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A NOTE ON THE EIGENVALUES OF n-CAYLEY GRAPHS

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Abstract. A graph Γ is called an n-Cayley graph over a group G if its automorphism contains a semi-regular subgroup isomorphic to G with n orbits. Every n-Cayley graph over a group G is completely determined by n^2 suitable subsets of G. If each of these subsets is a union of conjugacy classes of G, then it is called a quasi-abelian n-Cayley graph over G. In this paper, we determine the characteristic polynomial of quasi-abelian n-Cayley graphs. Then we exactly determine the eigenvalues and the number of closed walks of quasi-abelian semi-Cayley graphs. Furthermore, we construct some integral graphs.

1. Introduction

In this paper, all graphs are finite, without loops and multiple edges. Vertex set of a graph Γ and its edge set will be denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The spectrum of Γ is the spectrum of its adjacency matrix A, that is, the set of eigenvalues together with their multiplicities. If $\lambda_1, \ldots, \lambda_r$ are the distinct eigenvalues of Γ with multiplicities $m_1, \ldots, m_r$, respectively, then we shall write $\mathrm{Spec}(\Gamma) = \{\lambda_1^{[m_1]}, \ldots, \lambda_r^{[m_r]}\}$. A graph with integer eigenvalues is called integral graph. During the last forty years many mathematicians tried to construct and classify integral graphs; for a survey on integral graphs up to 2002, see [3]. The characteristic polynomial of Γ is the characteristic polynomial of A, that is the polynomial defined by $\chi_A(\lambda) = \det(\lambda I - A)$. The spectrum of a graph is one of the most important algebraic invariants, and it is known that numerous proofs in graph theory depend on the spectrum of graphs. The basic relationships between algebraic properties of these eigenvalues and the usual properties of graphs are available in [4].

Let G be a group and S be a subset of G not containing the identity element 1. The Cayley digraph of G with respect to S, $\mathrm{Cay}(G, S)$, is a digraph with vertex set G and edge set $\{(g, sg) \mid g \in G, s \in S\}$. If $S = S^{-1}$ then $\mathrm{Cay}(G, S)$ is undirected. If S is a union of conjugacy classes of G, then $\mathrm{Cay}(G, S)$ is called *quasi-abelian Cayley (di)graph* of G with respect to S, according to Wang and Xu [10]. Quasi-abelian

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Cayley (di)graphs have been considered in various contexts since 1964, for a survey up to 2002 see [11].

By a theorem of Sabidussi, a (di)graph Γ is a Cayley (di)graph of a group G if and only if there exists a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G. Recall that a subgroup G of the automorphism group $\mathrm{Aut}(\Gamma)$ of a graph Γ is called *semi-regular* if all its point-stabilizers are trivial, i.e., for every vertex v of $V(\Gamma)$ the only element of G fixing v is the identity element. Also a *regular* subgroup is a semi-regular subgroup with only one orbit. In other words, a regular subgroup is a transitive semi-regular subgroup. So there is a natural generalization of the Sabidussi's Theorem which was introduced in [1]. A (di)graph Γ is called an *n-Cayley (di)graph over a group G* if there exists an semi-regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G with n orbits. Clearly, every Cayley graph is an 1-Cayley graph. Furthermore, every Cayley graph of a group G having a subgroup H of index n is an n-Cayley graph over H [1, Lemma 8]. The undirected 2-Cayley graphs are called *semi-Cayley* graphs, according to Resmini and Jungnickel [9]. Generalized Petersen graphs are examples of semi-Cayley graphs over cyclic groups.

It is proved in [1, Lemma 2] that a (di)graph Γ is an n-Cayley (di)graph over G if and only if there exist n^2 subsets T_{ij}, $1 \le i, j \le n$, of G such that $\Gamma \cong \mathrm{Cay}(G; T_{ij} \mid 1 \le i, j \le n)$, where $\mathrm{Cay}(G; T_{ij} \mid 1 \le i, j \le n)$ is a graph with vertex set $G \times \{1, \ldots, n\}$ and edge set $\bigcup_{1 \le i,j \le n}\{((g, i), (tg, j)) \mid g \in G, t \in T_{ij}\}$. Hence we may denote an n-Cayley graph over a group G with $\mathrm{Cay}(G; T_{ij} \mid 1 \le i, j \le n)$ for some subsets T_{ij} of G. Note that $\mathrm{Cay}(G; T_{ij} \mid 1 \le i, j \le n)$ is undirected if and only if for all $1 \le i, j \le n$, $T_{ij} = T_{ji}^{-1}$. Furthermore, it is loop-free if and only if for all $1 \le i \le n$, $1 \notin T_{ii}$. Let $R_G = \{\rho_g \mid g \in G\}$, where $\rho_g : G \times \{1, \ldots, n\} \to G \times \{1, \ldots, n\}$ and $(x, i)^{\rho_g} = (xg, i)$. Then R_G is a semi-regular subgroup of the automorphism group of $\mathrm{Cay}(G; T_{ij} \mid 1 \le i, j \le n)$ with n orbits $G \times \{i\}$, $i = 1, \ldots, n$. We define quasi-abelian n-Cayley graphs in analogous way to quasi-abelian Cayley graphs as follows.

Definition 1.1. $\mathrm{Cay}(G; T_{ij} \mid 1 \le i, j \le n)$ is called a *quasi-abelian* n-Cayley graph if for all $1 \le i, j \le n$, T_{ij}'s are unions of conjugacy classes of G.

The characteristic polynomial of an n-Cayley graph over a group G is determined in terms of irreducible representations of G in [1]. We refer the reader to [2] for a survey on the eigenvalues of n-Cayley graphs. In this paper, we study quasi-abelian n-Cayley graphs over a group G and determine their spectrum in terms of irreducible characters of G.

2. A characterization of quasi-abelian n-Cayley graphs

Let $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \le i, j \le n)$. Recall that $R_G = \{\rho_g : V(\Gamma) \to V(\Gamma) \mid g \in G\}$, where $(x, i)^{\rho_g} = (xg, i)$ for all $x \in G$ and $i \in \{1, \ldots, n\}$, is a semi-regular subgroup of $\mathrm{Aut}(\Gamma)$. Furthermore, recall that Γ is called quasi-abelian if all T_{ij}'s are a union of conjugacy classes of G. In this section, we are going to determine quasi-abelian

n-Cayley graphs using their automorphism group. We generalize them to n-Cayley graphs, where $n \geq 2$ is an arbitrary integer.

Let $L_G = \{\psi_g : V(\Gamma) \to V(\Gamma) \mid g \in G\}$, $\forall x \in G, 1 \leq i \leq n,\ (x,i)^{\psi_g} = (gx,i)$. Then we have the following lemma.

Lemma 2.1. *Let* $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i,j \leq n)$*. Then* Γ *is a quasi-abelian* n*-Cayley graph over* G *if and only if* $L_G \leq \mathrm{Aut}(\Gamma)$.

Proof. Let Γ be quasi-abelian and $\psi_g \in L_G$. Then

$$\begin{aligned}((x,i),(y,j)) \in E(\Gamma) \Leftrightarrow\ & yx^{-1} \in T_{ij}\\ \Leftrightarrow\ & gyx^{-1}g^{-1} \in T_{ij} \text{ (since } T_{ij} \text{ is a union of conjugacy classes)}\\ \Leftrightarrow\ & ((gx,i),(gy,j)) \in E(\Gamma)\\ \Leftrightarrow\ & ((x,i)^{\psi_g},(y,j)^{\psi_g})) \in E(\Gamma),\end{aligned}$$

which proves that $L_G \leq \mathrm{Aut}(\Gamma)$. Conversely, suppose that $L_G \leq \mathrm{Aut}(\Gamma)$, $t \in T_{ij}$ for some $i,j \in \{1,2\}$ and $g \in G$. Then $((1,i),(t,j)) \in E(\Gamma)$. So $((1,i)^{\psi_g},(t,j)^{\psi_g}) \in E(\Gamma)$ which implies that $((g,i),(gt,j)) \in E(\Gamma)$. Hence $gtg^{-1} \in T_{ij}$ which means that $g^{-1}T_{ij}g = T_{ij}$ i.e T_{ij} is a union of conjugacy classes of G. This proves that Γ is a quasi-abelian graph over G. $\square$

Let $I_G = \{\theta_g : V(\Gamma) \to V(\Gamma) \mid g \in G\}$, $\forall x \in G, 1 \leq i \leq n,\ (x,i)^{\theta_g} = (g^{-1}xg,i)$. Then one can see that $R_GL_G = R_GI_G$. Furthermore, I_G is a subgroup of the automorphism group of quasi-abelian n-Cayley graphs over G as follows:

Lemma 2.2. *Suppose that* $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i,j \leq n)$*. Then* Γ *is quasi-abelian if and only if* $I_G \leq \mathrm{Aut}(\Gamma)$.

Proof. Let $\theta_g \in I_G$. Then

$$\begin{aligned}((x,i),(y,j)) \in E(\Gamma) \Leftrightarrow\ & yx^{-1} \in T_{ij}\\ \Leftrightarrow\ & g^{-1}yx^{-1}g \in T_{ij} \text{ (since } T_{ij} \text{ is a union of conjugacy classes)}\\ \Leftrightarrow\ & g^{-1}ygg^{-1}x^{-1}g \in T_{ij}\\ \Leftrightarrow\ & ((g^{-1}xg,i),(g^{-1}yg,j)) \in E(\Gamma)\\ \Leftrightarrow\ & ((x,i)^{\psi_g},(y,j)^{\psi_g})) \in E(\Gamma),\end{aligned}$$

which means that $\theta_g \in \mathrm{Aut}(\Gamma)$. This proves one direction. Conversely, suppose that $I_G \leq \mathrm{Aut}(\Gamma)$. Since for all $g,h \in G$, we have $\rho_g\theta_h = \theta_x\rho_y$, where x is an element of centralizer of h in G and $y = h^{-1}gh$, we have $R_GI_G = I_GR_G$, which means that $R_GI_G \leq \mathrm{Aut}(\Gamma)$. Since $R_GI_G = R_GL_G$, we conclude that $L_G \leq \mathrm{Aut}(\Gamma)$. Hence Lemma 2.1 implies that Γ is quasi-abelian. $\square$

Combining the above lemmas, one can obtain the following corollary.

Corollary 2.3. *Let* $\Gamma = \mathrm{Cay}(G; T_{ij} | 1 \leq i,j \leq n)$*. Then the following are equivalent.*
(i) Γ *is quasi-abelian.* *(ii)* $L_G \leq \mathrm{Aut}(\Gamma)$. *(iii)* $I_G \leq \mathrm{Aut}(\Gamma)$.
(iv) $R_GL_G \leq \mathrm{Aut}(\Gamma)$. *(v)* $R_GI_G \leq \mathrm{Aut}(\Gamma)$.

3. Spectrum of quasi-abelian n-Cayley graphs

In this section, we determine the characteristic polynomial of a quasi-abelian n-Cayley graph over a group G in terms of irreducible representations of G. Our notations in character theory of finite groups are standard and mainly taken from [8].

Let G be a finite group and $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_m\}$ be the set of all inequivalent irreducible characters of G. Let $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$ and for all $\chi \in \mathrm{Irr}(G)$ define a matrix $\chi(\Gamma) = \frac{1}{\chi(1)}(\sum_{t \in T_{ij}} \chi(t))_{1 \leq i,j \leq n}$. Note that $\chi(1)$ is a positive integer for all $\chi \in \mathrm{Irr}(G)$. Keeping the above notations, we have the following theorem.

THEOREM 3.1. *Let $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1 \leq i, j \leq n)$ be a finite n-Cayley digraph over a group G and each T_{ij} is a union of conjugacy classes of G. Then the spectrum of Γ is the union of spectrum of $\chi_k(\Gamma)$, where $k = 1, \ldots, m$. Moreover, if λ occurs with multiplicity $m_k(\lambda)$ in $\chi_k(\Gamma)$, then the multiplicity of λ in Γ is $\sum_{k=1}^m m_k(\lambda)\chi_k(1)^2$.*

Proof. Let A be the adjacency matrix of Γ. Then by a suitable ordering of vertices of Γ, we can write $A = (A_{ij})_{1 \leq i,j \leq n}$, where A_{ij} is the adjacency matrix of Cayley graph $\mathrm{Cay}(G, T_{ij})$. Since each T_{ij} is a union of conjugacy classes of G, each $\mathrm{Cay}(G, T_{ij})$ is a quasiabelian Cayley graph with eigenvalues $\frac{\sum_{t \in T_{ij}} \chi_k(t)}{\chi_k(1)}$, $k = 1, \ldots, m$ [6]. Also rows of the matrix $D_k = (\chi_k(x_i x_j^{-1}))_{1 \leq i,j \leq g}$, $1 \leq k \leq m$, where $G = \{x_1, x_2, \ldots, x_g\}$, are eigenvectors of A_{ij} corresponding to the eigenvalue $\frac{\sum_{t \in T_{ij}} \chi_k(t)}{\chi_k(1)}$ and dimension of D_k is $\chi_k(1)^2$ (see [6, pages 1-3]). So the matrices A_{ij}, $1 \leq i, j \leq n$, have a common eigenvector basis. Let elements of this common basis are column vectors $v_1, \ldots, v_g$. On the other hand, $A_{ij} = PB_{ij}P^{-1}$, where $P = [v_1, \ldots, v_g]$, $B_{ij} = \mathrm{diag}(\lambda_{ij}^1, \ldots, \lambda_{ij}^g)$ and λ_{ij}^l is the corresponding eigenvalue of v_l of A_{ij}. Now $A = (A_{ij})_{1 \leq i,j \leq n} = (PB_{ij}P^{-1})_{1 \leq i,j \leq n} = CDC^{-1}$, where $C = \mathrm{diag}(P, \ldots, P)$ and $D = (B_{ij})_{1 \leq i,j \leq n}$ are both $ng \times ng$ matrices. So the eigenvalues of A are the same as eigenvalues of D. Now there is a permutation matrix which establishes a similarity relation between D and the diagonal matrix $\mathrm{diag}(I_{d_1} \otimes A_1, \ldots, I_{d_m} \otimes A_m)$ where $A_k = \chi_k(\Gamma) \otimes I_{d_k}$, $d_k = \chi_k(1)$ and I_{d_k} is the $d_k \times d_k$ identity matrix. This completes the proof. □

Now we focus on semi-Cayley graphs. Recall that semi-Cayley graphs are undirected 2-Cayley graphs. So every semi-Cayley graph over a group G is of the form $\mathrm{Cay}(G; T_{11}, T_{22}, T_{12}, T_{21})$, where $T_{12} = T_{21}^{-1} \subseteq G$, $T_{11} = T_{11}^{-1}$, $T_{22} = T_{22}^{-1} \subseteq G \setminus \{1\}$. Let us denote $\mathrm{Cay}(G; T_{11}, T_{22}, T_{12}, T_{21})$ by $\mathrm{SC}(G; R, L, S)$, where $R = T_{11}$, $L = T_{22}$ and $S = T_{12} = T_{21}^{-1}$.

COROLLARY 3.2. *Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be a quasi-abelian semi-Cayley graph. Then the characteristic polynomial of Γ is $\prod_{k=1}^m (\lambda - \lambda_k^+)^{\chi_k(1)^2} (\lambda - \lambda_k^-)^{\chi_k(1)^2}$,*

$$\lambda_k^+ = \frac{\alpha_k + \beta_k + \sqrt{(\alpha_k - \beta_k)^2 + 4|\gamma_k|^2}}{2\chi_k(1)}, \quad \lambda_k^- = \frac{\alpha_k + \beta_k - \sqrt{(\alpha_k - \beta_k)^2 + 4|\gamma_k|^2}}{2\chi_k(1)}$$

where $\alpha_k = \sum_{r \in R} \chi_k(r)$, $\beta_k = \sum_{l \in L} \chi_k(l)$, $\gamma_k = \sum_{s \in S} \chi_k(s)$ and $k = 1, \ldots, m$.

In what follows, we keep the notation of Corollary 3.2.

Example 3.3. Let G be a finite non-abelian group, $R = G \setminus Z(G)$, $L = Z(G) \setminus \{1\}$, and $\Gamma = \mathrm{SC}(G; R, L, \{1\})$. Then the eigenvalues of Γ are

$$\frac{|G| - 1 \pm \sqrt{(|G| - 2|Z(G)| + 1)^2 + 4}}{2}$$

and

$$\frac{-\chi(1) \pm \sqrt{5\chi(1)^2 - 4\sum_{r\in R, l\in L} \chi(r)\chi(l)}}{2\chi(1)}, \quad 1 \neq \chi \in \mathrm{Irr}(G)$$

with multiplicity $\chi(1)^2$.

It is clear that Γ is a quasi-abelian semi-Cayley graph. Let $1 \neq \chi \in \mathrm{Irr}(G)$. Then by the row orthogonality relations, $\sum_{g\in G} \chi(g) = 0$, which implies that $\sum_{r\in R} \chi(r) + \sum_{l\in L} \chi(l) = -\chi(1)$. Now the result is a direct consequence of Corollary 3.2.

In the following example, for every odd integer $n \geq 3$, we construct an $(n + 1)$-regular integral and bipartite graph with $4n$ vertices.

Example 3.4. Let $n \geq 3$ be odd, $G = \langle a, b \mid a^n = b^2 = (ab)^2 = 1\rangle \cong D_{2n}$, and $\Gamma = \mathrm{SC}(G; R, L, \{1\})$, where $R = L = \{b, ba, ba^2, \ldots, ba^{n-1}\}$. Then $\mathrm{Spec}(\Gamma) = \{n \pm 1, -n \pm 1, (\pm 1)^{[2(n-1)]}\}$.

It is well-known that $\{b, ba, \ldots, ba^{n-1}\}$ is a conjugacy class of D_{2n}. The irreducible characters of D_{2n} are

$$\chi_0 : b^i a^j \mapsto 1, \quad \chi_1 : b^r a^s \mapsto (-1)^r, \quad \chi_j : a^s \mapsto 2\cos(\frac{2\pi j s}{n}),\ ba^s \mapsto 0, \quad 1 \leq j \leq \frac{n-1}{2}.$$

Now the result is a direct consequence of Corollary 3.2.

A conjugacy class C of a finite group G is called rational, if an element c of C has order r and if $(s, r) = 1$ then c^s belongs to C.

Corollary 3.5. *Let $\Gamma = \mathrm{SC}(G; \mathrm{Cl}(g), \mathrm{Cl}(g), \mathrm{Cl}(h))$. If $\mathrm{Cl}(g)$ and $\mathrm{Cl}(h)$ are both rational then Γ is integral. In particular, if $\mathrm{Cl}(g) = \mathrm{Cl}(h)$, then Γ is integral if and only if $\mathrm{Cl}(g)$ is rational.*

Proof. By Corollary 3.2, eigenvalues of Γ are $\lambda_k^{\pm} = \frac{\alpha_k \pm |\gamma_k|}{\chi_k(1)}$ both with multiplicity $\chi_k(1)^2$, $k = 1, \ldots, m$. Furthermore, $\alpha_k = |\mathrm{Cl}(g)|\chi_k(g)$ and $|\gamma_k| = |\mathrm{Cl}(h)||\chi_k(h)|$. Suppose that $\mathrm{Cl}(g)$ and $\mathrm{Cl}(h)$ are rational. Then, by [8, Theorem 22.16], $\chi_k(g), \chi_k(h) \in \mathbb{Z}$. Hence λ_k^+ and λ_k^- are both rational numbers. On the other hand, both are algebraic integers (recall that a complex number λ is an algebraic integer if and only if λ is an eigenvalue of some matrix, all of whose entries are integers [8, Definition 22.1]). Hence [8, Proposition 22.5] implies that λ_k^+ and λ_k^- are both integers, which means that Γ is an integral graph.

Now suppose that $\mathrm{Cl}(g) = \mathrm{Cl}(h)$. If $\mathrm{Cl}(g)$ is rational then Γ is integral, by previous paragraph. Conversely, suppose that Γ is integral.Then $\frac{|\mathrm{Cl}(g)|}{\chi_k(1)}(\chi_k(g) \pm |\chi_k(g)|)$, $k = 1, \ldots, m$, are integers. Thus $\chi_k(g) + |\chi_k(g)|$ and $\chi_k(g) - |\chi_k(g)|$ are rational, and so $\chi_k(g)$ is rational for all k. Hence $\mathrm{Cl}(g)$ is rational [7, Problem 2.12], which completes the proof. □

Keeping the notation of Corollary 3.2, one can compute the number of closed walks in quasi-abelian semi-Cayley graphs as follows.

COROLLARY 3.6. *Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be quasi-abelian and $\omega_n(\Gamma)$ be the number of closed walks of length n in Γ. Then*

$$\omega_n(\Gamma) = \frac{1}{2^{n-1}} \sum_{k=1}^{m} \sum_{i=0}^{[n/2]} \frac{1}{\chi_k(1)^{n-2}} (\alpha_k + \beta_k)^{n-2i} ((\alpha_k - \beta_k)^2 + 4|\gamma_k|^2)^i,$$

where $\alpha_k = \sum_{r \in R} \chi_k(r)$, $\beta_k = \sum_{l \in L} \chi_k(l)$, $\gamma_k = \sum_{s \in S} \chi_k(s)$.

Proof. By [4, Lemma 2.5], it is enough to note that $\omega_n(\Gamma) = \sum_{k=1}^{m} \chi_k(1)^2 \big((\lambda_k^+)^n + (\lambda_k^-)^n\big)$ and use Corollary 3.2. □

It is clear that every n-Cayley graph over an abelian group is a quasi-abelian n-Cayley graph. Furthermore, every finite abelian group G has $|G|$ irreducible characters and for any character χ of G we have $\chi(1) = 1$. Hence one can find the eigenvalues and the number of closed walks of semi-Cayley graphs over abelian groups as a direct consequence of Corollaries 3.2 and 3.6.

Let Γ be a graph with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the Estrada index of Γ is defined as $EE(\Gamma) = \sum_{i=1}^{n} e^{\lambda_i}$. This quantity which has many applications in chemistry and computer science, was defined in 2000 by Ernesto Estrada [5].

COROLLARY 3.7. *Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be a quasi-abelian semi-Cayley graph. The $EE(\Gamma)$ is equal to*

$$2 \sum_{k=1}^{m} \chi_k(1)^2 e^{\frac{\alpha_k + \beta_k}{2\chi_k(1)}} \cosh\Big(\frac{\sqrt{(\alpha_k - \beta_k)^2 + 4|\gamma_k|^2}}{2\chi_k(1)}\Big).$$

In particular, if $R = L$ then $EE(\Gamma) = 2 \sum_{k=1}^{m} \chi_k(1)^2 e^{\frac{\alpha_k}{\chi_k(1)}} \cosh(\frac{|\gamma_k|}{\chi_k(1)})$.

Proof. It is enough to note that $e^{x+y} + e^{x-y} = 2e^x \cosh(y)$ and use Corollary 3.2. □

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