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RESULTS ON AMALGAMATION ALONG A SEMIDUALIZING IDEAL

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Abstract. Let R be a commutative Noetherian ring and let I be a semidualizing ideal of R. In this paper, it is shown that the G_I-projective, G_I-injective, and G_I-flat dimensions agree with Gpd $_{R\bowtie I}(-)$, Gid $_{R\bowtie I}(-)$, and Gfd $_{R\bowtie I}(-)$, respectively. Also, it is proved that for a non-negative integer n if $\sup\{\mathcal{GP}_I-\mathrm{pd}_R(M) \mid M \in \mathcal{M}(R)\} \leq n$ (or $\sup\{\mathcal{GI}_I-\mathrm{id}_R(M) \mid M \in \mathcal{M}(R)\} \leq n$), then for every projective $(R \bowtie I)$-module P we have $\mathrm{id}_{R\bowtie I}(P) \leq n$, and for every injective $(R \bowtie I)$-module E we have $\mathrm{pd}_{R\bowtie I}(E) \leq n$.

1. Introduction

Throughout this paper R is a commutative Noetherian ring and all modules are unital. Recall that for an R-module M the idealization $R \ltimes M$ (also called trivial extension) introduced by Nagata in 1956 [13, Page 2], is a new ring where the module M can be viewed as an ideal such that its square is 0. In [4], D'Anna and Fontana considered a different type of construction obtained involving a ring R and an ideal $I \subset R$ that is denoted by $R \bowtie I$, called amalgamated duplication, and it is defined $R \bowtie I = \{(r, r+i) \mid r \in R, i \in I\}$, as a subring of $R \times R$. The properties of the ring $R \bowtie I$ were studied extensively in [1,3–5,14,17]. Also, in [15], the authors focused on the properties of $R \bowtie I$, when I is a semidualizing ideal of R, i.e., I is an ideal of R and I is a semidualizing R-module. The notion of a "semidualizing module" was first introduced by Foxby [8], and then Vasconcelos [18] and Golod [9] rediscovered these modules using different terminology for different purposes.

In [11], the authors showed that how a semidualizing module C gives rise to three new relative homological dimensions which are called G_C-projective, G_C-injective, and G_C-flat dimension. Also, they investigated the properties of these dimensions and they suggested the view point that one should change ring from R to $R \ltimes C$ and they showed that the G_C-projective, G_C-injective, and G_C-flat dimensions always

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agree with the ring changed Gorenstein dimensions Gpd $_{R\ltimes C}(-)$, Gid $_{R\ltimes C}(-)$, and Gfd $_{R\ltimes C}(-)$, respectively.

This paper builds on work of Holm and Jörgensen [11] for the ring $R \bowtie I$, where I is a semidualizing ideal, instead of idealization. In particular, it is shown that for a semidualizing ideal I the G_I-projective, G_I-injective, and G_I-flat dimensions agree with Gpd $_{R\bowtie I}(-)$, Gid $_{R\bowtie I}(-)$, and Gfd $_{R\bowtie I}(-)$, respectively. Also, we give some homological properties of $(R \bowtie I)$-modules, where I is a semidualizing ideal of the ring R. In particular, it is proved that for a non-negative integer n if $\sup\{\mathcal{GP}_I - \mathrm{pd}_R(M) \mid M \in \mathcal{M}(R)\} \leq n$ (or $\sup\{\mathcal{GI}_I - \mathrm{id}_R(M) \mid M \in \mathcal{M}(R)\} \leq n$), then for every projective $(R \bowtie I)$-module P we have $\mathrm{id}_{R\bowtie I}(P) \leq n$, and for every injective $(R \bowtie I)$-module E we have $\mathrm{pd}_{R\bowtie I}(E) \leq n$.

2. Background material

Throughout this paper $\mathcal{M}(R)$ denotes the category of R-modules. We use the term "subcategory" to mean a "full, additive subcategory $\mathcal{X} \subseteq \mathcal{M}(R)$ such that, for all R-modules M and N, if $M \cong N$ and $M \in \mathcal{X}$, then $N \in \mathcal{X}$". Write $\mathcal{P}(R)$, $\mathcal{F}(R)$ and $\mathcal{I}(R)$ for the subcategories of projective, flat and injective R-modules, respectively.

Definition 2.1. An *R-complex* is a sequence $Y = \cdots \xrightarrow{\partial^Y_{n+1}} Y_n \xrightarrow{\partial^Y_n} Y_{n-1} \xrightarrow{\partial^Y_{n-1}} \cdots$ of R-modules and R-homomorphisms such that $\partial^Y_{n-1}\partial^Y_n = 0$ for each integer n. Let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$. The R-complex Y is $\mathrm{Hom}_R(\mathcal{X}, -)$-exact if for each X in $\mathcal{X}$, the complex $\mathrm{Hom}_R(X, Y)$ is exact, and similarly for $\mathrm{Hom}_R(-, \mathcal{X})$-exact.

The notion of semidualizing modules, defined next, goes back at least to Foxby [8], but was rediscovered by others.

Definition 2.2. A finitely generated R-module C is called *semidualizing* if the natural homothety homomorphism $\chi^R_C : R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^{\geq 1}_R(C, C) = 0$.

Definition 2.3. Let C be a semidualizing R-module. An R-module is *C-projective* (resp. *C-flat* or *C-injective*) if it is isomorphic to a module of the form $P \otimes_R C$ for some projective R-module P (resp. $F \otimes_R C$ for some flat R-module F or $\mathrm{Hom}_R(C, I)$ for some injective R-module I). We let $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ denote the categories of C-projective, C-flat and C-injective R-modules, respectively.

The next two classes were also introduced by Foxby [8].

Definition 2.4. Let C be a semidualizing R-module. The *Auslander class* with respect to C is the class $\mathcal{A}_C(R)$ of R-modules M such that:
(i) $\mathrm{Tor}^R_i(C, M) = 0 = \mathrm{Ext}^i_R(C, C \otimes_R M)$ for all $i \geq 1$, and

(ii) the natural map $\gamma^M_C : M \to \mathrm{Hom}_R(C, C) \otimes_R M$ is an isomorphism.
The *Bass class* with respect to C is the class $\mathcal{B}_C(R)$ of R-modules M such that:

(i) $\mathrm{Ext}^i_R(C,M)=0=\mathrm{Tor}^R_i(C,\mathrm{Hom}_R(C,M))$ for all $i\geq 1$, and

(ii) the natural evaluation map $\xi^C_M : C\otimes_R \mathrm{Hom}_R(C,M)\to M$ is an isomorphism.

The notion of precovers and preenvelopes, defined next, are from [6].

DEFINITION 2.5. Let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$. An *$\mathcal{X}$-precover* of an R-module M is an R-module homomorphism $X\xrightarrow{\varphi} M$, where $X\in\mathcal{X}$, and such that the map $\mathrm{Hom}_R(X',\varphi)$ is surjective for every $X'\in\mathcal{X}$. If every R-module admits $\mathcal{X}$-precover, then the class $\mathcal{X}$ is *precovering*. The notions of *$\mathcal{X}$-preenvelope* and *preenveloping* are defined dually.

DEFINITION 2.6. Let C be a semidualizing R-module. In [12], it is shown that the class $\mathcal{P}_C(R)$ is precovering. So, one can iteratively take precovers to construct an *augmented proper $\mathcal{P}_C$-projective resolution* for any R-module M, that is, a complex $X^+=\cdots\longrightarrow C\otimes_R P_1\longrightarrow C\otimes_R P_0\longrightarrow M\longrightarrow 0$ which is $\mathrm{Hom}_R(\mathcal{P}_C(R),-)$-exact. The truncated complex $X=\cdots\longrightarrow C\otimes_R P_1\longrightarrow C\otimes_R P_0\longrightarrow 0$ is a *proper $\mathcal{P}_C$-projective resolution* of M.

Dually, in [12] it is proved that the class $\mathcal{I}_C(R)$ is enveloping. So, for an R-module N one can construct an *augmented proper $\mathcal{I}_C$-injective resolution*, that is, a complex $Y^+=0\longrightarrow N\longrightarrow \mathrm{Hom}_R(C,I^0)\longrightarrow \mathrm{Hom}_R(C,I^1)\longrightarrow\cdots$ which is $\mathrm{Hom}_R(-,\mathcal{I}_C(R))$-exact. Also, in [12] it is shown that the class $\mathcal{F}_C(R)$ is covering. Similarly for an R-module M one can construct an *augmented proper $\mathcal{F}_C$-flat resolution.*

FACT 2.7. *Note that X^+ and Y^+ need not be exact. In [16, Corollary 2.4], it is proved that if M is in $\mathcal{B}_C(R)$ (resp. $\mathcal{A}_C(R)$), then every augmented proper $\mathcal{P}_C$-projective resolution (resp. $\mathcal{I}_C$-injective resolution) of M is exact.*

DEFINITION 2.8. Let C be a semidualizing R-module and let M be an R-module. The *$\mathcal{P}_C$-projective dimension* of M is $\mathcal{P}_C-\mathrm{pd}_R(M)=\inf\{\sup\{n\mid X_n\neq 0\}\mid X$ is a proper $\mathcal{P}_C$-projective resolution of $M\}$. The *$\mathcal{F}_C$-projective dimension*, denoted $\mathcal{F}_C-\mathrm{pd}_R(-)$ is defined similarly and the *$\mathcal{I}_C$-injective dimension*, denoted $\mathcal{I}_C-\mathrm{id}_R(-)$ is defined dually.

FACT 2.9 ([16, Theorem 2.11]). *Let C be a semidualizing R-module. Then for every R-module M, we have the following statements.*
(i) $\mathrm{pd}_R(M)=\mathcal{P}_C\text{-}\mathrm{pd}_R(C\otimes_R M)$ *and* $\mathcal{P}_C\text{-}\mathrm{pd}_R(M)=\mathrm{pd}_R(\mathrm{Hom}_R(C,M))$.

(ii) $\mathcal{I}_C\text{-}\mathrm{id}_R(M)=\mathrm{id}_R(C\otimes_R M)$ *and* $\mathrm{id}_R(M)=\mathcal{I}_C\text{-}\mathrm{id}_R(\mathrm{Hom}_R(C,M))$.

DEFINITION 2.10 ([11]). Let C be a semidualizing R-module. A *complete $\mathcal{I}_C\mathcal{I}$-resolution* is a complex Y of R-modules satisfying the following:
(i) Y is exact and $\mathrm{Hom}_R(I,Y)$ is exact for each $I\in\mathcal{I}_C(R)$, and

(ii) $Y_i\in\mathcal{I}_C(R)$ for all $i\geq 0$ and Y_i is injective for all $i<0$.

An R-module M is *G_C-injective* if there exists a complete $\mathcal{I}_C\mathcal{I}$-resolution Y such that $M\cong\mathrm{Coker}(\partial^Y_1)$; in this case Y is a *complete $\mathcal{I}_C\mathcal{I}$-resolution* of M. The class of all G_C-injective R-modules is denoted by $\mathcal{GI}_C(R)$. In the case $C = R$, we use the more common terminology "complete injective resolution" and "Gorenstein injective module" and the notation $\mathcal{GI}(R)$.

A *complete $\mathcal{PP}_C$-resolution* is a complex X of R-modules such that:

(i) X is exact and $\mathrm{Hom}_R(X, P)$ is exact for each $P \in \mathcal{P}_C(R)$, and

(ii) X_i is projective for all $i \geq 0$ and $X_i \in \mathcal{P}_C(R)$ for all $i < 0$.

An R-module M is *G_C-projective* if there exists a complete $\mathcal{PP}_C$-resolution X such that $M \cong \mathrm{Coker}(\partial_1^X)$; in this case X is a *complete $\mathcal{PP}_C$-resolution* of M. The class of all G_C-projective R-modules is denoted by $\mathcal{GP}_C(R)$. In the case $C = R$, we use the more common terminology "complete projective resolution" and "Gorenstein projective module" and the notation $\mathcal{GP}(R)$.

A *complete $\mathcal{FF}_C$-resolution* is a complex Z of R-modules such that:

(i) Z is exact and $Z \otimes_R I$ is exact for each $I \in \mathcal{I}_C(R)$, and

(ii) Z_i is flat for all $i \geq 0$ and $Z_i \in \mathcal{F}_C(R)$ for all $i < 0$.

An R-module M is *G_C-flat* if there exists a complete $\mathcal{FF}_C$-resolution Z such that $M \cong \mathrm{Coker}(\partial_1^Z)$; in this case Z is a *complete $\mathcal{FF}_C$-resolution* of M. The class of all G_C-flat R-modules is denoted by $\mathcal{GF}_C(R)$. In the case $C = R$, we use the more common terminology "complete flat resolution" and "Gorenstein flat module" and the notation $\mathcal{GF}(R)$.

Fact 2.11 ([11]). *Let C be a semidualizing module of the ring R. Then the following statements hold:*

(i) $\mathcal{P}(R) \subseteq \mathcal{GP}_C(R)$ and $\mathcal{P}_C(R) \subseteq \mathcal{GP}_C(R)$.

(ii) $\mathcal{I}(R) \subseteq \mathcal{GI}_C(R)$ and $\mathcal{I}_C(R) \subseteq \mathcal{GI}_C(R)$.

(iii) $\mathcal{F}(R) \subseteq \mathcal{GF}_C(R)$ and $\mathcal{F}_C(R) \subseteq \mathcal{GF}_C(R)$.

Definition 2.12. Let C be a semidualizing module of the ring R and let M be an R-module. A *$\mathcal{GP}_C$-resolution* of M is a complex of R-modules in $\mathcal{GP}_C(R)$ of the form $X = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \longrightarrow 0$ such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \geq 1$. The *$\mathcal{GP}_C$-projective dimension* of M is the quantity $\mathcal{GP}_C - \mathrm{pd}_R(M) = \inf\{\sup\{n \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{GP}_C\text{-resolution of } M\}$.

In particular, $\mathcal{GP}_C - \mathrm{pd}_R(0) = -\infty$. The modules of $\mathcal{GP}_C$-projective dimension zero are the non-zero modules in $\mathcal{GP}_C(R)$. The *$\mathcal{GF}_C$-resolution* and *$\mathcal{GF}_C$-projective dimension* are defined similarly.

Dually, an *$\mathcal{GI}_C$-coresolution* of M is a complex of R-modules in $\mathcal{GI}_C(R)$ of the form $X = 0 \longrightarrow X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots$ such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \leq -1$. The *$\mathcal{GI}_C$-injective dimension* of M is the quantity $\mathcal{GI}_C - \mathrm{id}_R(M) = \inf\{\sup\{n \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{GI}_C\text{-coresolution of } M\}$.

In particular, $\mathcal{GI}_C - \mathrm{id}_R(0) = -\infty$. The modules of $\mathcal{GI}_C$-injective dimension zero are the non-zero modules in $\mathcal{GI}_C(R)$.

3. Amalgamation along a semidualizing ideal and relative Gorenstein homological dimensions

The first aim of this section is to show that for a semidualizing ideal I of the ring R, i.e., I is an ideal of R and I is a semidualizing R-module, the G_I-projective, G_I-injective, and G_I-flat dimensions agree with $\mathrm{Gpd}_{R\bowtie I}(-)$, $\mathrm{Gid}_{R\bowtie I}(-)$, and $\mathrm{Gfd}_{R\bowtie I}(-)$, respectively.

First, we deal with some applications of a general construction, introduced in [4], called amalgamated duplication of a ring along an ideal.

Let R be a commutative ring with unit element 1 and let I be an ideal of R. Set $R \bowtie I = \{(r, s) \mid r, s \in R, s - r \in I\}$. It is easy to check that $R \bowtie I$ is a subring, with unit element $(1, 1)$, of $R \times R$ (with the usual componentwise operations) and that $R \bowtie I = \{(r, r + i) \mid r \in R, i \in I\}$. In the following, we recall some main properties of the ring $R \bowtie I$ from [3] which will be important later on.

Proposition 3.1. *Let R be a ring and let I be an ideal of R. Then the following statements hold.*
(i) By introducing a multiplicative structure in the R-module direct sum $R \oplus I$ by setting $(r, i)(s, j) = (rs, rj + si + ij)$, the map $f : R \oplus I \to R \bowtie I$ defined by $f((r, i)) = (r, r + i)$ is a ring isomorphism and R-isomorphism too. Moreover, there is an exact sequence of R-modules $0 \longrightarrow R \xrightarrow{\varphi} R \bowtie I \xrightarrow{\psi} I \longrightarrow 0$ where $\varphi(r) = (r, r)$ for all $r \in R$, and $\psi((r, s)) = s - r$, for all $(r, s) \in R \bowtie I$. Notice that this sequence splits; hence we also have the short exact sequence of R-modules $0 \longrightarrow I \xrightarrow{\psi'} R \bowtie I \xrightarrow{\varphi'} R \longrightarrow 0$, where $\psi'(i) = (0, i)$ and $\varphi'((r, s)) = r$, for every $i \in I$ and $(r, s) \in R \bowtie I$.

(ii) R and $R \bowtie I$ have the same Krull dimension. Also, if R is a Noetherian ring, then $R \bowtie I$ is a finitely generated R-module.

In [1, 3–5, 14, 17], the properties of the ring $R \bowtie I$ were studied extensively. In addition, in [15], the authors focused on the properties of $R \bowtie I$, where I is a semidualizing ideal. Some of these results are collected in the following proposition.

Proposition 3.2 ([15, Lemmas 3.7 and 3.1(v)]). *Let I be an ideal of the ring R. Then the following statements hold.*
(i) If E is a (faithfully) injective R-module, then $\mathrm{Hom}_R(R \bowtie I, E)$ is a (faithfully) injective $(R \bowtie I)$-module.

(ii) Every injective $(R \bowtie I)$-module is a direct summand of the R-module $\mathrm{Hom}_R(R \bowtie I, E)$, where E is a injective R-module.

(iii) If I is a semidualizing ideal of the ring R, then for every injective R-module E we have the following equivalence of $(R \bowtie I)$-module $\mathrm{Hom}_{R\bowtie I}(\mathrm{Hom}_R(R \bowtie I, E), -) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(I, E), -)$.

Using the same method of the proof of Proposition 3.2, we obtain the following dual.

Proposition 3.3. *Let I be an ideal of the ring R. Then the following statements hold.*
(i) If P is a projective R-module, then $(R \bowtie I) \otimes_R P$ is a projective $(R \bowtie I)$-module.

(ii) Every projective $(R \bowtie I)$-module is a direct summand of the R-module $(R \bowtie I) \otimes_R P$, where P is a projective R-module.

(iii) If I is a semidualizing ideal of the ring R, then for every projective R-module Q we have the following equivalence of $(R \bowtie I)$-module $\mathrm{Hom}_{R\bowtie I}(-, (R \bowtie I) \otimes_R Q) \cong \mathrm{Hom}_R(-, I \otimes_R Q)$.

Corollary 3.4. *Let I be a semidualizing ideal of the ring R and let M be an R-module. Then the following statements hold for any integer n.*
(i) $\mathrm{Ext}^n_R(\mathrm{Hom}_R(I, J), M) = 0$ *for any injective R-module J if and only if for any injective $(R \bowtie I)$-module U we have* $\mathrm{Ext}^n_{R\bowtie I}(U, M) = 0$.

(ii) $\mathrm{Ext}^n_R(M, I \otimes_R P) = 0$ *for any projective R-module P if and only if for any projective $(R \bowtie I)$-module S we have* $\mathrm{Ext}^n_{R\bowtie I}(M, S) = 0$.

Proof. The item (i) follows from Proposition 3.2 while the item (ii) is a consequence of Proposition 3.3. □

Proposition 3.5. *Let I be an ideal of the ring R and let M be an R-module. If E is a faithfully injective R-module, then* $\mathrm{Gid}\ _{R\bowtie I}(\mathrm{Hom}_R(M, E)) = \mathrm{Gfd}\ _{R\bowtie I}(M)$.

Proof. By Proposition 3.2 (i), $L = \mathrm{Hom}_R(R \bowtie I, E)$ is a faithfully injective $(R \bowtie I)$-module. Therefore, [2, Theorem 6.4.2] implies that $\mathrm{Gid}\ _{R\bowtie I}(\mathrm{Hom}_{R\bowtie I}(M, L)) = \mathrm{Gfd}\ _{R\bowtie I}(M)$. In the following sequence, the first isomorphism follows from adjointness and the second one follows from tensor cancellation.

$$\begin{aligned}\mathrm{Hom}_{R\bowtie I}(M, L) &= \mathrm{Hom}_{R\bowtie I}(M, \mathrm{Hom}_R(R \bowtie I, E)) \\ &\cong \mathrm{Hom}_R((R \bowtie I) \otimes_{R\bowtie I} M, E) \cong \mathrm{Hom}_R(M, E).\end{aligned}$$ □

Proposition 3.6 ([7, Proposition 2.2]). *Let I be a semidualizing ideal of the ring R and let M be an R-module which is Gorenstein injective over $R \bowtie I$. Then there exists a short exact sequence of R-modules* $0 \to M' \to \mathrm{Hom}_R(I, E) \to M \to 0$, *where E is an injective R-module and M' is Gorensrein injective $(R \bowtie I)$-module, which stays exact under applying the functor* $\mathrm{Hom}_R(\mathrm{Hom}_R(I, J), -)$, *for any injective R-module J.*

The dual proof of Proposition 3.6 (this time using Proposition 3.3), is as follows.

Proposition 3.7. *Let I be a semidualizing ideal of the ring R and let M be an R-module which is Gorenstein projective as $(R \bowtie I)$-module. Then there exists a short exact sequence of R-modules* $0 \to M \to I \otimes_R P \to M' \to 0$, *where P is a projective R-module and M' is Gorenstein projective as $(R \bowtie I)$-module. Furthermore, the sequence stays exact applying the functor* $\mathrm{Hom}_R(-, I \otimes_R Q)$ *for any projective R-module Q.*

Lemma 3.8. *Let I be a semidualizing ideal of the ring R and let M be a G_I-injective R-module. Then there exists the short exact sequence of $(R \bowtie I)$-modules* $0 \to$

$M' \to U \to M \to 0$, *where* $\mathrm{id}_{R\bowtie I}(U) = 0$ *and* $\mathcal{GI}_I - \mathrm{id}_R(M') = 0$. *Furthermore, the sequence stays exact over applying the functor* $\mathrm{Hom}_{R\bowtie I}(V, -)$ *for any injective* $(R \bowtie I)$*-module* V.

Proof. By definition there exists a short exact sequence of R-modules $0 \to N \to \mathrm{Hom}_R(I, E) \to M \to 0$, where E is injective and N is G_I-injective, and stays exact by applying the functor $\mathrm{Hom}_R(\mathrm{Hom}_R(I, J), -)$ for every injective R-module J. By Proposition 3.1 (i), we have the following short exact sequence of R-modules $(*) : 0 \to I \to R \bowtie I \to R \to 0$. By applying the functor $\mathrm{Hom}_R(-, E)$ to the sequence $(*)$, we get the exact sequence of $(R \bowtie I)$-modules $(**) : 0 \to E \to \mathrm{Hom}_R(R \bowtie I, E) \to \mathrm{Hom}_R(I, E) \to 0$. Now we have the following commutative diagram of $(R \bowtie I)$-modules with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & \mathrm{Hom}_R(R \bowtie I, E) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & \mathrm{Hom}_R(I, E) & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

By Proposition 3.2 (i), $\mathrm{Hom}_R(R \bowtie I, E)$ is an injective $(R \bowtie I)$-module. Also using Snake lemma on the diagram embeds the vertical arrows into exact sequences, which implies the short exact sequence of R-modules $0 \to E \to M' \to N \to 0$. Therefore $M' \cong E \oplus N$ as R-modules. But N is G_I-injective and E is by Fact 2.11 (ii). So M' is also G_I-injective. Furthermore the lower row in the diagram stays exact under $\mathrm{Hom}_R(\mathrm{Hom}_R(I, J), -)$ for every injective R-module J. Also, the sequence $(**)$ splits as R-modules, so the surjection $\mathrm{Hom}_R(R \bowtie I, E) \to \mathrm{Hom}_R(I, E)$ splits, which implies that the upper row in the diagram also stays exact under $\mathrm{Hom}_R(\mathrm{Hom}_R(I, J), -)$. Now using Proposition 3.2 (iii) we see that the upper row in the diagram stays exact under $\mathrm{Hom}_{R\bowtie I}(\mathrm{Hom}_R(R \bowtie I, J), -)$ for every injective R-module J. This proves that the sequence stays exact under $\mathrm{Hom}_{R\bowtie I}(V, -)$, for every injective $(R \bowtie I)$-module V. □

By a similar argument, the following result obtained.

LEMMA 3.9. *Let* I *be a semidualizing ideal of the ring* R *and let* M *be a* G_I*-projective* R*-module. Then there exists the short exact sequence of* $(R \bowtie I)$*-modules* $0 \to M \to P \to M' \to 0$, *where* $\mathrm{pd}_{R\bowtie I}(P) = 0$ *and* $\mathcal{GP}_I - \mathrm{pd}_R(M') = 0$. *Furthermore, the sequence stays exact over applying the functor* $\mathrm{Hom}_{R\bowtie I}(-, S)$ *for any projective* $(R \bowtie I)$*-module* S.

In [11], Holm and Jörgensen investigated the properties of relative Gorenstein homological dimensions, G_C-projective, G_C-injective, and G_C-flat dimensions, where C is a semidualizing R-module and they showed that the G_C-projective, G_C-injective, and G_C-flat dimensions always agree with the ring changed Gorenstein dimensions Gpd $_{R\ltimes C}(-)$, Gid $_{R\ltimes C}(-)$, and Gfd $_{R\ltimes C}(-)$, respectively. In the following, we study these result for amalgamation instead of idealization.

PROPOSITION 3.10. *Let* I *be a semidualizing ideal of the ring* R. *Then for every* R*-module* M *the following statements holds.*
(i) M *is a* G_I*-injective* R*-module if and only if* M *is a Gorenstein injective* $(R \bowtie I)$*-module.*

(ii) M is a G_I-projective R-module if and only if M is a Gorenstein projective $(R \bowtie I)$-module.

(iii) M is a G_I-flat R-module if and only if M is a Gorenstein flat $(R \bowtie I)$-module.

Proof. (i) Assume that M is G_I-injective R-module. Then Lemma 3.8 implies that M is Gorenstein injective as $(R \bowtie I)$-module. Conversely, if M is Gorenstein injective over $R \bowtie I$, then Proposition 3.6 and Corollary 3.4 (i) gives the existence of a complete $\mathcal{I}_C\mathcal{I}$-resolution.

(ii) Similar, with using Proposition 3.7 and Lemma 3.9 and Corollary 3.4 (ii).

(iii) By item (i) and Propositions 3.5, we only need to show that for every faithfully injective R-module E we have M is G_I-flat if and only if $\mathrm{Hom}_R(M, E)$ is G_I-injective, which is proved in the proof of [11, Proposition 2.15]. □

THEOREM 3.11. *Let I be a semidualizing R-module of the ring R and let M be an R-module. Then the following equalities hold.*
(i) $\mathcal{GI}_I - \mathrm{id}_R(M) = \mathrm{Gid}\ _{R\bowtie I}(M)$,

(ii) $\mathcal{GP}_I - \mathrm{pd}_R(M) = \mathrm{Gpd}\ _{R\bowtie I}(M)$,

(iii) $\mathcal{GP}_I - \mathrm{fd}_R(M) = \mathrm{Gfd}\ _{R\bowtie I}(M)$.

Proof. We only prove the first equality. The proofs of other items are similar. By Proposition 3.10 (i) we have $\mathcal{GI}_I - \mathrm{id}_R(M) \geq \mathrm{Gid}\ _{R\bowtie I}(M)$. For the opposite, assume that $\mathrm{Gid}\ _{R\bowtie I}(M) = n$. Pick an injective resolution $\mathbf{E}$ of M as R-module, $\mathbf{E} : 0 \to M \to E_0 \to E_{-1} \to \cdots \to E_{1-n} \to K_{-n} \to 0$. By [15, Theorem 3.8] the modules E_i are Gorenstein injective as $(R \bowtie I)$-module, and therefore [10, Theorem (2.22)] implies that the R-module K_{-n} is Gorenstein injective as $(R \bowtie I)$-module. Now Proposition 3.10 implies that K_{-n} is a G_I-injective R-module. On the other hand, Fact 2.11(*ii*) implies that the modules E_i are G_I-injective R-modules, which shows that $\mathcal{GI}_I - \mathrm{id}_R(M) \leq n$. □

Here, we investigate some homological properties on amalgamation along a semidualizing ideal I.

LEMMA 3.12. *Let I be a semidualizing ideal of the ring R, P be a projective R-module, and let E be an injective R-module. Then the following statements hold.*
(i) $\mathrm{id}_{R\bowtie I}((R \bowtie I) \otimes_R P) \leq \mathrm{id}_R(I \otimes_R P)$.

(ii) $\mathrm{pd}_{R\bowtie I}(\mathrm{Hom}_R(R \bowtie I, E)) \leq \mathrm{pd}_R(\mathrm{Hom}_R(I, E))$.

Proof. (i) Consider the following injective resolution of the R-module $I \otimes_R P$,

$$\mathbf{E} : 0 \to I \otimes_R P \to E^0 \to E^1 \to \cdots .$$

By [12, Corollary 6.1], $I \otimes_R P \in \mathcal{B}_I(R)$. Therefore, using Proposition 3.1 (i), we have $\mathrm{Ext}^{i\geq 1}_R(R \bowtie I, I \otimes_R P) \cong \mathrm{Ext}^{i\geq 1}_R(R \oplus I, I \otimes_R P) = 0$. So, the sequence $\mathbf{E}$ stays exact by applying the functor $\mathrm{Hom}_R(R \bowtie I, -)$. On the other hand, Proposition 3.2 (i) implies that $\mathrm{Hom}_R(R \bowtie I, E^i)$ is an injective $(R \bowtie I)$-module for every $i \geq 0$, which shows that $\mathrm{Hom}_R(R \bowtie I, \mathbf{E})$ is an injective resolution of the $(R \bowtie I)$-module

$\mathrm{Hom}_R(R \bowtie I, I \otimes_R P)$. But, $\mathrm{Hom}_R(R \bowtie I, I \otimes_R P) \cong \mathrm{Hom}_R(R \bowtie I, I) \otimes_R P$, by [6, Theorem 3.2.14], and $\mathrm{Hom}_R(R \bowtie I, I) \otimes_R P \cong (R \bowtie I) \otimes_R P$ as $(R \bowtie I)$-module by [5, Theorem 4.1].

(ii) Consider the projective resolution of the R-module $\mathrm{Hom}_R(I, E)$ as follows, $\mathbf{P} : \cdots \to P_1 \to P_0 \to \mathrm{Hom}_R(I, E) \to 0$. By [12, Corollary 6.1], $\mathrm{Hom}_R(I, E) \in \mathcal{A}_C(R)$. Therefore using Proposition 3.1 (i), we have $\mathrm{Tor}^R_{i\geq 1}(R \bowtie I, \mathrm{Hom}_R(I, E)) \cong \mathrm{Tor}^R_{i\geq 1}(R \oplus I, \mathrm{Hom}_R(I, E)) = 0$. So, the sequence $\mathbf{P}$ stays exact by applying the functor $(R \bowtie I) \otimes_R -$. Also, Proposition 3.3 (i) implies that $(R \bowtie I) \otimes_R P_i$ is a projective $(R \bowtie I)$-module for every $i \geq 0$, which shows that $(R \bowtie I) \otimes_R \mathbf{P}$ is a projective resolution of the $(R \bowtie I)$-module $(R \bowtie I) \otimes_R \mathrm{Hom}_R(I, E)$. On the other hand, we have:

$$(R \bowtie I) \otimes_R \mathrm{Hom}_R(I, E) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(R \bowtie I, I), E) \cong \mathrm{Hom}_R(R \bowtie I, E).$$

Note that in the above sequence the first isomorphism follows from [6, Theorem 3.2.11], since $R \bowtie I$ is a finitely generated R-module by Proposition 3.1 (ii), and the second one follows from [5, Theorem 4.1]. □

Theorem 3.13. *Let I be a semidualizing ideal of the ring R. Assume that* $\sup\{\mathcal{GP}_I - \mathrm{pd}_R(M) \mid M \in \mathcal{M}(R)\} \leq n$, *(or* $\sup\{\mathcal{GI}_I - \mathrm{id}_R(M) \mid M \in \mathcal{M}(R)\} \leq n$*), where n is a non-negative integer. Then for every projective $(R \bowtie I)$-module P and every injective $(R \bowtie I)$-module E the following statements hold.*
(i) $\mathrm{id}_{R\bowtie I}(P) \leq n$. *(ii)* $\mathrm{pd}_{R\bowtie I}(E) \leq n$.

Proof. Let P be a projective $(R \bowtie I)$-module and let E be an injective $(R \bowtie I)$-module. By Proposition 3.2 (ii) and Proposition 3.3 (ii), E is a direct summand of the R-module $\mathrm{Hom}_R(R \bowtie I, E')$ for some injective R-module E' and P is a direct summand of the R-module $(R \bowtie I) \otimes_R Q$ for some projective R-module Q. Now we show that $\mathrm{id}_{R\bowtie I}((R \bowtie I) \otimes_R Q) \leq n$ and $\mathrm{pd}_{R\bowtie I}(\mathrm{Hom}_R(R \bowtie I, E')) \leq n$.

First assume that $\sup\{\mathcal{GP}_I - \mathrm{pd}_R(M) \mid M \in \mathcal{M}(R)\} \leq n$.

(i) Let Q be a projective R-module and let M be an R-module. Then by [19, Proposition 2.12], $\mathrm{Ext}^{i>n}_R(M, I \otimes_R Q) = 0$, which implies that $\mathrm{id}_R(I \otimes_R Q) \leq n$. Now, Lemma 3.12($i$) implies that $\mathrm{id}_{R\bowtie I}((R \bowtie I) \otimes_R Q) \leq n$.

(ii) By [20, Lemma 3.4(1)], $\mathcal{P}_I - \mathrm{pd}_R(E) = \mathcal{GP}_I - \mathrm{pd}_R(E)$ for any injective R-module E. Therefore Fact 2.9 (i) implies that $\mathrm{pd}_R(\mathrm{Hom}_R(I, E)) = \mathcal{P}_I - \mathrm{pd}_R(E) \leq n$. Now Lemma 3.12 (ii) implies that $\mathrm{pd}_{R\bowtie I}(\mathrm{Hom}_R(R \bowtie I, E)) \leq n$.

Now suppose that $\sup\{\mathcal{GI}_I - \mathrm{id}_R(M) \mid M \in \mathcal{M}(R)\} \leq n$.

(i) By [20, Lemma 3.4(2)], $\mathcal{I}_I - \mathrm{id}_R(Q) = \mathcal{GI}_I - \mathrm{id}_R(Q)$ for any projective R-module Q. So, Fact 2.9 (ii) implies that $\mathrm{id}_R(I \otimes_R Q) \leq n$. Hence, $\mathrm{id}_{R\bowtie I}((R \bowtie I) \otimes_R Q) \leq n$ by Lemma 3.12 (i).

(ii) Let M be an R-module. Then $\mathrm{Ext}^{i>n}_R(\mathrm{Hom}_R(I, E), M) = 0$ for any injective R-module E, by the dual of [19, Proposition 2.12]. So, $\mathrm{pd}_R(\mathrm{Hom}_R(I, E)) \leq n$. Now, Lemma 3.12 (ii) implies that $\mathrm{pd}_{R\bowtie I}(\mathrm{Hom}_R(R \bowtie I, E)) \leq n$. □

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