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NEW CONGRUENCES MODULO SMALL POWERS OF 2 FOR OVERPARTITIONS INTO ODD PARTS

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Abstract. In this article, we establish several infinite families of Ramanujan-type congruences modulo 16, 32 and 64 for $\overline{p}_o(n)$, the number of overpartitions of n in which only odd parts are used.

1. Introduction

An overpartition of the nonnegative integer n is a partition of n where the first occurrence of parts of each size may be overlined. For example, there are 8 overpartitions of the integer 3:

$$3, \quad \overline{3}, \quad 2+1, \quad \overline{2}+1, \quad 2+\overline{1}, \quad \overline{2}+\overline{1}, \quad 1+1+1, \quad \overline{1}+1+1.$$

We denote the number of overpartitions of n by $\overline{p}(n)$. The generating function of $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

where

$$(a;q)_n = \begin{cases} 1, & \text{for } n=0; \\ \prod_{k=1}^{n} (1-aq^{k-1}), & \text{for } n>0 \end{cases}$$

is q-shifted factorial, $(a;q)_\infty = \lim_{n\to\infty} (a;q)_n$, $|q|<1$ and let $f_k := (q^k;q^k)_\infty$. Many mathematicians have extensively studied overpartitions to obtain properties analogous to ordinary partitions, see, for example [4–6, 9].

In this context, we consider the number of overpartitions into odd parts. Let $\overline{p}_o(n)$ denote the number of such partitions. It is evident that

$$\sum_{n=0}^{\infty} \overline{p}_o(n) q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^3}{f_1^2 f_4}. \tag{1}$$

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The generating function of $\overline{p}_o(n)$ appears in the following series-product identity of Lebesque [8],

$$\sum_{j=0}^{\infty} \frac{(-1;q)_j q^{j(j+1)/2}}{(q;q)_j} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.$$

We assert that the sequence $\{\overline{p}_o(n)\}_{n\geq 0}$ is known and can be seen in [11, A080054]. Hirschhorn and Sellers [7] considered $\overline{p}_o(n)$ in arithmetic point of view and obtained many congruences modulo 8 and 16 for $\overline{p}_o(n)$, for example they proved that $\overline{p}_o(2^\alpha(8n+5)) \equiv 0 \pmod 8$ and

$$\overline{p}_o(8n+7) \equiv 0 \pmod{16}, \tag{2}$$

for all nonnegative integers α and n, while Chen [3] showed that

$$\sum_{n=0}^{\infty} \overline{p}_o(16n+14)q^n = 112\frac{f_2^{27}}{f_1^{25}f_4^2} + 256q\frac{f_2^3 f_4^{14}}{f_1^{17}} \equiv 48\frac{f_2^{11}}{f_1} \pmod{64}. \tag{3}$$

which implies that $\overline{p}_o(16n+14) \equiv 0 \pmod{16}$. Using elementary theory of modular forms, Chen [3] extented these congruences to modulo 32 and 64. In particular, Chen showed the following theorem.

Theorem 1.1. *Let t be an integer, $p \equiv 1 \pmod 8$ be a prime. Then for all integers n with $n \not\equiv -\frac{7}{8} \pmod p$,*

$$\overline{p}_o(p^{2t+1}(16n+14)) \equiv 0 \pmod{32},$$
$$\overline{p}_o(p^{4t+3}(16n+14)) \equiv 0 \pmod{64}.$$

Suppose that $p_1, p_2 \equiv 1 \pmod 8$ are two distinct primes. Then for all nonnegative integers n satisfying $n \not\equiv -\frac{7}{8} \pmod{p_1}$ and $n \not\equiv -\frac{7}{8} \pmod{p_2}$, $\overline{p}_o(p_1p_2(16n+14)) \equiv 0 \pmod{64}$.

Recently, C. Ray and R. Barman [10] obtained identities for $\overline{p}_o(n)$ and as a consequence derived many congruences modulo 8 and 16 for $\overline{p}_o(n)$.

With this motivation, we prove several infinite families of congruences modulo 16, 32 and 64 for the partition function $\overline{p}_o(n)$.

2. Congruences modulo small powers of 2 for $\overline{p}_o(n)$

Jacobi's triple product identity can be stated in terms of the Ramanujan's theta function [1, p. 34] as follows:

$$(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}, \quad |ab| < 1. \tag{4}$$

The following lemma plays a vital role in proving our main results.

Lemma 2.1. *The following 2-dissections hold:*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8},$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{5}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2} \tag{6}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{7}$$

Proof. Lemma 2.1 is an immediate consequence of dissection formulas of Ramanujan, collected in Berndt's book [1, Entry 25, p. 40]. □

The main results are the following.

THEOREM 2.2. *For all integers $n \geq 0$ and $\alpha \in \{2, 3, 4, 5\}$,*

$$\overline{p}_o(2^{\alpha+1}n) \equiv \overline{p}_o(2^\alpha n) \pmod{2^{\alpha+2}}. \tag{8}$$

THEOREM 2.3. *If n cannot be represented as a sum of a triangular number and twice a triangular number, then for any nonnegative integer α,*

$$\overline{p}_o(2^\alpha(8n+3)) \equiv 0 \pmod{32}. \tag{9}$$

COROLLARY 2.4. *For any positive integer k, let $p_j \geq 3$, $1 \leq j \leq k$ be primes. If $(-2/p_j) = -1$ for every j, then for all nonnegative integers α and n with $p_k \nmid n$,*

$$\overline{p}_o(2^\alpha(8p_1^2p_2^2\cdots p_{k-1}^2p_kn + 3p_1^2p_2^2\cdots p_{k-1}^2p_k^2)) \equiv 0 \pmod{32}. \tag{10}$$

THEOREM 2.5. *For all integers $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}_o(2^\alpha(8n+7)) \equiv 0 \pmod{16}. \tag{11}$$

THEOREM 2.6. *Let $t \geq 0$ be an integer and $p \equiv 1 \pmod 8$ be a prime. Then for all nonnegative integers α and n with $8n \not\equiv -7 \pmod p$,*

$$\overline{p}_o(2^\alpha p^{2t+1}(8n+7)) \equiv 0 \pmod{32},$$

$$\overline{p}_o(2^\alpha p^{4t+3}(8n+7)) \equiv 0 \pmod{64}$$

and

$$\overline{p}_o(2^\alpha p_1p_2(8m+7)) \equiv 0 \pmod{64}$$

where $p_1, p_2 \equiv 1 \pmod 8$ are two distinct primes and m is any integer satisfying $8m \not\equiv -7 \pmod{p_1}$ and $8m \not\equiv -7 \pmod{p_2}$.

Note that Theorem 1.1 is the special case of Theorem 2.6 for $\alpha = 2$.

THEOREM 2.7. *If n cannot be represented as a sum of a triangular number and four times a triangular number, then for any nonnegative integer α,*

$$\overline{p}_o(2^\alpha(8n+5)) \equiv 0 \pmod{32}. \tag{12}$$

COROLLARY 2.8. *For any positive integer k, let $p_j \geq 3$, $1 \leq j \leq k$ be primes. If $(-4/p_j) = -1$ for every j, then for all nonnegative integers α and n with $p_k \nmid n$,*

$$\overline{p}_o(2^\alpha(8p_1^2p_2^2\cdots p_{k-1}^2p_kn + 5p_1^2p_2^2\cdots p_{k-1}^2p_k^2)) \equiv 0 \pmod{32}.$$

By the binomial theorem, it is easy to see that for all positive integers k and m,

$$f_m^{2^k} \equiv f_{2m}^{2^{k-1}} \pmod{2^k}. \tag{13}$$

Proof (Proof of Theorem 2.2). Consider the generating function (1),

$$\sum_{n=0}^{\infty}\overline{p}_o(n)q^n = \frac{f_2^3}{f_1^2 f_4}. \tag{14}$$

Substituting (5) in (14) and extracting the terms involving even and odd powers of q, we find that

$$\sum_{n=0}\overline{p}_o(2n)q^n = \frac{f_4^5}{f_1^2 f_2 f_8^2} \tag{15}$$

and

$$\sum_{n=0}\overline{p}_o(2n+1)q^n = 2\frac{f_2 f_8^2}{f_1^2 f_4}. \tag{16}$$

Again, employing (5) in (15),

$$\sum_{n=0}^{\infty}\overline{p}_o(2n)q^n = \frac{f_4^5 f_8^3}{f_2^6 f_{16}^2} + 2q\frac{f_4^7 f_{16}^2}{f_2^6 f_8^3},$$

which yields

$$\sum_{n=0}^{\infty}\overline{p}_o(4n)q^n = \frac{f_2^5 f_4^3}{f_1^6 f_8^2} \tag{17}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_o(4n+2)q^n = 2\frac{f_2^7 f_8^2}{f_1^6 f_4^3}. \tag{18}$$

Employing (5) in (17) and extracting the terms involving even powers of q,

$$\sum_{n=0}^{\infty}\overline{p}_o(8n)q^n = \frac{f_2^3 f_4^{13}}{f_1^{10} f_8^6} + 12q\frac{f_2^7 f_4 f_8^2}{f_1^{10}}. \tag{19}$$

By (6),

$$\frac{f_2^5 f_4^3}{f_1^6 f_8^2} = \frac{f_2^3 f_4^{13}}{f_1^{10} f_8^6} - 4q\frac{f_2^7 f_4 f_8^2}{f_1^{10}}. \tag{20}$$

In view of (19) and (20),

$$\sum_{n=0}^{\infty}\overline{p}_o(8n)q^n = \frac{f_2^5 f_4^3}{f_1^6 f_8^2} + 16q\frac{f_2^7 f_4 f_8^2}{f_1^{10}}. \tag{21}$$

From (17) and (21),

$$\sum_{n=0}^{\infty}\overline{p}_o(8n)q^n = \sum_{n=0}^{\infty}\overline{p}_o(4n)q^n + 16q\frac{f_2^7 f_4 f_8^2}{f_1^{10}}.$$

Let

$$\sum_{n=0}^{\infty}a(n)q^n = q\frac{f_2^7 f_4 f_8^2}{f_1^{10}} \equiv q\frac{f_2^3 f_4 f_8^2}{f_1^2} \pmod{8}.$$

Substituting (5) in the previous line and extracting the terms involving even powers of q,

$$\sum_{n=0}^{\infty}a(2n)q^n \equiv 2q\frac{f_2^3 f_4 f_8^2}{f_1^2} \pmod{8}.$$

Last three identities yield,

$$\sum_{n=0}^{\infty}\overline{p}_o(2^{\alpha+1}n)q^n \equiv \sum_{n=0}^{\infty}\overline{p}_o(2^{\alpha}n)q^n + 2^{\alpha+2}q\frac{f_2^3 f_4 f_8^2}{f_1^2} \pmod{128} \tag{22}$$

for all integers $\alpha \geq 2$. Congruence (8) follows from (22). □

Proof (Proof of Theorem 2.3). Using (5), we can rewrite (16) as

$$\sum_{n=0}^{\infty} \overline{p}_o(2n+1)q^n = 2\frac{f_8^7}{f_2^4 f_4 f_{16}^2} + 4q\frac{f_4 f_8 f_{16}^2}{f_2^4} \tag{23}$$

which yeilds

$$\sum_{n=0}^{\infty} \overline{p}_o(4n+1)q^n = 2\frac{f_4^7}{f_1^4 f_2 f_8^2} \tag{24}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_o(4n+3)q^n = 4\frac{f_2 f_4 f_8^2}{f_1^4}. \tag{25}$$

Invoking (7) in (25) and extracting the terms involving odd and even powers of q,

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+7)q^n = 16\frac{f_2^3 f_4^6}{f_1^9} \equiv 16\frac{f_2^{11}}{f_1} \pmod{64}. \tag{26}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+3)q^n = 4\frac{f_2^{15}}{f_1^{13} f_4^2} = 4\frac{f_1^3 f_4^2}{f_2}\frac{f_2^{16}}{f_1^{16} f_4^4} \equiv 4\frac{f_1^3 f_4^2}{f_2} \pmod{32}. \tag{27}$$

From [2, Theorem 1.3.9, p.14], (4) and (27),

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+3)q^n = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} (-1)^k(2k+1)q^{\frac{k(k+1)}{2}+m(m+1)} \pmod{32}. \tag{28}$$

Substituting (5) and (7) in (18) and extracting the terms involving even and odd powers of q,

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+2)q^n = 2\frac{f_2^{11} f_4^3}{f_1^{12} f_8^2} + 16q\frac{f_2 f_4^5 f_8^2}{f_1^8} \tag{29}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+6)q^n = 4\frac{f_2^{13} f_8^2}{f_1^{12} f_4^3} + 8\frac{f_4^{11}}{f_1^8 f_8^2 f_2} \equiv 12\frac{f_2 f_4 f_8^2}{f_1^4} \pmod{32}. \tag{30}$$

In view of (25) and (30), we see that

$$\overline{p}_o(8n+6) \equiv 3\,\overline{p}_o(4n+3) \pmod{32}. \tag{31}$$

Substituting (5) and (7) in (17) and extracting the terms involving odd powers of q,

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+4)q^n = 2\frac{f_2^{19} f_8^2}{f_1^{14} f_4^7} + 4\frac{f_2^5 f_4^7}{f_1^{10} f_8^2}. \tag{32}$$

Employing (5) and (7) in (32) and extracting terms involving even and odd powers of q,

$$\sum_{n=0}^{\infty} \overline{p}_o(16n+4)q^n = 6\frac{f_2^{35}}{f_1^{28} f_4^5 f_8^2} + 160q\frac{f_2^{11} f_4^{11}}{f_1^{20} f_8^2} + 112q\frac{f_2^{25} f_8^2}{f_1^{24} f_4^3} + 256q^2\frac{f_2 f_4^{13} f_8^2}{f_1^{16}} \tag{33}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{p}_o(16n+12)q^n &= 12\frac{f_2^{37} f_8^2}{f_1^{28} f_4^{11}} + 320q\frac{f_2^{13} f_4^5 f_8^2}{f_1^{20}} + 56\frac{f_2^{23} f_4^3}{f_1^{24} f_8^2} + 128q\frac{f_4^{19}}{f_1^{16} f_8^2 f_2} \\ &\equiv 12\frac{f_2^{21} f_8^2 f_1^4}{f_4^{11}} + 56\frac{f_2^3 f_8^2}{f_4} \pmod{64}\end{aligned} \tag{34}$$

$$\equiv 4\frac{f_2 f_4 f_8^2}{f_1^4} \pmod{32}. \tag{35}$$

From (25) and (35),

$$\overline{p}_o(16n+12) \equiv \overline{p}_o(4n+3) \pmod{32}. \tag{36}$$

Employing (5) in (22) and extracting odd powers of q,

$$\sum_{n=0}^{\infty} \overline{p}_o(2^{\alpha+1}(2n+1))q^n \equiv \sum_{n=0}^{\infty} \overline{p}_o(2^{\alpha}(2n+1))q^n + 2^{\alpha+2}\frac{f_2 f_4^7}{f_1^2 f_8^2} \pmod{128}. \tag{37}$$

Again, extracting the terms involving even and odd powers of q from the last expression,

$$\sum_{n=0}^{\infty} \overline{p}_o(2^{\alpha+1}(4n+1))q^n \equiv \sum_{n=0}^{\infty} \overline{p}_o(2^{\alpha}(4n+1))q^n + 2^{\alpha+2}\frac{f_2^7 f_4^3}{f_1^4 f_8^2} \pmod{128} \tag{38}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_o(2^{\alpha+1}(4n+3))q^n \equiv \sum_{n=0}^{\infty} \overline{p}_o(2^{\alpha}(4n+3))q^n + 2^{\alpha+3}\frac{f_2^9 f_8^2}{f_1^4 f_4^3} \pmod{128} \tag{39}$$

for all $\alpha \geq 2$. In view of (31), (36) and (39), we see that

$$\overline{p}_o(2^{\alpha}(4n+3)) \equiv 3^{\beta}\, \overline{p}_o(4n+3) \pmod{32} \tag{40}$$

for all $\alpha > 0, n \geq 0$ and $\beta = \begin{cases} 1, & if \quad \alpha = 1; \\ 0, & \text{otherwise}. \end{cases}$

Congruence (9) follows from (28) and (40). □

Proof (Proof of Corollary 2.4). By (28)

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+3)q^{8n+3} \equiv 4\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^k(2k+1)q^{(2k+1)^2+2(2m+1)^2} \pmod{32},$$

which implies that if $8n+3$ is not of the form $(2k+1)^2+2(2m+1)^2$, then $\overline{p}_o(8n+3) \equiv 0 \pmod{32}$. Let $k \geq 1$ be an integer and let $p_i \geq 3,\ 1 \leq i \leq k$ be primes with $(\frac{-2}{p_i}) = -1$. If N is of the form x^2+2y^2, then $v_{p_i}(N)$ is even since $(\frac{-2}{p_i}) = -1$. Let

$$\begin{aligned} N &= 8\left(p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 3\frac{p_1^2 p_2^2 \cdots p_{k-1}^2 p_k^2 - 1}{8}\right) + 3 \\ &= 8p_1^2 p_2^2 \cdots p_{k-1}^2 p_k n + 3p_1^2 p_2^2 \cdots p_{k-1}^2 p_k^2. \end{aligned}$$

If $p_k \nmid n$, then $v_{p_k}(N)$ is an odd number and hence N is not of the form x^2+2y^2.

Thus,

$$\overline{p}_o(8p_1^2 p_2^2 \ldots p_{k-1}^2 p_k n + 3p_1^2 p_2^2 \ldots p_{k-1}^2 p_k^2) \equiv 0 \pmod{32}. \tag{41}$$

Congruence (10) follows from (41) and (40). □

Proof (Proof of Theorems 2.5 and 2.6). Applying (6) in (34) and extracting the terms involving odd powers of q,

$$\sum_{n=0}^{\infty} \overline{p}_o(32n+28)q^n \equiv -48\frac{f_1^{23} f_4^6}{f_2^{13}} \equiv 16\frac{f_2^{11}}{f_1} \pmod{64}. \tag{42}$$

From (39),

$$\overline{p}_o(2^{\alpha+1}(8n+7)) \equiv \overline{p}_o(2^{\alpha}(8n+7)) \pmod{128} \tag{43}$$

for all $\alpha \geq 2$ and $n \geq 0$. In view of (26), (3), (42) and (43),

$$\overline{p}_o(2^\alpha(8n+7)) \equiv 3^\beta\ \overline{p}_o(8n+7) \pmod{64} \tag{44}$$

for all $\alpha > 0, n \geq 0$ and $\beta = \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$

Congruence (11) follows from (2) and (44). Theorem 2.6 follows from Theorem 1.1 and (44). □

Proof (Proof of Theorem 2.7). By employing (7) in (23) and extracting the terms involving odd powers of q,

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+5)q^n = 8\frac{f_2^9 f_4^2}{f_1^{11}} \equiv 8\frac{f_1 f_8^2}{f_2} \pmod{32}. \tag{45}$$

Using (13) in (29),

$$\sum_{n=0}^{\infty} \overline{p}_o(8n+2)q^n \equiv 2\frac{f_1^4 f_2^3 f_4^3}{f_8^2} + 16q\frac{f_2 f_{16}^2}{f_4} \pmod{32}. \tag{46}$$

Using (6) in the last expression and extracting the terms involving odd powers of q,

$$\sum_{n=0}^{\infty} \overline{p}_o(16n+10)q^n \equiv -8f_1^5 f_2 f_4^2 + 16\frac{f_1 f_8^2}{f_2} \equiv 8\frac{f_1 f_8^2}{f_2} \pmod{32}. \tag{47}$$

Using (13) in (33), $\sum_{n=0}^{\infty} \overline{p}_o(16n+4)q^n \equiv 6\frac{f_1^4 f_2^3 f_4^3}{f_8^2} + 48q\frac{f_2 f_{16}^2}{f_4} \pmod{32}$. Extracting coefficients of odd powers of q from (38),

$$\overline{p}_o(2^{\alpha+1}(8n+5)) \equiv \overline{p}_o(2^\alpha(8n+5)) \pmod{64} \tag{48}$$

for all $\alpha \geq 2$ and $n \geq 0$. From (45)- (48) it is evident that

$$\overline{p}_o(2^\alpha(8n+5)) \equiv 3^\beta\ \overline{p}_o(8n+5) \pmod{32} \tag{49}$$

for all $\alpha > 0, n \geq 0$ and $\beta = \begin{cases} 0 & \text{if } \alpha = 1, \\ 1 & \text{otherwise.} \end{cases}$

Furthermore,

$$\begin{aligned} \frac{f_1 f_8^2}{f_2} &= (q;q^2)_\infty (q^8;q^8)_\infty (q^4;q^8)_\infty (-q^4;q^8)_\infty (q^{16};q^{16})_\infty \\ &= (q;q^2)_\infty (q^4;q^4)_\infty (-q^4;q^8)_\infty (q^{16};q^{16})_\infty \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{\lceil k/2 \rceil} q^{\frac{k(k+1)}{2} + 2m(m+1)}. \end{aligned}$$

Last equality follows from (4). Thus, congruence (12) follows from (45), (49) and the last equality. □

Proof of Corollary 2.8 is similar to Corollary 2.4, so we skip the details.

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