$$
$$\leq \left(\int_T \alpha(t)\|A_t\|^p\,d\mu(t)\right)^{\frac{1}{p}} \left(\int_T \alpha(t)\|B_t\|^q\,d\mu(t)\right)^{\frac{1}{q}} = |||(A_t)|||_p|||(B_t)|||_q < \infty,$$

and
$$\left\|\int_T \alpha(t)A_t\,d\mu(t) \circ \int_T \alpha(t)B_t\,d\mu(t)\right\| \leq \left\|\int_T \alpha(t)A_t\,d\mu(t)\right\| \left\|\int_T \alpha(t)B_t\,d\mu(t)\right\|$$
$$\leq \int_T \alpha(t)\|A_t\|\,d\mu(t) \int_T \alpha(t)\|B_t\|\,d\mu(t)$$
$$\leq \left(\int_T \alpha(s)\,d\mu(s)\right)^{\frac{1}{q}}\left(\int_T \alpha(t)\|A_t\|^p\,d\mu(t)\right)^{\frac{1}{p}}\left(\int_T \alpha(s)\,d\mu(s)\right)^{\frac{1}{p}}\left(\int_T \alpha(t)\|B_t\|^q\,d\mu(t)\right)^{\frac{1}{q}}$$
$$\leq \left(\int_T \alpha(s)\,d\mu(s)\right) |||(A_t)|||_p|||(B_t)|||_q < \infty.$$

For $p = 1$, we have

$$\int_T \alpha(t)\|A_t \circ B_t\|d\mu(t) \leq \int_T \alpha(t)\|A_t\|\ \|B_t\|d\mu(t)$$
$$\leq \left(\int_T \alpha(t)\|A_t\|d\mu(t)\right)(\operatorname{ess\,sup}\|B_t\|) = |||(A_t)|||_1|||(B_t)|||_\infty < \infty, \text{ and}$$
$$\left\|\int_T \alpha(t)A_t d\mu(t) \circ \int_T \alpha(t)B_t d\mu(t)\right\| \leq \left\|\int_T \alpha(t)A_t d\mu(t)\right\| \left\|\int_T \alpha(t)B_t d\mu(t)\right\|$$
$$\leq \int_T \alpha(t)\|A_t\|d\mu(t) \int_T \alpha(t)\|B_t\|d\mu(t) \leq \int_T \alpha(t)\|A_t\|d\mu(t) \int_T \alpha(t)|||B_t|||_\infty d\mu(t)$$
$$\leq \left(\int_T \alpha(s)d\mu(s)\right) |||(A_t)|||_1|||(B_t)|||_\infty < \infty.$$

Similarly, for $p = \infty$ we get

$$\int_T \alpha(t)\|A_t \circ B_t\|d\mu(t) \leq |||(A_t)|||_\infty|||(B_t)|||_1 < \infty, \text{ and}$$

$$\left\| \int_T \alpha(t) A_t d\mu(t) \circ \int_T \alpha(t) B_t d\mu(t) \right\| \leq \left(\int_T \alpha(s) d\mu(s) \right) |||(A_t)|||_\infty |||(B_t)|||_1 < \infty.$$

Now, we show that the inequality (5) holds.

$$\begin{aligned}
&\int_T \alpha(s)\, d\mu(s) \int_T \alpha(t)\big(A_t \circ B_t\big)\, d\mu(t) - \left(\int_T \alpha(t) A_t\, d\mu(t) \right) \circ \left(\int_T \alpha(s) B_s\, d\mu(s) \right) \\
&= \int_T \int_T \alpha(s)\alpha(t)\big(A_t \circ B_t\big)\, d\mu(t)\, d\mu(s) - \int_T \int_T \alpha(t)\alpha(s)\big(A_t \circ B_s\big)\, d\mu(t)\, d\mu(s) \\
&= \frac{1}{2} \int_T \int_T \alpha(s)\alpha(t)\big[(A_t - A_s) \circ (B_t - B_s)\big]\, d\mu(t)\, d\mu(s) \geq 0. \qquad \square
\end{aligned}$$

Corollary 2.3. *Let T be a compact Hausdorff space equipped with a Radon measure μ, and $\alpha : T \to [0, \infty)$ be an integrable function. If (A_t) and (B_t) are fields in $\mathcal{L}^2_\alpha(T, L^2(H))$ with the weakly synchronous property for Hadamard product, then*

$$\operatorname{tr}\left(\int_T \alpha(t) A_t\, d\mu(t) \circ \int_T \alpha(t) B_t\, d\mu(t) \right) \leq \int_T \alpha(s)\, d\mu(s) \int_T \alpha(t) \operatorname{tr}(A_t \circ B_t)\, d\mu(t)$$

Proof. It is easy to show that

$$\begin{aligned}
&\int_T \alpha(s)\, d\mu(s) \int_T \alpha(t) \operatorname{tr}(A_t \circ B_t)\, d\mu(t) - \operatorname{tr}\left(\int_T \alpha(t) A_t\, d\mu(t) \circ \int_T \alpha(t) B_t\, d\mu(t) \right) \\
&= \operatorname{tr}\left(\int_T \alpha(s)\, d\mu(s) \int_T \alpha(t)(A_t \circ B_t)\, d\mu(t) - \int_T \alpha(t) A_t\, d\mu(t) \circ \int_T \alpha(t) B_t\, d\mu(t) \right) \\
&= \operatorname{tr}\left(\frac{1}{2} \int_T \int_T \alpha(t)\alpha(s)(A_t - A_s) \circ (B_t - B_s)\, d\mu(t)\, d\mu(s) \right) \\
&= \frac{1}{2} \int_T \int_T \alpha(t)\alpha(s) \operatorname{tr}\big((A_t - A_s) \circ (B_t - B_s)\big)\, d\mu(t)\, d\mu(s) \geq 0,
\end{aligned}$$

since $\operatorname{tr}\big((A_t - A_s) \circ (B_t - B_s)\big) \geq 0$, a.e. $[\mu \times \mu]$. $\square$

In the following, we introduce a semi-inner product on $\mathcal{L}^2_\alpha(T, \mathcal{A})$, where $\mathcal{A}$ is the Banach $*$-algebra $L^2(H)$. First we state a simple proposition as follows:

Proposition 2.4. *For all $A, B \in L^2(H)$ the following Schwarz inequality holds:*

$$|\operatorname{tr}(A \circ B^*)|^2 \leq \operatorname{tr}(A \circ A^*) \operatorname{tr}(B \circ B^*) = \|1_H \circ A\|_2^2 \|1_H \circ B\|_2^2 \leq \|A\|_2^2 \|B\|_2^2. \tag{6}$$

Proof. It is obvious that if A is a bounded linear operator on a Hilbert space H and A^* is the adjoint operator of A, then

$$\operatorname{tr}(A \circ A^*) = \operatorname{tr}(1_H \circ A \circ A^*) = \operatorname{tr}((1_H \circ A)(1_H \circ A)^*) = \|1_H \circ A\|_2^2 \geq 0.$$

This shows that $\langle A, B \rangle := \operatorname{tr}(A \circ B^*)$ is a semi-inner product on H. Using the Schwarz's inequality, we obtain the desired inequality (6). $\square$

We know that $A \circ A^*$ may not be positive, but it is always $\operatorname{tr}(A \circ A^*) \geq 0$. Therefore to define a semi-inner product and to obtain Schwarz and Chebyshev type inequalities, we can consider that $\mathcal{L}^2_\alpha(T, L^2(H))$ consists of all square integrable field of operators such that

$$|||(A_t)||| := \left(\int_T \alpha(t) \|A_t\|_2^2\, d\mu(t) \right)^{\frac{1}{2}} < \infty.$$

The following result concerning a semi-inner product on $\mathcal{L}^2_\alpha(T, L^2(H))$ may be stated.

THEOREM 2.5. *(i) The map* $\langle\cdot,\cdot\rangle : \mathcal{L}^2_\alpha(T, L^2(H)) \times \mathcal{L}^2_\alpha(T, L^2(H)) \to \mathbb{C}$,

$$\langle (A_t),(B_t)\rangle := \int_T \alpha(s)\,d\mu(s) \int_T \alpha(t)\mathrm{tr}\big(A_t \circ B_t^*\big)\,d\mu(t) - \mathrm{tr}\left(\int_T \alpha(t)A_t\,d\mu(t) \circ \int_T \alpha(t)B_t^*\,d\mu(t)\right), \tag{7}$$

is a semi-inner product on $\mathcal{L}^2_\alpha(T, L^2(H))$ *and the following inequality holds*

$$\left|\int_T \alpha(s)\,d\mu(s) \int_T \alpha(t)\mathrm{tr}\big(A_t \circ B_t\big)\,d\mu(t) - \mathrm{tr}\left(\int_T \alpha(t)A_t\,d\mu(t) \circ \int_T \alpha(t)B_t\,d\mu(t)\right)\right|$$
$$\le \int_T \alpha(s)\,d\mu(s)\ |||(A_t)|||\ |||(B_t)||| - \left\|1_H \circ \int_T \alpha(t)A_t\,d\mu(t) \circ \int_T \alpha(t)B_t\,d\mu(t)\right\|_2. \tag{8}$$

(ii) If $(A_t),(B_t) \in \mathcal{L}^2_\alpha(T, L^2(H))$ *and* $A, B \in L^2(H)$, *then*

$$\int_T \alpha(s)\,d\mu(s) \int_T \alpha(t)(A_t - A)\circ(B_t - B)^*\,d\mu(t) - \int_T \alpha(t)\big(A_t - A\big)\,d\mu(t) \circ \int_T \alpha(t)(B_t - B)^*\,d\mu(t)$$
$$= \int_T \alpha(s)\,d\mu(s) \int_T \alpha(t)A_t \circ B_t^*\,d\mu(t) - \int_T \alpha(t)A_t\,d\mu(t) \circ \int_T \alpha(t)B_t^*\,d\mu(t). \tag{9}$$

Proof. (i) Let $(A_t),(B_t) \in \mathcal{L}^2_\alpha(T, L^2(H))$; we must state that integrands (7) are Bochner integrable on Ω. These functions are strongly measurable and from the inequality (6), we have

$$\int_T \alpha(t)\mathrm{tr}\big(A_t \circ B_t^*\big)\,d\mu(t) \le \int_T \alpha(t)\|A_t\|_2\|B_t\|_2\,d\mu(t)$$
$$\le \left(\int_T \alpha(t)\|A_t\|_2^2\,d\mu(t)\right)^{\frac12} \left(\int_T \alpha(t)\|B_t\|_2^2\,d\mu(t)\right)^{\frac12} = |||(A_t)|||\ |||(B_t)||| < \infty.$$

We also have

$$\mathrm{tr}\left(\int_T \alpha(t)A_t\,d\mu(t) \circ \int_T \alpha(t)B_t^*\,d\mu(t)\right) \le \left\|\int_T \alpha(t)A_t\,d\mu(t)\right\|_2 \left\|\int_T \alpha(t)B_t^*\,d\mu(t)\right\|_2$$
$$\le \int_T \alpha(t)\|A_t\|_2\,d\mu(t) \int_T \alpha(t)\|B_t^*\|_2\,d\mu(t) \le \left(\int_T \alpha(s)\,d\mu(s)\right)^{\frac12} \left(\int_T \alpha(t)\|A_t\|_2^2\,d\mu(t)\right)^{\frac12}$$
$$\times \left(\int_T \alpha(s)\,d\mu(s)\right)^{\frac12} \left(\int_T \alpha(t)\|B_t^*\|_2^2\,d\mu(t)\right)^{\frac12} \le \int_T \alpha(s)\,d\mu(s)\ |||(A_t)|||\ |||(B_t)||| < \infty.$$

It is easy to show that $\langle\cdot,\cdot\rangle$ is a semi-inner product on $\mathcal{L}^2_\alpha(T, L^2(H))$. Therefore the following Schwarz inequality holds:

$$|\langle (A_t),(B_t)\rangle|^2 \le \langle (B_t),(B_t)\rangle\ \langle (A_t),(A_t)\rangle. \tag{10}$$

On the other hand, we have

$$\begin{aligned}\langle (A_t),(A_t)\rangle =&\int_T \alpha(s)\,d\mu(s)\int_T \alpha(t)\mathrm{tr}\big(A_t\circ A_t^*\big)\,d\mu(t) - \mathrm{tr}\left(\int_T \alpha(t)A_t\,d\mu(t)\circ\int_T \alpha(t)A_t^*\,d\mu(t)\right)\\ =&\int_T \alpha(s)\,d\mu(s)\int_T \alpha(t)\|1_H\circ A_t\|_2^2\,d\mu(t) - \left\|1_H\circ\int_T \alpha(t)A_t\,d\mu(t)\right\|_2^2\\ \le&\int_T \alpha(s)\,d\mu(s)|||(A_t)|||^2 - \left\|1_H\circ\int_T \alpha(t)A_t\,d\mu(t)\right\|_2^2, \qquad (11)\end{aligned}$$

and similarly

$$\langle (B_t),(B_t)\rangle \le \int_T \alpha(s)\,d\mu(s)\;|||(B_t)|||^2 - \left\|1_H\circ\int_T \alpha(t)B_t\,d\mu(t)\right\|_2^2. \qquad (12)$$

Using inequalities (11), (12), the Schwarz's inequality (10) and the elementary inequality for real numbers $(m^2-n^2)(p^2-q^2)\le(mp-nq)^2$, we obtain the desired inequality in (8).

(ii) Now, if $(A_t),(B_t)\in\mathcal{L}^2_\alpha(T,L^2(H))$ and $A,B\in L^2(H)$, then we have

$$\begin{aligned}&\int_T \alpha(s)\,d\mu(s)\int_T \alpha(t)(A_t-A)\circ(B_t-B)^*\,d\mu(t)\\ &-\int_T \alpha(t)(A_t-A)\,d\mu(t)\circ\int_T \alpha(t)(B_t-B)^*\,d\mu(t)\\ =&\int_T \alpha(s)\,d\mu(s)\int_T \alpha(t)\Big(A_t\circ B_t^*-A_t\circ B^*-A\circ B_t^*+A\circ B^*\Big)\,d\mu(t)\\ &-\left(\int_T \alpha(t)A_t\,d\mu(t)-A\int_T \alpha(t)\,d\mu(t)\right)\circ\left(\int_T \alpha(t)B_t^*\,d\mu(t)-B^*\int_T \alpha(t)\,d\mu(t)\right)\\ =&\int_T \alpha(s)\,d\mu(s)\int_T \alpha(t)A_t\circ B_t^*\,d\mu(t)-\int_T \alpha(t)A_t\,d\mu(t)\circ\int_T \alpha(t)B_t^*\,d\mu(t). \qquad \square\end{aligned}$$

THEOREM 2.6. *Let $(A_t),(B_t)\in\mathcal{L}^2_\alpha(T,L^2(H))$ and there exist some vectors $A,A',B,B'\in L^2(H)$ such that*

$$\begin{aligned}\left|\left|\left|(A_t)-\frac{A'+A}{2}\right|\right|\right| &= \left|\left|\left|\left(A_t-\frac{A'+A}{2}\right)\right|\right|\right| \le \frac{1}{2}|||A'-A|||,\\ \left|\left|\left|(B_t)-\frac{B'+B}{2}\right|\right|\right| &= \left|\left|\left|\left(B_t-\frac{B'+B}{2}\right)\right|\right|\right| \le \frac{1}{2}|||B'-B|||.\end{aligned} \qquad (13)$$

Then the following inequality holds:

$$\begin{aligned}&\left|\int_T \alpha(s)\,d\mu(s)\int_T \alpha(t)\mathrm{tr}\big(A_t\circ B_t\big)\,d\mu(t)-\mathrm{tr}\left(\int_T \alpha(t)A_t\,d\mu(t)\circ\int_T \alpha(t)B_t\,d\mu(t)\right)\right|\\ \le&\frac{1}{4}\int_T \alpha(s)\,d\mu(s)\;|||A'-A|||\;|||B'-B|||\\ &-\left\|\int_T \alpha(t)\;1_H\circ\left(A_t-\frac{A'+A}{2}\right)\circ\left(B_t-\frac{B'+B}{2}\right)\,d\mu(t)\right\|_2. \qquad (14)\end{aligned}$$

The coefficient $\frac{1}{4}$ in the inequality (14) is sharp in the sense that it cannot be replaced by a smaller quantity.

Proof. Using (9) with $A_t = B_t$ and $\frac{A'+A}{2}$ instead of A and B, we get

$$\begin{aligned}
\langle (A_t), (A_t)\rangle = &\int_T \alpha(s)\, d\mu(s) \int_T \alpha(t)\mathrm{tr}\big(A_t \circ A_t^*\big)\, d\mu(t) \\
&- \mathrm{tr}\left(\int_T \alpha(t) A_t\, d\mu(t) \circ \int_T \alpha(t) A_t^*\, d\mu(t)\right) \\
= &\int_T \alpha(s)\, d\mu(s) \int_T \alpha(t)\mathrm{tr}\left(\left(A_t - \frac{A'+A}{2}\right) \circ \left(A_t - \frac{A'+A}{2}\right)^*\right) d\mu(t) \\
&- \mathrm{tr}\left(\int_T \alpha(t)\left(A_t - \frac{A'+A}{2}\right) d\mu(t) \circ \int_T \alpha(t)\left(A_t - \frac{A'+A}{2}\right)^* d\mu(t)\right) \\
= &\int_T \alpha(s)\, d\mu(s) \int_T \alpha(t) \left\| 1_H \circ \left(A_t - \frac{A'+A}{2}\right)\right\|_2^2 d\mu(t) \\
&- \left\| 1_H \circ \int_T \alpha(t)\left(A_t - \frac{A'+A}{2}\right) d\mu(t)\right\|_2^2 \\
\leq &\int_T \alpha(s)\, d\mu(s) \left|\left|\left| (A_t) - \frac{A'+A}{2}\right|\right|\right|^2 - \left\| 1_H \circ \int_T \alpha(t)\left(A_t - \frac{A'+A}{2}\right) d\mu(t)\right\|_2^2 .
\end{aligned}$$

Therefore

$$\langle (A_t), (A_t)\rangle \leq \frac{1}{4}\, |||A' - A|||^2 \int_T \alpha(s)\, d\mu(s) - \left\| \int_T \alpha(t)\, 1_H \circ \left(A_t - \frac{A'+A}{2}\right) d\mu(t)\right\|_2^2 . \tag{15}$$

Similarly,

$$\langle (B_t), (B_t)\rangle \leq \frac{1}{4}\, |||B' - B|||^2 \int_T \alpha(s)\, d\mu(s) - \left\| \int_T \alpha(t)\, 1_H \circ \left(B_t - \frac{B'+B}{2}\right) d\mu(t)\right\|_2^2 . \tag{16}$$

By Schwarz inequality (10) and inequalities (15), (16), we deduce (14).

Now, suppose that (14) holds with the constants $C > 0$ in the inequality. That is,

$$\begin{aligned}
&\left| \int_T \alpha(s)\, d\mu(s) \int_T \alpha(t)\mathrm{tr}(A_t \circ B_t^*)\, d\mu(t) - \mathrm{tr}\left(\int_T \alpha(t) A_t\, d\mu(t) \circ \int_T \alpha(t) B_t^*\, d\mu(t)\right)\right| \\
\leq &C|||A' - A|||\ |||B' - B||| \int_T \alpha(s)\, d\mu(s). \qquad (17)
\end{aligned}$$

Let $\{e_\alpha\}$ be an orthonormal basis of a Hilbert space H, and $P_{H_1} : H \to H_1$ be the orthogonal projection onto subspace H_1 generated by e_{α_1}. It is obvious that $\mathrm{tr}(P_{H_1}) = 1$ and $P_{H_1} \circ P_{H_1} = P_{H_1}$. If we choose $\alpha : T = [0,1] \to \mathbb{R}, \alpha(t) = 1$, and

$$A_t = B_t = \begin{cases} -P_{H_1} & 0 \leq t \leq \frac{1}{2} \\ P_{H_1} & \frac{1}{2} < t \leq 1 \end{cases} \tag{18}$$

then clearly (A_t) belongs to $\mathcal{L}^2_\alpha(T, L^2(H))$, but is not continuous. Since $\sup_{t\in T} \|A_t - A_{t_0}\| < \epsilon$ for t sufficiently closed to t_0, then $\|P_{H_1} + A_{t_0}\| \leq \sup_{t\in T} \|A_t - A_{t_0}\| < \epsilon$, and $\|P_{H_1} - A_{t_0}\| \leq \sup_{t\in T} \|A_t - A_{t_0}\| < \epsilon$. Therefore $\|A_{t_0}\| \leq \frac{1}{2}\|P_{H_1} + A_{t_0}\| + \frac{1}{2}\|P_{H_1} - A_{t_0}\| < \epsilon$, implying that $A_{t_0} = 0$, which is a contradiction with $|||(A_t)||| =$

$\sup_{t\in T} \|A_t\| = 1$.

For $A' = B' = P_{H_1}, A = B = -P_{H_1}$ and Lebesgue measure μ on Ω, the condition (15) holds. By (17) we deduce $C \geq \frac{1}{4}$, and the proof is completed. $\square$

Corollary 2.7. *Let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be operators in $L^2(H)$ and $\omega_1, \omega_2, \ldots, \omega_n$ be positive numbers. Then*

$$\begin{aligned}&\left|\sum_{i=1}^n \omega_i \sum_{i=1}^n \omega_i \mathrm{tr}(A_i \circ B_i) - \mathrm{tr}\left(\sum_{i=1}^n \omega_i A_i \circ \sum_{i=1}^n \omega_i B_i\right)\right| \\ &\qquad \leq \sum_{i=1}^n \omega_i \sum_{i=1}^n \omega_i \|A_i\|_2 \|B_i\|_2 - \left\|1_H \circ \sum_{i=1}^n \omega_i A_i \circ \sum_{i=1}^n \omega_i B_i\right\|_2.\end{aligned}$$

Proof.

$$\begin{aligned}&\left|\sum_{i=1}^n \omega_i \sum_{i=1}^n \omega_i \mathrm{tr}(A_i \circ B_i) - \mathrm{tr}\left(\sum_{i=1}^n \omega_i A_i \circ \sum_{i=1}^n \omega_i B_i\right)\right| \\ &\quad \leq \sum_{i=1}^n \omega_i \sum_{i=1}^n \omega_i \|1_H \circ A_i\|_2 \|1_H \circ B_i\|_2 - \left\|1_H \circ \sum_{i=1}^n \omega_i A_i\right\|_2 \left\|1_H \circ \sum_{i=1}^n \omega_i B_i\right\|_2 \\ &\quad \leq \sum_{i=1}^n \omega_i \sum_{i=1}^n \omega_i \|A_i\|_2 \|B_i\|_2 - \left\|1_H \circ \sum_{i=1}^n \omega_i A_i \circ \sum_{i=1}^n \omega_i B_i\right\|_2. \qquad \square\end{aligned}$$

Example 2.8. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis of a separable Hilbert space H, and H_n be the subspace of H generated by the vectors $e_1, e_2, \ldots, e_n$. Suppose that $P_{H_n} : H \to H_n$ is the orthogonal projection onto subspace H_n, and $\alpha : T = [0,1] \to \mathbb{R}, \alpha(t) = 1$. It is obvious that $\mathrm{tr}(P_{H_n}) = \mathrm{rank}(H_n) = n$ and $P_{H_n} \circ P_{H_m} = P_{H_{m\wedge n}}$, where $m \wedge n = \min\{m, n\}$. If (A_t) belong to $\mathcal{L}^2(T, L^2(H))$, and $m \geq 1$, then

$$\left|\int_0^1 t^{m-1}\mathrm{tr}(P_{H_n} \circ A_t)\, dt - \frac{1}{m}\mathrm{tr}\left(P_{H_n} \circ \int_0^1 A_t\, dt\right)\right| \leq \sqrt{\frac{n}{2m-1} - \frac{n}{m^2}}\, |||(A_t)|||,$$

and $$\left|\int_0^1 e^t\, \mathrm{tr}(P_{H_n} \circ A_t)\, dt - (e-1)\mathrm{tr}\left(P_{H_n} \circ \int_0^1 A_t\, dt\right)\right| \leq \sqrt{\frac{n(e-1)(3-e)}{2}}\, |||(A_t)|||.$$

From inequality (12), for $B_t = t^{m-1} P_{H_n}$, we have

$$\begin{aligned}&\left|\int_0^1 t^{m-1}\mathrm{tr}(P_{H_n} \circ A_t)\, dt - \mathrm{tr}\left(\int_0^1 t^{m-1} P_{H_n}\, dt \circ \int_0^1 A_t\, dt\right)\right|^2 \\ &\leq \left[\int_0^1 \mathrm{tr}(t^{2m-2} P_{H_n})\, dt - \mathrm{tr}\left(\int_0^1 t^{m-1} P_{H_n}\, dt \circ \int_0^1 t^{m-1} P_{H_n}\, dt\right)\right] \\ &\quad \times \left[\int_0^1 \mathrm{tr}(A_t^* \circ A_t)\, dt - \mathrm{tr}\left(\int_0^1 A_t^*\, dt \circ \int_0^1 A_t\, dt\right)\right] \\ &\leq \left(\frac{n}{2m-1} - \frac{n}{m^2}\right) \int_0^1 \mathrm{tr}(A_t^* \circ A_t)\, dt,\end{aligned}$$

therefore

$$\limsup_{n\to\infty}\frac{1}{\sqrt{n}}\left|\int_0^1 t^{m-1}\mathrm{tr}(P_{H_n}\circ A_t)\,dt\ -\frac{1}{m}\mathrm{tr}\left(P_{H_n}\circ\int_0^1 A_t\,dt\right)\right|\le\sqrt{\frac{1}{2m-1}-\frac{1}{m^2}}|||(A_t)|||,$$

and consequently,

$$\left|\int_0^1 t^{m-1}\mathrm{tr}(P_{H_n}\circ A_t)\,dt\ -\frac{1}{m}\mathrm{tr}\left(P_{H_n}\circ\int_0^1 A_t\,dt\right)\right|=O(\sqrt{n}).$$

Now, for $B_t=\exp(t)P_{H_n}$, from inequality (12), we get

$$\begin{aligned}&\left|\int_0^1 e^t\ \mathrm{tr}(P_{H_n}\circ A_t)\,dt-\mathrm{tr}\left(\int_0^1 e^t\ P_{H_n}\,dt\circ\int_0^1 A_t\,dt\right)\right|^2\\
&\le\left[\int_0^1\mathrm{tr}(e^{2t}\ P_{H_n})\,dt-\mathrm{tr}\left(\int_0^1 e^t\ P_{H_n}\,dt\circ\int_0^1 e^t\ P_{H_n}\,dt\right)\right]\\
&\quad\times\left[\int_0^1\mathrm{tr}(A_t^*\circ A_t)\,dt-\mathrm{tr}\left(\int_0^1 A_t^*\,dt\circ\int_0^1 A_t\,dt\right)\right]\\
&\le\left(\frac{n(e-1)(3-e)}{2}\right)\int_0^1\mathrm{tr}(A_t^*\circ A_t)\,dt,\end{aligned}$$

therefore

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{\sqrt{n}}&\left|\int_0^1 e^t\ \mathrm{tr}(P_{H_n}\circ A_t)\,dt\ -(e-1)\mathrm{tr}\left(P_{H_n}\circ\int_0^1 A_t\,dt\right)\right|\\
&\le\sqrt{\frac{(e-1)(3-e)}{2}}|||(A_t)|||,\end{aligned}$$

and consequently

$$\left|\int_0^1 e^t\ \mathrm{tr}(P_{H_n}\circ A_t)\,dt\ -(e-1)\mathrm{tr}\left(P_{H_n}\circ\int_0^1 A_t\,dt\right)\right|=O(\sqrt{n}).$$

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