

INEQUALITIES FOR DOUBLE INTEGRALS OF SCHUR CONVEX FUNCTIONS ON SYMMETRIC AND CONVEX DOMAINS

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Abstract. In this paper, by making use of Green's identity for double integrals, we establish some integral inequalities for Schur convex functions defined on domains $D \subset \mathbb{R}^2$ that are symmetric, convex and have nonempty interiors. Examples for squares and disks are also provided.

1. Introduction

For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $x_{[1]} \geq \dots \geq x_{[n]}$ denote the components of x in decreasing order, and let $x_{\downarrow} = (x_{[1]}, \dots, x_{[n]})$ denote the decreasing rearrangement of x . For $x, y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, & k = 1, \dots, n-1; \\ \sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}. \end{cases}$$

When $x \prec y$, x is said to be *majorized* by y (y *majorizes* x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex. Perhaps "Schur-increasing" would be more appropriate, but the term "Schur-convex" is by now well entrenched in the literature, [3, p.80].

A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *Schur-convex* on \mathcal{A} if

$$x \prec y \text{ on } \mathcal{A} \Rightarrow \phi(x) \leq \phi(y). \quad (1)$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y , then ϕ is said to be *strictly Schur-convex* on \mathcal{A} . If $\mathcal{A} = \mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

2020 Mathematics Subject Classification: 26D15

Keywords and phrases: Schur convex functions; double integral inequalities.

For fundamental properties of Schur convexity see the monograph [3] and the references therein. For some recent results, see [1, 2, 4–6].

The following result is known in the literature as *Schur-Ostrowski theorem* [3, p.84]:

THEOREM 1.1. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \rightarrow \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \text{ is symmetric on } I^n, \quad (2)$$

and for all $i \neq j$, with $i, j \in \{1, \dots, n\}$,

$$(z_i - z_j) \left[\frac{\partial \phi(z)}{\partial x_i} - \frac{\partial \phi(z)}{\partial x_j} \right] \geq 0 \text{ for all } z \in I^n, \quad (3)$$

where $\frac{\partial \phi}{\partial x_k}$ denotes the partial derivative of ϕ with respect to its k -th argument.

With the aid of (2), condition (3) can be replaced by the condition

$$(z_1 - z_2) \left[\frac{\partial \phi(z)}{\partial x_1} - \frac{\partial \phi(z)}{\partial x_2} \right] \geq 0 \text{ for all } z \in I^n. \quad (4)$$

This simplified condition is sometimes more convenient to verify.

The above condition is not sufficiently general for all applications because the domain of ϕ may not be a Cartesian product.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) \mathcal{A} is *symmetric* in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π of the coordinates.

(ii) \mathcal{A} is convex and has a nonempty interior.

We have the following result, [3, p. 85].

THEOREM 1.2. *If ϕ is continuously differentiable on the interior of \mathcal{A} and continuous on \mathcal{A} , then necessary and sufficient conditions for ϕ to be Schur-convex on \mathcal{A} are*

$$\phi \text{ is symmetric on } \mathcal{A} \quad (5)$$

and
$$(z_1 - z_2) \left[\frac{\partial \phi(z)}{\partial x_1} - \frac{\partial \phi(z)}{\partial x_2} \right] \geq 0 \text{ for all } z \in \mathcal{A}. \quad (6)$$

It is well known that any symmetric convex function defined on a symmetric convex set \mathcal{A} is Schur convex, [3, p. 97]. If the function $\phi : \mathcal{A} \rightarrow \mathbb{R}$ is symmetric and quasi-convex, namely $\phi(\alpha u + (1 - \alpha)v) \leq \max\{\phi(u), \phi(v)\}$ for all $\alpha \in [0, 1]$ and $u, v \in \mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on \mathcal{A} [3, p.98].

In this paper we establish some integral inequalities for Schur convex functions defined on domains $D \subset \mathbb{R}^2$ that are symmetric, convex and have nonempty interiors. Examples for squares and disks are also provided.

2. Main results

For a function $f : D \rightarrow \mathbb{C}$ having partial derivatives on the domain $D \subset \mathbb{R}^2$ we define $\Lambda_{\partial f, D} : D \rightarrow \mathbb{C}$ as

$$\Lambda_{\partial f, D}(x, y) := (x - y) \left(\frac{\partial f(x, y)}{\partial x} - \frac{\partial f(x, y)}{\partial y} \right).$$

Let ∂D be a simple, closed counterclockwise curve in the xy -plane, bounding a region D . Let L and M be scalar functions defined at least on an open set containing D . Assume L and M have continuous first partial derivatives. Then the following equality is well known as the Green theorem:

$$\iint_D \left(\frac{\partial M(x, y)}{\partial x} - \frac{\partial L(x, y)}{\partial y} \right) dx dy = \oint_{\partial D} (L(x, y) dx + M(x, y) dy). \quad (7)$$

By applying this equality for real and imaginary parts, we can also state it for complex valued functions P and Q . Moreover, if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a, b]$, with x, y differentiable on (a, b) then we can calculate the path integral as

$$\oint_{\partial D} (L(x, y) dx + M(x, y) dy) = \int_a^b [L(x(t), y(t)) x'(t) + M(x(t), y(t)) y'(t)] dt.$$

We have the following identity of interest.

LEMMA 2.1. *Let ∂D be a simple, closed counterclockwise curve in the xy -plane, bounding a region D . Assume that the function $f : D \rightarrow \mathbb{C}$ has continuous partial derivatives on the domain D . Then*

$$\begin{aligned} & \frac{1}{2} \oint_{\partial D} [(x - y) f(x, y) dx + (x - y) f(x, y) dy] - \iint_D f(x, y) dx dy \\ &= \frac{1}{2} \iint_D \Lambda_{\partial f, D}(x, y) dx dy. \end{aligned} \quad (8)$$

Proof. Consider the functions $M(x, y) := (x - y) f(x, y)$ and $L(x, y) := (x - y) f(x, y)$ for $(x, y) \in D$.

We have

$$\frac{\partial}{\partial x} [(x - y) f(x, y)] = f(x, y) + (x - y) \frac{\partial f(x, y)}{\partial x}$$

and

$$\frac{\partial}{\partial y} [(y - x) f(x, y)] = f(x, y) + (y - x) \frac{\partial f(x, y)}{\partial y}$$

for $(x, y) \in D$.

If we add these two equalities, then we get

$$\frac{\partial M(x, y)}{\partial x} - \frac{\partial L(x, y)}{\partial y} = 2f(x, y) + \Lambda_{\partial f, D}(x, y) \quad (9)$$

for $(x, y) \in D$.

If we integrate this equality on D , then we obtain

$$\begin{aligned} & \iint_D \left(\frac{\partial M(x, y)}{\partial x} - \frac{\partial L(x, y)}{\partial y} \right) dx dy \\ &= 2 \iint_D f(x, y) dx dy + \iint_D \Lambda_{\partial f, D}(x, y) dx dy. \end{aligned} \quad (10)$$

From Green's identity we also have

$$\begin{aligned} & \iint_D \left(\frac{\partial M(x, y)}{\partial x} - \frac{\partial L(x, y)}{\partial y} \right) dx dy = \oint_{\partial D} (L(x, y) dx + M(x, y) dy) \\ &= \oint_{\partial D} [(x - y) f(x, y) dx + (x - y) f(x, y) dy]. \end{aligned} \quad (11)$$

By employing (10) and (11) we deduce the desired equality (8). \square

COROLLARY 2.2. *With the assumptions of Lemma 2.1 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a, b]$, with x, y differentiable on (a, b) , then*

$$\begin{aligned} & \frac{1}{2} \int_a^b (x(t) - y(t)) f(x(t), y(t)) (x'(t) + y'(t)) dt - \iint_D f(x, y) dx dy \\ &= \frac{1}{2} \iint_D \Lambda_{\partial f, D}(x, y) dx dy. \end{aligned} \quad (12)$$

We have the following result for Schur convex functions defined on symmetric convex domains of \mathbb{R}^2 .

THEOREM 2.3. *Let $D \subset \mathbb{R}^2$ be symmetric, convex and has a nonempty interior. If ϕ is continuously differentiable on the interior of D , continuous and Schur convex on D and ∂D is a simple, closed counterclockwise curve in the xy -plane bounding D , then*

$$\iint_D \phi(x, y) dx dy \leq \frac{1}{2} \oint_{\partial D} [(x - y) \phi(x, y) dx + (x - y) \phi(x, y) dy]. \quad (13)$$

If ϕ is Schur concave on D , then the sign of inequality reverses in (13).

The proof follows by Lemma 2.1 and Theorem 1.1.

COROLLARY 2.4. *Let $D \subset \mathbb{R}^2$ be symmetric, convex and has a nonempty interior. If ϕ is continuously differentiable on the interior of D , continuous and convex or quasi-convex on D and ∂D is a simple, closed counterclockwise curve in the xy -plane bounding D , then the inequality (13) is valid.*

REMARK 2.5. With the assumptions of Theorem 2.3 and if the curve ∂D is described by the function $r(t) = (x(t), y(t))$, $t \in [a, b]$, with x, y differentiable on (a, b) , then

$$\iint_D \phi(x, y) dx dy \leq \frac{1}{2} \int_a^b (x(t) - y(t)) \phi(x(t), y(t)) (x'(t) + y'(t)) dt. \quad (14)$$

Let $a < b$. Put $A = (a, a)$, $B = (b, a)$, $C = (b, b)$, $D = (a, b) \in \mathbb{R}^2$ the vertices of the square $ABCD = [a, b]^2$. Consider the counterclockwise segments

$$AB : \begin{cases} x = (1-t)a + tb, & t \in [0, 1] \\ y = a \end{cases}$$

$$BC : \begin{cases} x = b \\ y = (1-t)a + tb, & t \in [0, 1] \end{cases}$$

$$CD : \begin{cases} x = (1-t)b + ta \\ y = b, & t \in [0, 1] \end{cases}$$

and

$$DA : \begin{cases} x = a \\ y = (1-t)b + ta, & t \in [0, 1]. \end{cases}$$

Therefore $\partial(ABCD) = AB \cup BC \cup CD \cup DA$.

For any function f defined on $ABCD$, we have

$$\begin{aligned} & \oint_{AB} [(x-y)f(x,y)dx + (x-y)f(x,y)dy] \\ &= (b-a) \int_0^1 ((1-t)a + tb - a) f((1-t)a + tb, a) dt \\ &= (b-a)^2 \int_0^1 t f((1-t)a + tb, a) dt, \end{aligned}$$

$$\begin{aligned} & \oint_{BC} [(x-y)f(x,y)dx + (x-y)f(x,y)dy] \\ &= (b-a) \int_0^1 (b - (1-t)a - tb) f(b, (1-t)a + tb) dt \\ &= (b-a)^2 \int_0^1 (1-t) f(b, (1-t)a + tb) dt, \end{aligned}$$

$$\begin{aligned} & \oint_{CD} [(x-y)f(x,y)dx + (x-y)f(x,y)dy] \\ &= (a-b) \int_0^1 ((1-t)b + ta - b) f((1-t)b + ta, b) dt \\ &= (a-b)^2 \int_0^1 t f((1-t)b + ta, b) dt \\ &= (a-b)^2 \int_0^1 (1-t) f((1-t)a + tb, b) dt \text{ (by change of variable)}. \end{aligned}$$

and

$$\oint_{DA} [(x-y)f(x,y)dx + (x-y)f(x,y)dy]$$

$$\begin{aligned}
&= (a-b) \int_0^1 (a - (1-t)b - ta) f(a, (1-t)b + ta) dt \\
&= (a-b)^2 \int_0^1 (1-t) f(a, (1-t)b + ta) dt \\
&= (a-b)^2 \int_0^1 t f(a, (1-t)a + tb) dt \text{ (by change of variable)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\oint_{\partial(ABCD)} [(x-y) f(x, y) dx + (x-y) f(x, y) dy] \tag{15} \\
&= (b-a)^2 \int_0^1 t f((1-t)a + tb, a) dt + (b-a)^2 \int_0^1 (1-t) f(b, (1-t)a + tb) dt \\
&\quad + (b-a)^2 \int_0^1 (1-t) f((1-t)a + tb, b) dt + (b-a)^2 \int_0^1 t f(a, (1-t)a + tb) dt \\
&= (b-a)^2 \int_0^1 t [f((1-t)a + tb, a) + f(a, (1-t)a + tb)] dt \\
&\quad + (b-a)^2 \int_0^1 (1-t) [f(b, (1-t)a + tb) + f((1-t)a + tb, b)] dt.
\end{aligned}$$

Since the vast majority of examples of Schur convex functions are defined on the Cartesian product of intervals, we can state the following result of interest.

COROLLARY 2.6. *If ϕ is continuously differentiable on the interior of $D = [a, b]^2$, continuous on D and Schur convex, then*

$$\begin{aligned}
\frac{1}{(b-a)^2} \int_a^b \int_a^b \phi(x, y) dx dy &\leq \int_0^1 t \phi((1-t)a + tb, a) dt \tag{16} \\
&\quad + \int_0^1 (1-t) \phi((1-t)a + tb, b) dt.
\end{aligned}$$

Proof. From (13) we get

$$\begin{aligned}
&\frac{1}{(b-a)^2} \int_a^b \int_a^b \phi(x, y) dx dy \tag{17} \\
&\leq \int_0^1 t \left[\frac{\phi((1-t)a + tb, a) + \phi(a, (1-t)a + tb)}{2} \right] dt \\
&\quad + \int_0^1 (1-t) \left[\frac{\phi((1-t)a + tb, b) + \phi(b, (1-t)a + tb)}{2} \right] dt.
\end{aligned}$$

Since ϕ is symmetric on $D = [a, b]^2$, hence

$$\phi((1-t)a + tb, a) = \phi(a, (1-t)a + tb)$$

and

$$\phi((1-t)a + tb, b) = \phi(b, (1-t)a + tb)$$

for all $t \in [0, 1]$ and by (17) we get (16). \square

REMARK 2.7. By making the change of variable $x = (1-t)a + tb$, $t \in [0, 1]$, then $dx = (b-a)dt$, $t = \frac{x-a}{b-a}$ and by (16) we get

$$\begin{aligned} & \frac{1}{(b-a)^2} \int_a^b \int_a^b \phi(x, y) dx dy \\ & \leq \frac{1}{b-a} \int_a^b \frac{x-a}{b-a} \phi(x, a) dx + \frac{1}{b-a} \int_a^b \frac{b-x}{b-a} \phi(x, b) dx, \end{aligned} \quad (18)$$

or, equivalently,

$$\int_a^b \int_a^b \phi(x, y) dx dy \leq \int_a^b (x-a) \phi(x, a) dx + \int_a^b (b-x) \phi(x, b) dx. \quad (19)$$

3. Lower and upper Schur convexity

Start with the following extensions of Schur convex functions:

DEFINITION 3.1. Let D be symmetric, convex and has a nonempty interior in \mathbb{R}^2 and a symmetric function $f : D \rightarrow \mathbb{R}$ having continuous partial derivatives on $D \subset \mathbb{R}^2$.

(i) For $m \in \mathbb{R}$, f is called m -lower Schur convex on D if

$$m(x-y)^2 \leq \Lambda_{\partial f, D}(x, y) \text{ for all } (x, y) \in D. \quad (20)$$

(ii) For $M \in \mathbb{R}$, f is called M -upper Schur convex on D if

$$\Lambda_{\partial f, D}(x, y) \leq M(x-y)^2 \text{ for all } (x, y) \in D. \quad (21)$$

(iii) For $m, M \in \mathbb{R}$ with $m < M$, f is called (m, M) -Schur convex on D if

$$m(x-y)^2 \leq \Lambda_{\partial f, D}(x, y) \leq M(x-y)^2 \text{ for all } (x, y) \in D. \quad (22)$$

We have the following simple but useful result.

PROPOSITION 3.2. Let D be symmetric, convex and has a nonempty interior in \mathbb{R}^2 and a symmetric function $f : D \rightarrow \mathbb{R}$ having continuous partial derivatives on $D \subset \mathbb{R}^2$.

(i) For $m \in \mathbb{R}$, f is m -lower Schur convex on D iff $f_m : D \rightarrow \mathbb{R}$,

$$f_m(x, y) := f(x, y) - \frac{1}{2}m(x^2 + y^2)$$

is Schur convex on D .

(ii) For $M \in \mathbb{R}$, f is M -upper Schur convex on D iff $f_M : D \rightarrow \mathbb{R}$,

$$f_M(x, y) := \frac{1}{2}M(x^2 + y^2) - f(x, y)$$

is Schur convex on D .

(iii) For $m, M \in \mathbb{R}$ with $m < M$, f is (m, M) -Schur convex on D iff f_m and f_M are Schur convex on D .

Proof. (i) Observe that

$$\begin{aligned}\Lambda_{\partial f_m, D}(x, y) &= (x - y) \left(\frac{\partial f_m(x, y)}{\partial x} - \frac{\partial f_m(x, y)}{\partial y} \right) \\ &= (x - y) \left(\frac{\partial f(x, y)}{\partial x} - mx - \frac{\partial f(x, y)}{\partial y} + my \right) \\ &= (x - y) \left(\frac{\partial f(x, y)}{\partial x} - \frac{\partial f(x, y)}{\partial y} - m(x - y) \right) \\ &= \Lambda_{\partial f, D}(x, y) - m(x - y)^2,\end{aligned}$$

for all $(x, y) \in D$, which proves the statement.

The statements (ii) and (iii) follow in a similar way. \square

THEOREM 3.3. *Let ∂D be a simple, closed counterclockwise curve in the xy -plane, bounding a domain $D \subset \mathbb{R}^2$ that is symmetric, convex and has a nonempty interior.*

(i) *Assume that the function $f : D \rightarrow \mathbb{R}$ is m -lower Schur convex, then*

$$\begin{aligned}& \frac{1}{2}m \iint_D (x - y)^2 dx dy \\ & \leq \frac{1}{2} \oint_{\partial D} [(x - y) f(x, y) dx + (x - y) f(x, y) dy] - \iint_D f(x, y) dx dy.\end{aligned}\tag{23}$$

(ii) *Assume that the function $f : D \rightarrow \mathbb{R}$ is M -upper Schur convex, then*

$$\begin{aligned}& \frac{1}{2} \oint_{\partial D} [(x - y) f(x, y) dx + (x - y) f(x, y) dy] - \iint_D f(x, y) dx dy \\ & \leq \frac{1}{2}M \iint_D (x - y)^2 dx dy.\end{aligned}\tag{24}$$

(iii) *Assume that the function $f : D \rightarrow \mathbb{R}$ is (m, M) -Schur convex, then*

$$\begin{aligned}& \frac{1}{2}m \iint_D (x - y)^2 dx dy \\ & \leq \frac{1}{2} \oint_{\partial D} [(x - y) f(x, y) dx + (x - y) f(x, y) dy] - \iint_D f(x, y) dx dy \\ & \leq \frac{1}{2}M \iint_D (x - y)^2 dx dy.\end{aligned}\tag{25}$$

Proof. (i) Since $f_m(x, y) := f(x, y) - \frac{1}{2}m(x^2 + y^2)$ is Schur convex on D , then by (13) we get

$$\iint_D f_m(x, y) dx dy \leq \frac{1}{2} \oint_{\partial D} [(x - y) f_m(x, y) dx + (x - y) f_m(x, y) dy],$$

namely
$$\iint_D \left[f(x, y) - \frac{1}{2}m(x^2 + y^2) \right] dx dy \tag{26}$$

$$\begin{aligned}
& \leq \frac{1}{2} \oint_{\partial D} \left\{ (x-y) \left[f(x,y) - \frac{1}{2}m(x^2+y^2) \right] dx \right. \\
& \quad \left. + (x-y) \left[f(x,y) - \frac{1}{2}m(x^2+y^2) \right] dy \right\}. \\
\text{Since } & \iint_D \left[f(x,y) - \frac{1}{2}m(x^2+y^2) \right] dx dy = \iint_D f(x,y) dx dy \\
& \quad - \frac{1}{2}m \iint_D (x^2+y^2) dx dy \\
\text{and } & \frac{1}{2} \oint_{\partial D} \left\{ (x-y) \left[f(x,y) - \frac{1}{2}m(x^2+y^2) \right] dx \right. \\
& \quad \left. + (x-y) \left[f(x,y) - \frac{1}{2}m(x^2+y^2) \right] dy \right\} \\
& = \frac{1}{2} \oint_{\partial D} [(x-y)f(x,y) dx + (x-y)f(x,y) dy] \\
& \quad - \frac{1}{4}m \oint_{\partial D} [(x^2+y^2) dx + (x^2+y^2) dy],
\end{aligned}$$

hence, by (26), we get

$$\begin{aligned}
& \frac{1}{2}m \left\{ \frac{1}{2} \oint_{\partial D} [(x-y)(x^2+y^2) dx + (x-y)(x^2+y^2) dy] - \iint_D (x^2+y^2) dx dy \right\} \\
& \leq \frac{1}{2} \oint_{\partial D} [(x-y)f(x,y) dx + (x-y)f(x,y) dy] - \iint_D f(x,y) dx dy. \quad (27)
\end{aligned}$$

Further, if we use the identity (8) for the function $g(x,y) = x^2 + y^2$ we get

$$\begin{aligned}
& \frac{1}{2} \oint_{\partial D} [(x-y)(x^2+y^2) dx + (x-y)(x^2+y^2) dy] - \iint_D (x^2+y^2) dx dy \\
& = \frac{1}{2} \iint_D 2(x-y)^2 dx dy = \iint_D (x-y)^2 dx dy,
\end{aligned}$$

which together with (27) gives the desired result (23).

The statements (ii) and (iii) follow in a similar way and we omit the details. \square

If f is symmetric on D , for all $(x,y) \in D$, we have

$$\Lambda_{\partial f,D}(x,y) = (x-y) \left(\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y} \right) = (x-y) \left(\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(y,x)}{\partial x} \right).$$

$$\text{If } 0 < k \leq \left| \frac{\frac{\partial f(x,y)}{\partial x} - \frac{\partial f(y,x)}{\partial x}}{x-y} \right| \leq K < \infty \text{ for all } (x,y) \in D \text{ with } x \neq y, \quad (28)$$

then $0 \leq k(x-y)^2 \leq \Lambda_{\partial f,D}(x,y) \leq K(x-y)^2$ for all $(x,y) \in D$.

By making use of Theorem 3.3 we can state the following result.

COROLLARY 3.4. *Let ∂D be a simple, closed counterclockwise curve in the xy -plane, bounding a domain $D \subset \mathbb{R}^2$ that is symmetric, convex and has a nonempty interior. If f is continuously differentiable on the interior of D , continuous and symmetric on D and the partial derivative $\frac{\partial f}{\partial x}$ satisfies the condition (28), then we have the inequalities*

$$\begin{aligned} 0 &\leq \frac{1}{2}k \iint_D (x-y)^2 dx dy & (29) \\ &\leq \frac{1}{2} \oint_{\partial D} [(x-y) f(x,y) dx + (x-y) f(x,y) dy] - \iint_D f(x,y) dx dy \\ &\leq \frac{1}{2}K \iint_D (x-y)^2 dx dy. \end{aligned}$$

REMARK 3.5. If $D = [a, b]^2$ and since

$$\int_a^b \int_a^b (x-y)^2 dx dy = \int_a^b \frac{(b-x)^3 + (x-a)^3}{3} dx = \frac{1}{6} (b-a)^4$$

hence by (29) we get

$$\begin{aligned} 0 &\leq \frac{1}{12}k (b-a)^4 & (30) \\ &\leq \int_a^b (x-a) f(x,a) dx + \int_a^b (b-x) f(x,b) dx - \int_a^b \int_a^b f(x,y) dx dy \\ &\leq \frac{1}{12}K (b-a)^4, \end{aligned}$$

provided that f is *continuously differentiable* on the interior of $[a, b]^2$, continuous and symmetric on $[a, b]^2$ and the partial derivative $\frac{\partial f}{\partial x}$ satisfies the condition (28).

4. Examples for disks

We consider the closed disk $D(O, R)$ centered in $O(0, 0)$ and of radius $R > 0$, parameterized by

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta, \quad r \in [0, R], \theta \in [0, 2\pi], \end{cases}$$

and the circle $\mathcal{C}(O, R)$, parameterized by

$$\begin{cases} x = R \cos \theta \\ y = R \sin \theta, \quad \theta \in [0, 2\pi]. \end{cases}$$

Observe that, if $\phi : D(O, R) \rightarrow \mathbb{R}$, then

$$\oint_{\mathcal{C}(O, R)} [(x-y) \phi(x,y) dx + (x-y) \phi(x,y) dy]$$

$$\begin{aligned}
&= - \int_0^{2\pi} R(R \cos \theta - R \sin \theta) \sin \theta \phi(R \cos \theta, R \sin \theta) d\theta \\
&\quad + \int_0^{2\pi} R(R \cos \theta - R \sin \theta) \cos \theta \phi(R \cos \theta, R \sin \theta) d\theta \\
&= R^2 \int_0^{2\pi} \phi(R \cos \theta, R \sin \theta) (\cos \theta - \sin \theta)^2 d\theta.
\end{aligned}$$

Also, we have
$$\iint_{D(O,R)} \phi(x, y) dx dy = \int_0^R \int_0^{2\pi} \phi(r \cos \theta, r \sin \theta) r dr d\theta.$$

Using Theorem 2.3 we can state the following result.

PROPOSITION 4.1. *If ϕ is continuously differentiable on the interior of $D(O, R)$, continuous and Schur convex on $D(O, R)$, then*

$$\int_0^R \int_0^{2\pi} \phi(r \cos \theta, r \sin \theta) r dr d\theta \leq \frac{1}{2} R^2 \int_0^{2\pi} \phi(R \cos \theta, R \sin \theta) (\cos \theta - \sin \theta)^2 d\theta. \quad (31)$$

Now, observe that

$$\begin{aligned}
\iint_{D(O,R)} (x - y)^2 dx dy &= \int_0^R \int_0^{2\pi} (R \cos \theta - R \sin \theta)^2 r dr d\theta \\
&= \frac{1}{2} R^4 \int_0^{2\pi} (\cos \theta - \sin \theta)^2 d\theta = \frac{1}{2} R^4 \int_0^{2\pi} (1 - 2 \sin \theta \cos \theta) d\theta = \pi R^4.
\end{aligned}$$

By Corollary 3.4, the following holds.

PROPOSITION 4.2. *If ϕ is continuously differentiable on the interior of $D(O, R)$, continuous and Schur convex on $D(O, R)$ and the derivative $\frac{\partial f}{\partial x}$ satisfies the condition (28) on $D(O, R)$, then*

$$\begin{aligned}
\frac{1}{2} \pi k R^4 &\leq \frac{1}{2} R^2 \int_0^{2\pi} \phi(R \cos \theta, R \sin \theta) (\cos \theta - \sin \theta)^2 d\theta \\
&\quad - \int_0^R \int_0^{2\pi} \phi(r \cos \theta, r \sin \theta) r dr d\theta \leq \frac{1}{2} \pi K R^4.
\end{aligned}$$

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(received 18.07.2019; in revised form 06.05.2020; available online 07.12.2020)

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