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ON THE ERDŐS-GYÁRFÁS CONJECTURE FOR SOME CAYLEY GRAPHS

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Abstract. In 1995, Paul Erdős and András Gyárfás conjectured that for every graph X of minimum degree at least 3, there exists a non-negative integer m such that X contains a simple cycle of length 2^m. In this paper, we prove that the conjecture holds for Cayley graphs of order $2p^2$ and $4p$.

1. Introduction

In this paper all graphs will be simple and finite and all groups will be finite. For a graph X, we let $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ denote the vertex set, the edge set, the full group of automorphisms of X, respectively.

A graph X is said to be *vertex-transitive* if $\mathrm{Aut}(X)$ acts transitively on $V(X)$. The *minimum degree* of X is the minimum degree of its vertices. Also, a k-cycle is a cycle of length k.

Several questions on cycles in graphs have been posed by Erdős and his colleagues (see, e.g. [1]). In particular, in 1995 Erdős and Gyárfás [3] asked: If G is a graph with minimum degree at least three, does G have a cycle whose length is a power of 2? This is known as the Erdős-Gyárfás conjecture. In fact, Erdős and Gyárfás [3] said that "we are convinced now that this is false and no doubt there are graphs for every r every vertex of which has degree $\geq r$ and which contain no cycle of length 2^k, but we never found a counterexample even for $r = 3$".

Using the computer searches, Markström [6] verified the conjecture for cubic graphs of order at most 29, and found that the smallest cubic planar graph with no 4- or 8-cycles has 24 vertices. Note that this graph contains a 16-cycle. Shauger [8] proved the conjecture for $K_{1,m}$–free graphs of minimum degree at least $m+1$ or maximum degree at least $2m-1$. Daniel and Shauger [2] proved the conjecture for planar claw-free graphs. Also, in [5] it is proved that the conjecture holds for 3-connected

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cubic planar graphs (see also [7]). In [4] the authors proved that the conjecture holds for Cayley graphs on some special groups.

In this paper we study the conjecture for some families of Cayley graphs. Let G be a finite group and S a subset of G not containing the identity element 1. The *Cayley digraph* $X = \mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set $V(X) = G$ and edge set $E(X) = \{(g, sg) \mid g \in G, s \in S\}$. If $S^{-1} = S$, then $\mathrm{Cay}(G, S)$ can be viewed as undirected graph, identifying an undirected edge $\{g, h\}$ with two directed edges (g, h) and (h, g). This graph is called the *Cayley graph* on G with respect to S. It is well-known that $\mathrm{Aut}(X)$ contains the right regular representation R(G) of G, the acting group of G by right multiplication, and X is connected if and only if $G = \langle S\rangle$, that is, S generates G.

Let G be a finite group and let S and T be two subsets of G not containing the identity 1 of G. If there is an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$, then S and T are said to be equivalent, denoted by $S \cong T$. It is easy to see that $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, S^\alpha)$. Throughout this paper, we denote by $\mathbb{Z}_n$ the cyclic group of order n and by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. Also, an element of order 2 is called involution.

2. Main results

Suppose that $X = \mathrm{Cay}(G, S)$ where $|G| = 2p^2$. If G is an abelian group then by [4, Theorem 1.3], G has a 4-cycle. Also, if G is non-abelian and $p = 2$ then G is isomorphic to the dihedral group D_8 or quaternion group Q_8 and by [4] X contains a simple cycle whose length is a power of two. Thus we may suppose that $p > 2$. From the elementary group theory we know that up to isomorphism there are three non-abelian groups of order $2p^2$ defined as:

$$G = G_1(p) = \langle a, b \mid a^p = b^2 = 1, bab^{-1} = a^{-1}\rangle;$$

$$G = G_2(p) = \langle a, b, c \mid a^p = b^p = c^2 = 1 = [a, b], c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1}\rangle;$$

$$G = G_3(p) = \langle a, b, c \mid a^p = b^p = c^2 = 1, [a, b] = [a, c] = 1, c^{-1}bc = b^{-1}\rangle.$$

If $G = G_1(p)$ then by [4, Theorem 2.2] X has a cycle of length 4, 8 or 16. Thus we may suppose that $G \cong G_2(p)$ or $G \cong G_3(p)$.

Theorem 2.1. *Every connected Cayley graph $X = \mathrm{Cay}(G_2(p), S)$ contains a cycle of length 4 or 16.*

Proof. It is easy to see that $o(a^ib^j) = p$ where $0 \le i, j \le p$ and i, j are not zero simultaneously, and $o(a^ib^jc) = 2$, where $0 \le i, j \le p-1$. Since X is connected it follows that S contains an involution. Thus we may suppose that $a^ib^jc \in S$. Since $\mathrm{Aut}(G_2(p))$ is transitive on the set of involutions in $G_2(p)$ we may suppose that $c \in S$. Now we consider the following cases.

Case 1. S contains just involutions.

We may suppose that a^mb^nc belongs to S, where $0 \le m, n \le p-1$ and m, n are not zero simultaneously. Without loss of generality we may suppose that $n \neq 0$. Since the map $a \mapsto a$, $b \mapsto b^n$ and $c \mapsto a^m c$ is an automorphism of $G_2(p)$ one may suppose that $bc \in S$. Since X is connected graph S must contain another element of order 2, say $a^k b^l c$, where $0 \leq k, l \leq p-1$. If $l = 0$ then $a^k c \in S$. Since the map $a \mapsto a^k$, $b \mapsto b$ and $c \mapsto c$ is an automorphism of $G_2(p)$ one may suppose that $ac \in S$. Thus $\{c, bc, ac\} \subseteq S$. Now $(ab^{-2}, b^2c, b^{-1}, abc, a^{-1}, a^2c, a^{-2}b, a^2b^{-1}c, a^{-1}b, ac, 1, c, a, a^{-1}bc, ab^{-1}, a^{-1}b^2c, ab^{-2})$ is a 16-cycle in X. Thus we may suppose that $l \neq 0$. Again since the map $a \mapsto a$, $b \mapsto b^{-1}$ and $c \mapsto a^k c$ is an automorphism of $G_2(p)$ one may suppose that $b^{-1}c \in S$. Also, we know that $c \in S$. Thus $\{c, bc, b^{-1}c\} \subseteq S$. Now $(1, c, b, cb, 1)$ is a 4-cycle in X.

Case 2. S contains an element of order p.

We may suppose that $a^m b^n \in S$, where $0 \leq m, n \leq p-1$. First suppose that $m = 1$ and $n = 0$. Then $a \in S$ and so $\{c, a\} \subseteq S$. Now $(1, c, ac, a^{-1}, 1)$ is a 4-cycle in X. Now suppose that $m \neq 1$ and $n \neq 0$. It is easy to see that the map $a \mapsto a$, $b \mapsto a^m b^n$ and $c \mapsto c$ is an automorphism of $G_2(p)$. Thus we may suppose that $b \in S$. Thus $\{b, c\} \subseteq S$ and so $(1, c, bc, b^{-1}, 1)$ is a 4-cycle in X. □

Theorem 2.2. *Every connected Cayley graph* $X = \mathrm{Cay}(G_3(p), S)$ *contains a cycle of length* 4, 8 *or* 16.

Proof. It is easy to see that $o(a^i b^j c) = 2p$, where $0 < i \leq p-1$ and $0 \leq j \leq p-1$. We have $o(a^i b^j) = p$, where $0 \leq i, j \leq p-1$ and i, j are not zero simultaneously. Also, $o(b^i c) = 2$, where $0 \leq i \leq p-1$. Since X is connected it follows that S does not contain just involutions. Thus we may consider the following cases:

Case 1. S contains an involution and element of order p.

We may suppose that $a^i b^j \in S$. If $i = 0$ or $j = 0$ than $a \in S$ or $b \in S$. Since $S = S^{-1}$ it follows that $\{a, a^{-1}\} \subseteq S$ or $\{b, b^{-1}\} \subseteq S$. Also, since $\mathrm{Aut}(G_3(p))$ is transitive on the set of involutions in $G_3(p)$, one may assume that $c \in S$. Thus either $\{a, a^{-1}, c\} \subseteq S$ or $\{b, c, b^{-1}\} \subseteq S$. For the first case $(1, a, ac, c, 1)$ is a 4-cycle in X and for the second case $(1, c, b^{-1}c, b, 1)$ is a 4-cycle in X. Thus we may suppose that $i \neq 0$ and $j \neq 0$. The map $a \mapsto a^i$, $b \mapsto b^j$ and $c \mapsto c$ is an automorphism of $G_3(p)$ and so $\{ab, a^{-1}b^{-1}, c\} \subseteq S$. Now $(1, c, abc, ab^{-1}, a^2, ca^2, cab, ab, 1)$ is a 8-cycle in X.

Case 2. S contains an involution and an element of order $2p$.

We may suppose that $a^i b^j c \in S$, where $0 < i \leq p-1$ and $0 \leq j \leq p-1$. Since $\mathrm{Aut}(G_3(p))$ is transitive on elements of order $2p$, we may suppose that $ac \in S$. Also, since $S = S^{-1}$ it implies that $a^{-1}c \in S$. Suppose that $b^m c$ is an involution belongs to S. If $m = 0$ then $c \in S$. Thus $\{ac, a^{-1}c, c\} \subseteq S$ and $(1, c, a, ac, 1)$ is a 4-cycle in X. Thus we may suppose that $m \neq 0$. The map $a \mapsto a$, $b \mapsto b^m$ and $c \mapsto c$ is an automorphism of $G_3(p)$ and so we may suppose that $bc \in S$. Thus $\{ac, a^{-1}c, bc\} \subseteq S$. First suppose that $p > 3$. It is easy to see that $(ab, b^{-1}c, a^{-1}b, a^{-2}b^{-1}c, a^{-3}b, a^{-3}c, a^{-2}, a^{-2}bc, a^{-1}b^{-1}, a^{-1}b^2c, b^{-2}, ab^2c, ab^{-1}, bc, 1, ac, ab)$ is a 16-cycle in X. Now suppose that $p = 3$. Now $(b, c, a^2, a^2bc, b^2, b^2c, a^2b, ab^2c, b)$ is a 8-cycle in X.

Case 3. S contains an element of order p and $2p$.

In this case we may suppose that $a^ib^jc \in S$, where $0 < i \leq p-1$ and $0 \leq j \leq p-1$. First suppose that $j = 0$. Then $a^ic \in S$. Since $S = S^{-1}$ and the map $a \mapsto a^i$, $b \mapsto b$, $c \mapsto c$ is an automorphism of $G_3(p)$, it follows that $\{ac, a^{-1}c\} \subseteq S$. Also, suppose that a^mb^n where $0 \leq m, n \leq p-1$, is an element of order p which belongs to S. If $n = 0$ then $a^m \in S$. Now the map $a \mapsto a^m$, $b \mapsto b$, $c \mapsto c$ is an automorphism of $G_3(p)$ and so we may suppose that $a \in S$. Thus $\{ac, a^{-1}c, a\} \subseteq S$ and $(1, ac, c, a^{-1}c, 1)$ is a 4-cycle in X. If $n \neq 0$ then the map $a \mapsto a$, $b \mapsto b^n$, $c \mapsto c$ is an automorphism of $G_3(p)$ and so we may suppose that $a^mb \in S$. Therefore $\{ac, a^{-1}c, a^mb, a^{-m}b^{-1}\} \subseteq S$. Now it is easy to see that $(a^{-m+2}b^{-1}, a^{1-m}bc, a^{-m}b^{-1}, 1, a^mb, a^{m+1}b^{-1}c, a^{m+2}b, a^2, a^{-m+2}b^{-1})$ is a 8-cycle in X. Now suppose that $j \neq 0$. Since $S = S^{-1}$ and the map $a \mapsto a^i$, $b \mapsto b^j$, $c \mapsto c$ is an automorphism of $G_3(p)$, it follows that $\{abc, a^{-1}bc\} \subseteq S$. Also, suppose that a^mb^n is an element of order p which belongs to S. If $n = 0$ then $\{a, abc, a^{-1}bc\} \subseteq S$ and $(1, abc, bc, a^{-1}bc, 1)$ is a 4-cycle in X. Also, if $n \neq 0$ then the map $a \mapsto a$, $b \mapsto b^n$, $c \mapsto c$ is an automorphism of $G_3(p)$ and so we may suppose that $a^mb \in S$. Again since the map $a \mapsto a$, $b \mapsto b$, $c \mapsto b^{-1}c$ is an automorphism of $G_3(p)$ it follows that $\{ac, a^{-1}c\} \subseteq S$. Therefore $\{ac, a^{-1}c, a^mb, a^{-m}b^{-1}\} \subseteq S$. Now it is easy to see that $(a^{-m+2}b^{-1}, a^{1-m}bc, a^{-m}b^{-1}, 1, a^mb, a^{m+1}b^{-1}c, a^{m+2}b, a^2, a^{-m+2}b^{-1})$ is a 8-cycle in X.

Case 4. S contains just elements of order $2p$.

We may suppose that $a^ib^jc \in S$, where $0 < i \leq p-1$ and $0 \leq j \leq p-1$. Since $\mathrm{Aut}(G_3(p))$ is transitive on elements of order $2p$ we may suppose that $\{ac, a^{-1}c\} \subseteq S$. Also, suppose that $a^mb^nc \in S$, where $0 < m \leq p-1$ and $0 \leq n \leq p-1$. Since X is connected and S contains just elements of order $2p$ we may suppose that $n \neq 0$. Now again since the map $a \mapsto a$, $b \mapsto b^n$, $c \mapsto c$ is an automorphism of $G_3(p)$ we may suppose that $\{ac, a^{-1}c, a^mbc, a^{-m}bc\} \subseteq S$. If $m = 1$ then $(1, abc, a^2, ac, 1)$ is a 4-cycle in X. Thus we may suppose that $m > 1$. Now $(a^mb^{-1}c, a^{m-1}b, a^{2m-1}c, a^{2m}, a^mbc, 1, ac, a^{m+1}b, a^mb^{-1}c)$ is a 8-cycle in X. □

Now we consider the Cayley graphs of order $4p$. Suppose that $X = \mathrm{Cay}(H, S)$, where $|H| = 4p$. If G is an abelian group then by [4, Theorem 1.3], G has a 4-cycle. Also, if $p = 2$ then G is isomorphic to the dihedral group D_8 or quaternion group Q_8 and by [4], X contains a simple cycle whose length is a power of two. Thus we may suppose that $p > 2$. From the elementary group theory we know that up to isomorphism there are three non-abelian groups of order $4p$ defined as:

$$H = H_1(p) = \langle a, b \mid a^{2p} = b^2 = 1, bab^{-1} = a^{-1}\rangle;$$
$$H = H_2(p) = \langle a, b \mid a^{2p} = 1, b^2 = a^p, b^{-1}ab = a^{-1}\rangle;$$
$$H = H_3(p) = \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^r, r^2 \equiv -1(p)\rangle.$$

If $H = H_1(p)$ then by [4, Theorem 2.2] X has a cycle of length 4, 8 or 16. Thus we may suppose that $H \cong H_2(p)$ or $H \cong H_3(p)$.

Theorem 2.3. *Every connected Cayley graph $X = \mathrm{Cay}(H_2(p), S)$ contains a 4-cycle.*

Proof. Clearly $H = H_2(p) = \{a^i, ba^i \mid 0 \leq i \leq 2p-1\}$. Since H cannot be generated by elements in $\langle a\rangle$, one may assume that $ba^i \in S$. Furthermore, a and ba^i $(0 \leq i \leq$

$2p-1$) have the same relations as a and b. This implies there is an automorphism of H which maps a to a and ba^i to b. Thus one may assume that $b, b^{-1} \in S$. Now we consider the following cases.
Case 1. $a^m \in S$, where $m \neq 0$.

First suppose that $(m, 2p) = 1$. Now the map $a \mapsto a^m$, $b \mapsto b$ is an automorphism of $H_2(p)$ and so we may suppose that $\{a, a^{-1}\} \subseteq S$. Now it is easy to see that $(1, a^{-1}, ab, b, 1)$ is a 4-cycle in X. Now suppose that $(m, 2p) \neq 1$. Since $H = \langle S \rangle$, it follows that either $a^i \in S$ where $(i, 2p) = 1$ or $ba^j \in S$ where $(j, 2m) = 1$. For the former case with the similar arguments as before $\{a, a^{-1}\} \subseteq S$ and $(1, a^{-1}, ab, b, 1)$ is a 4-cycle in X. For the latter case the map $a \mapsto a^j$, $b \mapsto b$ is an automorphism of $H_2(p)$ and so we may suppose that $\{b, b^{-1}, ba, a^{-1}b^{-1}\} \subseteq S$. Now $(1, b, b^2, a^{-1}b, 1)$ is a 4-cycle in X.

Case 2. $ba^m \in S$.

First suppose that $(m, 2p) = 1$. In this case again the map $a \mapsto a^m$, $b \mapsto b$ is an automorphism of $H_2(p)$ and so we may suppose that $\{ba, a^{-1}b^{-1}\} \subseteq S$. Now $(1, b, b^2, a^{-1}b, 1)$ is a 4-cycle in $H_2(p)$. Now suppose that $(m, 2p) \neq 1$. If $m = p$ then $\{b, b^{-1}, ba^m, a^{-m}b^{-1}\} \subseteq S$. Since $H = \langle S \rangle$ one may suppose that either $a^i \in S$ where $(i, 2p) = 1$ or $ba^j \in S$ where $(j, 2p) = 1$. For the former case the map $a \mapsto a^i$, $b \mapsto b$ is an automorphism of $H_2(p)$ and so $\{b, b^{-1}, a, a^{-1}\} \subseteq S$ and $(1, a^{-1}, ab, b, 1)$ is a 4-cycle in X. Also, for the latter case the map $a \mapsto a^j$, $b \mapsto b$ is an automorphism of H and so $\{b, b^{-1}, ba, a^{-1}b^{-1}\} \subseteq S$. Now $(1, b, b^2, a^{-1}b, 1)$ is a 4-cycle in X. Therefore we may suppose that $m = 2$. Since $H = \langle S \rangle$ we may suppose that either $a^i \in S$ where $(i, 2p) = 1$ or $ba^j \in S$ where $(j, 2p) = 1$. Now with the similar arguments as before we get a 4-cycle in X. □

THEOREM 2.4. *Every connected Cayley graph $X = \mathrm{Cay}(H_3(p), S)$ contains a 4-cycle.*

Proof. Clearly $H = H_3(p) = \{a^i, ba^i, b^2a^i, b^3a^i \mid 0 \leq i \leq p-1\}$. Furthermore $o(ba^i) = o(b^3a^i) = 4$ and $o(b^2a^i) = 2$. Now we consider the following cases.
Case 1. $a^i \in S$, where $i \neq 0$.

In this case the map $a \mapsto a^i$, $b \mapsto b$ is an automorphism of $H_3(p)$ and so we may suppose that $\{a, a^{-1}\} \subseteq S$. Since $G = \langle S \rangle$, it follows that either $ba^i \in S$ or $b^3a^i \in S$. In both cases the map $a \mapsto a$, $b \mapsto b^ta^i$ ($t \in \{1, 3\}$) is an automorphism of $H_3(p)$. Thus $\{b, b^{-1}\} \subseteq S$ and $(1, b, b^2, b^3, 1)$ is a 4-cycle in X.

Case 2. $a^i \notin S$.

Since $G = \langle S \rangle$, one may assume that either $ba^i \in S$ or $b^3a^i \in S$. Also, the map $a \mapsto a$, $b \mapsto b^ta^i$ ($t \in \{1, 3\}$) is an automorphism of $H_3(p)$. Thus $\{b, b^{-1}\} \subseteq S$ and $(1, b, b^2, b^3, 1)$ is a 4-cycle in X. □

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