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ON CUBIC INTEGRAL EQUATIONS OF URYSOHN-STIELTJES TYPE

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Abstract. In this paper, we establish an existence theorem for a cubic Urysohn-Stieltjes integral equation in the Banach space $C([0,1])$. The equation under consideration is a general form of numerous integral equations encountered in the theory of radioactive transfer, in the kinetic theory of gases and in the theory of neutron transport. Our main tools are the measure of noncompactness (related to monotonicity) and a fixed point theorem due to Darbo.

1. Introduction

Cubic integral equations arise in several useful applications and appear in modeling different problems in the real world [1,2]. The aim of this paper is to investigate the existence of monotonic solutions of the so-called cubic integral equation of Urysohn-Stieltjes type, namely

$$x(t) = f(t) + g(t, x(t)) + x^2(t) \int_0^1 u(t, s, x(s), x(\lambda s))\, d_s h(t, s),\ t \in I = [0, 1]. \tag{1}$$

If $d_s h(t,s) = \frac{t}{t+s}$, (1) takes the form

$$x(t) = f(t) + g(t, x(t)) + x^2(t) \int_0^1 \frac{t}{t+s} u(t, s, x(s))\, ds,\ t \in I. \tag{2}$$

Equation (2) is a general form of the famous equation in transport theory, the so-called Chandrasekhar H-equation [9,10,15,16].

The classical theory of integral operators and equations can be generalized with the help of Stieltjes integrals having kernels depending on one or two variables. This approach was developed in several papers and books (see [3,6–8,11,12,14,17] and the references therein). Using the measure of noncompactness (related to monotonicity) defined by J. Banaś and L. Olszowy in [4], and Darbo's fixed point theorem we establish the existence of solutions to (1) in $C(I)$ and these solutions are nondecreasing on the interval I.

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2. Auxiliary facts and results

We denote by $(E, \|\cdot\|)$ a real Banach space and by $B(x,r)$ the closed ball of radius r and center x. Also, we denote by B_r the closed ball $B(\theta, r)$, where θ is the zero element of E. Let $\emptyset \neq X \subset E$ and the symbols $\overline{X}$ and Conv$X$ stand for the closure and convex closed hull of the set X, respectively. We denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets of E.

Definition 2.1 ([5]). A mapping $\mu : \mathcal{M}_E \to \mathbb{R}_+$ is said to be a regular measure of noncompactness in E if it satisfies the following conditions:
(i) $X \in \mathfrak{N}_E$ if and only if $\mu(X) = 0$.

(ii) $X \subset Y$ implies $\mu(X) \leq \mu(Y)$.

(iii) $\mu(X) = \mu(\overline{X}) = \mu(\text{Conv}X)$.

(iv) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $0 \leq \lambda \leq 1$.

(v) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \ldots$, and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Now, we state the fixed point theorem due to Darbo [13].

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of the Banach space E and let $\mathcal{H} : \Omega \to \Omega$ be a contraction with respect to the measure of noncompactness μ. Then $\mathcal{H}$ has a fixed point in the set Ω.*

In what follows, we will work in the Banach space $C(I)$ which consists of all real valued functions defined and continuous on I equipped with the norm $\|x\| = \sup_{t\in I} |x(t)|$. We now describe the measure of noncompactness in $C(I)$ that we will use in the next section (see [4]). We fix a nonempty and bounded subset X of $C(I)$. For $x \in X$ and $\varepsilon \geq 0$, the modulus of continuity of the function x, denoted by $\omega(x,\varepsilon)$, is given by $\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : s, t \in I,\ |t-s| \leq \varepsilon\}$. We put $\omega(X,\varepsilon) = \sup_{x\in X} \omega(x,\varepsilon)$ and $\omega_0(X) = \lim_{\varepsilon\to 0} \omega(X,\varepsilon)$. Next, we let $i(x) = \sup\{|x(t) - x(s)| - (x(t) - x(s)) : s, t \in I,\ s \leq t\}$ and $i(X) = \sup_{x\in X} i(x)$. Note that $i(X) = 0$ if and only if all functions belong to X are nondecreasing on I. Now, we define the function μ on the family $\mathfrak{M}_{C(I)}$ as follows: $\mu(X) = \omega_0(X) + i(X)$. The function μ is a measure of noncompactness in the space $C(I)$ [4]. Moreover, the kernel $\ker\mu$ consists of all sets X belonging to $\mathfrak{M}_{C(I)}$ such that all functions from X are equicontinuous and nondecreasing on the interval I.

Next, we state some auxiliary facts related to functions of bounded variation and the Stieltjes integral (see [8] and the references therein). Let x be a real valued function defined on the interval I. The variation of the function x on the interval I, denoted by $\bigvee_0^1 x$, is defined by

$$\bigvee_0^1 x = \sup_P \left\{ \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| : P = \{0 = t_0 < t_1 < \ldots < t_n = 1\} \text{ is a partition of } I \right\}.$$

If $\bigvee_0^1 x$ is finite, then we say that the function x is of bounded variation on I. We have the following properties:

(i) $\bigvee_0^1 x = \bigvee_0^1 (-x)$ (ii) $\bigvee_0^1 (x+y) \leq \bigvee_0^1 x + \bigvee_0^1 y$

(iii) $\bigvee_0^1 (x-y) \leq \bigvee_0^1 x + \bigvee_0^1 y$ (iv) $\left|\bigvee_0^1 x - \bigvee_0^1 y\right| \leq \bigvee_0^1 (x-y)$.

For more properties of functions of bounded variation see [14,17].

Next, let $k : I^2 \to \mathbb{R}$ be a function and let the symbol $\bigvee_{t=a}^{b} k(t,s)$ indicate the variation of the function $t \to k(t,s)$ on the interval $[a,b] \subset I$. Now, let us assume that x and $\phi : I \to \mathbb{R}$ are bounded functions. Then under some extra conditions, we can define the Stieltjes integral $\int_0^1 x(t)\, d\phi(t)$ of the function x with respect to the function ϕ. In this case, we say that x is Stieltjes integrable on the interval I with respect to the function ϕ. If x is continuous and ϕ is of bounded variation on the interval I, then x is Stieltjes integrable with respect to ϕ on I. Moreover, under the assumption that x and ϕ are of bounded variation on the interval I, the Stieltjes integral $\int_0^1 x(t)\, d\phi(t)$ exists if and only if the functions x and ϕ have no common points of discontinuity.

Further, we recall some properties of the Stieltjes integral which will be used later (see [14,17]).

Lemma 2.3. *If x is Stieltjes integrable on I with respect to a function ϕ of bounded variation then*

$$\left|\int_0^1 x(t)\, d\phi(t)\right| \leq \Big(\sup_{0\leq t\leq 1} |x(t)|\Big) \bigvee_0^1 \phi.$$

Moreover, the following inequality holds

$$\left|\int_0^1 x(t)\, d\phi(t)\right| \leq \int_0^1 |x(t)|\, d\Big(\bigvee_0^t \phi\Big).$$

Corollary 2.4. *If x is Stieltjes integrable function with respect to a nondecreasing function ϕ then*

$$\left|\int_0^1 x(t)\, d\phi(t)\right| \leq \Big(\sup_{0\leq t\leq 1} |x(t)|\Big) (\phi(1) - \phi(0)).$$

Lemma 2.5. *Let x_1 and x_2 be two Stieltjes integrable functions on I with respect to a nondecreasing function ϕ and such that $x_1(t) \leq x_2(t)$ for $t \in I$. Then*

$$\int_0^1 x_1(t)\, d\phi(t) \leq \int_0^1 x_2(t)\, d\phi(t).$$

Corollary 2.6. *Let x be Stieltjes integrable function on I with respect to a nondecreasing function φ and such that $x(t) \geq 0$ for all $t \in I$. Then*

$$\int_0^1 x(t)\, d\phi(t) \geq 0.$$

Lemma 2.7. *Let ϕ_1 and ϕ_2 be two nondecreasing functions on I with $(\phi_2 - \phi_1)$ a nondecreasing function. If x is Stieltjes integrable on I and $x(t) \geq 0$ for $t \in I$ then*

$$\int_0^1 x(t)\, d\phi_1(t) \leq \int_0^1 x(t)\, d\phi_2(t).$$

Throughout the paper, we will consider a Stieltjes integral of the form $\int_0^1 x(s)\, d_s k(t,s)$, where $k : I^2 \to \mathbb{R}$ and the symbol d_s denotes that the integration is taken with respect to s. Finally, we state the following two propositions (see [8]).

Proposition 2.8. *Suppose that the function $k : I^2 \to \mathbb{R}$ satisfies the following assumptions:*
(i) For all $t_1, t_2 \in I$ with $t_1 < t_2$ the function $s \mapsto (k(t_2,s) - k(t_1,s))$ is nondecreasing on I.

(ii) Both the function $t \mapsto k(t,0)$ and the function $t \mapsto k(t,1)$ are continuous on I.
Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for $t_1, t_2 \in I$ with $t_1 < t_2$ and $t_2 - t_1 \leq \delta$, we have $\bigvee_{s=0}^{1} (k(t_2,s) - k(t_1,s)) \leq \varepsilon$.

Proposition 2.9. *Suppose that the function $k : I^2 \mapsto \mathbb{R}$ satisfies the same assumptions as in Proposition 2.8. Moreover, assume that for each $t \in I$, the function $s \mapsto k(t,s)$ is of bounded variation on I. Then, the function $t \to \bigvee_{s=0}^{1} k(t,s)$ is continuous on I.*

Remark 2.10. Let the function $s \to k(t,s)$ be nondecreasing on I for each $t \in I$. Moreover, assume assumptions (i) and (ii) in Proposition 2.8 are satisfied. From the fact that every nondecreasing function is of bounded variation, the compactness of the interval I and Proposition 2.9, there exists a constant $\mathrm{T} > 0$ such that $\bigvee_{s=0}^{1} k(t,s) \leq \mathrm{T}$ for every $t \in I$.

3. Main theorem

In this section, we will study the equation (1) assuming that the following assumptions are satisfied:
(a1) The function $f : I \to \mathbb{R}$ is continuous, nondecreasing and nonnegative on I.

(a2) The function $g : I \times \mathbb{R} \to \mathbb{R}$ is continuous and $g : I \times \mathbb{R}_+ \to \mathbb{R}_+$. Moreover, there exists a nonnegative constant c such that

$$|g(t,x) - g(t,y)| \leq c|x-y| \ \forall t \in I,\ (x,y) \in \mathbb{R}^2. \tag{3}$$

Moreover, the superposition operator G generated by the function g (defined by $(Gx)(t) = g(t, x(t))$, where $x = x(t)$ is an arbitrary function defined on I) satisfies for any nonnegative function x the condition $i(Gx) \le c\, i(x)$, where c is the same constant appearing in (3).

(a3) The function $u : I^2 \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $u : I^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+$ and for arbitrary fixed $s \in I$ and $x, y \in \mathbb{R}_+$ the function $t \mapsto u(t, s, x, y)$ is nondecreasing on I.

(a4) The function u satisfies the following assumptions:

(i) There exists a continuous nondecreasing function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $|u(t, s, x, y)| \le \psi(|x|, |y|)$ for each $(t, s) \in I^2$ and $(x, y) \in \mathbb{R}$.

(ii) For any $\nu > 0$ there exists a continuous nondecreasing function $\varphi_\nu : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi_\nu(0) = 0$, such that $|u(t_2, s, x, y) - u(t_1, s, x, y)| \le \varphi_\nu(t_2 - t_1)$, for each $s \in I$, $(x, y) \in \mathbb{R}^2$ with $\max\{|x|, |y|\} \le \nu$ and for all $(t_1, t_2) \in I^2$ with $t_1 < t_2$.

(a5) The function $h : I^2 \to \mathbb{R}$ satisfies the following assumptions:

(a) The function $s \mapsto h(t, s)$ is nondecreasing on I for each $t \in I$.

(b) For all $t_1, t_2 \in I$ with $t_1 < t_2$ the function $s \mapsto (h(t_2, s) - h(t_1, s))$ is nondecreasing on I.

(c) Both the function $t \mapsto h(t, 0)$ and the function $t \mapsto h(t, 1)$ are continuous on I.

(a6) The inequality $\|f\| + cr + m + r^2\psi(r, r)\mathrm{T} \le r$ has a positive solution r_0 such that $c + 2r_0\psi(r_0, r_0)\mathrm{T} < 1$, where $m = \max_{t \in I} g(t, 0)$ and $\mathrm{T} = \sup\{\bigvee_{s=0}^{1} h(t, s) : t \in I\}$ (see Remark 2.10).

Theorem 3.1. *Suppose that assumptions (a1)–(a6) are satisfied. Then the equation* (1) *has at least one solution $x \in C(I)$ being nondecreasing on I.*

Proof. Let the function $M : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$M(\varepsilon) = \sup\left\{ \bigvee_{s=0}^{1} (h(t_2, s) - h(t_1, s)) : t_1, t_2 \in I,\ t_1 < t_2,\ t_2 - t_1 \le \varepsilon \right\}.$$

Then, by Proposition 2.8, we have $M(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We denote by $\mathfrak{F}$ the operator associated with the right-hand side of (1), so (1) becomes $x = \mathfrak{F}x$, where

$$(\mathfrak{F}x)(t) = f(t) + g(t, x(t)) + x^2(t) \int_0^1 u(t, s, x(s), x(\lambda s))\, d_s h(t, s),\ t \in I.$$

Notice that solving (1) is equivalent to finding a fixed point of the operator $\mathfrak{F}$ defined on the space $C(I)$.

Now, we will prove that if $x \in C(I)$ then $\mathfrak{F}x \in C(I)$. To prove this, it suffices to show that if $x \in C(I)$ then $\mathcal{U}x \in C(I)$, where $(\mathcal{U}x)(t) = \int_0^1 u(t, s, x(s), x(\lambda s))\, d_s h(t, s)$. We fix $\varepsilon > 0$ and take $t_1, t_2 \in I$ with $t_1 < t_2$ and $t_2 - t_1 \le \varepsilon$. Let $x \in C(I)$. Then there exists $\nu > 0$ with $\max\{\|x\|, \|x\|\} \le \nu$. Now, we have

$$|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)|$$

$$
\begin{aligned}
&=\left|\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right|\\
&\leq\left|\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)\right|\\
&\quad+\left|\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right|\\
&\leq\int_0^1 |u(t_2,s,x(s),x(\lambda s))-u(t_1,s,x(s),x(\lambda s))|\,d_s\Big(\bigvee_{p=0}^{s} h(t_2,p)\Big)\\
&\quad+\int_0^1 |u(t_1,s,x(s),x(\lambda s))|\,d_s\Big(\bigvee_{p=0}^{s}(h(t_2,p)-h(t_1,p))\Big)\\
&\leq\varphi_\nu(t_2-t_1)\bigvee_{p=0}^{1} h(t_2,p)+\psi(\|x\|,\|x\|)\bigvee_{p=0}^{1}(h(t_2,p)-h(t_1,p))\\
&\leq\varphi_\nu(\varepsilon)\mathrm{T}+\psi(\|x\|,\|x\|)M(\varepsilon).
\end{aligned}
$$

The above estimate gives us that $\omega(\mathcal{U}x,\varepsilon)\leq\varphi_\nu(\varepsilon)\mathrm{T}+\psi(\|x\|,\|x\|)M(\varepsilon)$. Thus, we have $\omega(\mathcal{U}x,\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore, $\mathcal{U}x\in C(I)$, and consequently, $\mathfrak{F}x\in C(I)$.

Next, we prove that the operator $\mathfrak{F}$ is continuous on the space $C(I)$. In order to prove this it suffices (to see the full proof for $\mathfrak{F}$ see the argument after the definition of $B^+_{r_0}$) to show that the operator $\mathcal{U}$ is continuous on $C(I)$. Fix $\varepsilon>0$ and take an arbitrary $x\in C(I)$ such that $\|x-y\|\leq\varepsilon$. Then, for fixed $t\in I$, we have

$$
\begin{aligned}
|(\mathcal{U}x)(t)-(\mathcal{U}y)(t)|&=\left|\int_0^1 u(t,s,x(s),x(\lambda s))\,d_sh(t,s)-\int_0^1 u(t,s,y(s),y(\lambda s))\,d_sh(t,s)\right|\\
&\leq\int_0^1 |u(t,s,x(s),x(\lambda s))-u(t,s,y(s),y(\lambda s))|\,d_s\Big(\bigvee_{p=0}^{s} h(t,p)\Big)\\
&\leq\beta(\varepsilon)\bigvee_{p=0}^{1} h(t,p)\leq\beta(\varepsilon)\mathrm{T},
\end{aligned}
$$

where, $\beta(\varepsilon)=\sup\{|u(t,s,x_1,y_1)-u(t,s,x_2,y_2)|: t,s\in I, x_1,x_2,y_1,y_2\in[-\|x\|-\varepsilon, \|x\|+\varepsilon],\ |x_1-x_2|\leq\varepsilon,\ |y_1-y_2|\}$. Notice that $\beta(\varepsilon)\to 0$ as $\varepsilon\to 0$, because the function $u(t,s,x,y)$ is uniformly continuous on the set $I^2\times[-\|x\|-\varepsilon,\|x\|+\varepsilon]^2$. Thus the last inequality guarantees that the operator $\mathcal{U}$ is continuous and consequently the operator $\mathfrak{F}$ is continuous.

Now using our assumptions, for arbitrary $x\in C(I)$, we obtain

$$
\begin{aligned}
|(\mathfrak{F}x)(t)|&=\left|f(t)+g(t,x(t))+x^2(t)\int_0^1 u(t,s,x(s),x(\lambda s))\,d_sh(t,s)\right|\\
&\leq|f(t)|+|g(t,x(t))-g(t,0)|+|g(t,0)|\\
&\quad+|x^2(t)|\int_0^1 |u(t,s,x(s),x(\lambda s))|\,d_s\Big(\bigvee_{p=0}^{s} h(t,p)\Big)
\end{aligned}
$$

$$\leq \|f\| + c\|x\| + m + \|x\|^2 \int_0^1 |u(t,s,x(s),x(\lambda s))|\, d_s\Big(\bigvee_{p=0}^{s} h(t,p)\Big)$$
$$\leq \|f\| + c\|x\| + m + \|x\|^2; \psi(\|x\|,\|x\|)\Big(\bigvee_{p=0}^{1} h(t,p)\Big)$$
$$\leq \|f\| + c\|x\| + m + \|x\|^2 \psi(\|x\|,\|x\|)\mathrm{T}.$$

If $\|x\| \leq r_0$, then by assumption (a6), we get $\|f\| + cr_0 + m + r_0^2\psi(r_0,r_0)\mathrm{T} \leq r_0$. Therefore, $\mathfrak{F}$ maps the closed ball B_{r_0} into itself.

In the following, we consider the operator $\mathfrak{F}$ on the set $B_{r_0}^+$ defined by $B_{r_0}^+ = \{x \in B_{r_0} : x(t) \geq 0,\ \text{for } t \in I\}$. Note that the set $B_{r_0}^+$ is a nonempty, closed, bounded and convex subset of $C(I)$. From assumptions (a1)–(a5), we infer that $\mathfrak{F}$ maps the set $B_{r_0}^+$ into itself. Note that the operator $\mathfrak{F}$ is continuous on $B_{r_0}^+$ (we basically established it before but here we present the full argument). Fix $\varepsilon > 0$ and take arbitrary $x, y \in B_{r_0}^+$ with $\|x-y\| \leq \varepsilon$. Then, for $t \in I$, we have

$$|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)|$$
$$\leq |g(t,x(t)) - g(t,y(t))|$$
$$+ \left| x^2(t)\int_0^1 u(t,s,x(s),x(\lambda s))\, d_s h(t,s) - y^2(t)\int_0^1 u(t,s,y(s),y(\lambda s))\, d_s h(t,s)\right|$$
$$\leq c|x(t)-y(t)|$$
$$+ \left| x^2(t)\int_0^1 u(t,s,x(s),x(\lambda s))\, d_s h(t,s) - y^2(t)\int_0^1 u(t,s,x(s),x(\lambda s))\, d_s h(t,s)\right|$$
$$+ \left| y^2(t)\int_0^1 u(t,s,x(s),x(\lambda s))\, d_s h(t,s) - y^2(t)\int_0^1 u(t,s,y(s),y(\lambda s))\, d_s h(t,s)\right|$$
$$\leq c\|x-y\| + |x^2(t) - y^2(t)| \int_0^1 |u(t,s,x(s),x(\lambda s))|\, d_s h(t,s)$$
$$+ |y^2(t)| \int_0^1 |u(t,s,x(s),x(\lambda s)) - u(t,s,y(s),y(\lambda s))|\, d_s h(t,s)$$
$$\leq c\|x-y\| + \|x-y\|(\|x\|+\|y\|)\psi(\|x\|,\|x\|) \bigvee_{p=0}^{1} h(t,p) + \|y\|^2\beta(\varepsilon) \bigvee_{p=0}^{1} h(t,p)$$
$$\leq c\varepsilon + 2r_0\ \varepsilon\psi(r_0,r_0)\mathrm{T} + r_0^2\beta(\varepsilon)\mathrm{T}.$$

Therefore, $\|\mathfrak{F}x - \mathfrak{F}y\| \leq c\varepsilon + 2r_0\ \varepsilon\psi(r_0,r_0)\mathrm{T} + r_0^2\beta(\varepsilon)\mathrm{T}$ and this implies that the operator $\mathfrak{F}$ is continuous on the set $B_{r_0}^+$.

In the following, we consider $\emptyset \neq X \subset B_{r_0}^+$. We fix an arbitrary number $\varepsilon > 0$ and choose $x \in X$ and $t_1, t_2 \in I$ with $t_2 \geq t_1$ and $|t_2 - t_1| \leq \varepsilon$. Then, in view of our assumptions, we get

$$|(\mathfrak{F}x)(t_2) - (\mathfrak{F}x)(t_1)|$$
$$\leq |f(t_2) - f(t_1)| + |g(t_2,x(t_2)) - g(t_1,x(t_1))|$$

$$
\begin{aligned}
&+\left|x^2(t_2)\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-x^2(t_1)\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right|\\
\le&\omega(f,\varepsilon)+|g(t_2,x(t_2))-g(t_1,x(t_2))|+|g(t_1,x(t_2))-g(t_1,x(t_1))|\\
&+\left|x^2(t_2)\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-x^2(t_2)\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)\right|\\
&+\left|x^2(t_2)\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)-x^2(t_1)\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)\right|\\
&+\left|x^2(t_1)\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)-x^2(t_1)\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right|\\
\le&\omega(f,\varepsilon)+\gamma_{r_0}(g,\varepsilon)+c\omega(x,\varepsilon)\\
&+|x^2(t_2)|\int_0^1 |u(t_2,s,x(s),x(\lambda s))-u(t_1,s,x(s),x(\lambda s))|\,d_sh(t_2,s)\\
&+|x(t_2)-x(t_1)|\,|x(t_2)+x(t_1)|\int_0^1 |u(t_1,s,x(s),x(\lambda s))|\,dsh(t_2,s)\\
&+|x^2(t_1)|\int_0^1 |u(t_1,s,x(s),x(\lambda s))|\,d_s(h(t_2,s)-h(t_1,s))\\
\le&\omega(f,\varepsilon)+\gamma_{r_0}(g,\varepsilon)+c\omega(x,\varepsilon)+\|x\|^2\varphi_\nu(t_2-t_1)\bigvee_{p=0}^{1}h(t_2,p)\\
&+2\|x\|\omega(x,\varepsilon)\psi(\|x\|,\|x\|)\bigvee_{p=0}^{1}h(t_2,p)+\|x\|^2\psi(\|x\|,\|x\|)\left(\bigvee_{p=0}^{1}(h(t_2,p)-h(t_1,p))\right)\\
\le&\omega(f,\varepsilon)+\gamma_{r_0}(g,\varepsilon)+(c+2r_0\psi(r_0,r_0)\mathrm{T})\,\omega(x,\varepsilon)+r_0^2\left(\varphi_\nu(\varepsilon)\mathrm{T}+\psi(r_0,r_0)M(\varepsilon)\right),
\end{aligned}
$$

where, $\gamma_{r_0}(g,\varepsilon)=\sup\{|g(t,x)-g(s,x)| : s,t\in I,\ x\in[0,r_0],\ |t-s|\le\varepsilon\}$. Since the function g is uniformly continuous on the set $I\times[0,r_0]$, then from the last inequality, we obtain

$$\omega_0(\mathfrak{F}X)\le(c+2r_0\psi(r_0,r_0)\mathrm{T})\omega_0(X). \tag{4}$$

Again fix an arbitrary $x\in X$ and $t_1,\ t_2\in I$ such that $t_1\le t_2$. Then we have

$$
\begin{aligned}
&|(\mathfrak{F}x)(t_2)-(\mathfrak{F}x)(t_1)|-((\mathfrak{F}x)(t_2)-(\mathfrak{F}x)(t_1))\\
=&\Bigg|f(t_2)+g(t_2,x(t_2))+x^2(t_2)\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)\\
&-f(t_1)-g(t_1,x(t_1))-x^2(t_1)\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\Bigg|\\
&-\Bigg(f(t_2)+g(t_2,x(t_2))+x^2(t_2)\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)\\
&-f(t_1)-g(t_1,x(t_1))-x^2(t_1)\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\leq & |f(t_2)-f(t_1)|-(f(t_2)-f(t_1)) \\
& +(|g(t_2,x(t_2))-g(t_1,x(t_1))|-(g(t_2,x(t_2))-g(t_1,x(t_1)))) \\
& +\left|x^2(t_2)\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)\right. \\
& \left.-x^2(t_1)\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right| \\
& +\left(x^2(t_2)\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)\right. \\
& \left.-x^2(t_1)\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right) \\
\leq & i(Gx)+|x^2(t_2)|\left|\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)\right. \\
& \left.-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right| \\
& +|x^2(t_2)-x^2(t_1)|\left|\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right| \\
& -x^2(t_2)\left(\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)\right. \\
& \left.-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right) \\
& -(x^2(t_2)-x^2(t_1))\int_0^1 u(t_1,s,x(s),x(\lambda s))d_sh(t_1,s) \\
\leq & i(Gx)+x^2(t_2)\left|\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)\right. \\
& \left.-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right| \\
& +(|x(t_2)-x(t_1)|-(x(t_2)-x(t_1)))\,(x(t_2)+x(t_1)) \\
& \times\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s) \\
& -x^2(t_2)\left(\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right) \\
\leq & i(Gx)+2r_0 i(x)\psi(\|x\|,\|x\|)\bigvee_{p=0}^{1}h(t_1,p) \\
& +x^2(t_2)\left[\left|\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right|\right.
\end{aligned}
$$

$$-\left(\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\right)\Bigg]. \quad (5)$$

We claim that $\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\geq 0$. Notice that

$$\begin{aligned}&\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\\ &=\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)\\ &\quad+\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s), \quad (6)\end{aligned}$$

so
$$\begin{aligned}&\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)\\ &=\int_0^1 (u(t_2,s,x(s),x(\lambda s))-u(t_1,s,x(s),x(\lambda s)))\,d_sh(t_2,s).\end{aligned}$$

Thus, by assumption (a4) and Corollary 2.6, we get

$$\int_0^1 u(t_2,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sg(t_2,s)\geq 0. \quad (7)$$

Also
$$\begin{aligned}&\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\\ &=\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_s(h(t_2,s)-h(t_1,s)).\end{aligned}$$

Recall that $h(t_1,s)$, $h(t_2,s)$ and $(h(t_2,s)-h(t_1,s))$ are nondecreasing functions (assumption (a5)), and $u(t_1,s,x,y)\geq 0$ (assumption (a4)). Thus, by Lemma 2.7, we infer that

$$\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_2,s)-\int_0^1 u(t_1,s,x(s),x(\lambda s))\,d_sh(t_1,s)\geq 0. \quad (8)$$

Now from (6), (7) and (8) our claim is proved. Therefore, from (5), we obtain $i(\mathfrak{F}x)\leq i(Gx)+2r_0 i(x)\psi(r_0,r_0)\mathrm{T}$ or $i(\mathfrak{F}x)\leq (c+2r_0\psi(r_0,r_0)\mathrm{T})i(x)$ and consequently

$$i(\mathfrak{F}X)\leq (c+2r_0\psi(r_0,r_0)\mathrm{T})i(X). \quad (9)$$

Finally, (4) and (9) imply that $\omega_0(\mathfrak{F}X)+i(\mathfrak{F}X)\leq (c+2r_0\psi(r_0,r_0)\mathrm{T})(\omega_0(X)+i(X))$ or $\mu(\mathfrak{F}X)\leq (c+2r_0\psi(r_0,r_0)\mathrm{T})\mu(X)$. Now, from the fact that $(c+2r_0\psi(r_0,r_0)\mathrm{T})<1$, we can apply Theorem 2.2. Therefore, the quation (1) has at least one solution $x\in C(I)$ being nondecreasing I. □

Now we give an example of a function $h: I^2\to\mathbb{R}$ which satisfies assumption (a5). Let the function h be defined by

$$h(t,s)=\begin{cases}0, & \text{for } t=0,\ s\in I,\\ t\ln\left(\frac{t+s}{t}\right), & \text{for } 0<t\leq 1,\ s\in I.\end{cases}$$

The function $s \mapsto h(t,s)$ is nondecreasing for each $t \in I$ since $\frac{d}{ds}\left[t \ln\left(\frac{t+s}{t}\right)\right] = \frac{1}{t+s} \geq 0$, $t, s \in I$. Therefore $\bigvee_{s=0}^{1} h(t,s) \leq \ln 2$.

To show that the function h satisfies assumptions part (b) and (c) of (a5) we fix $t_1, t_2 \in I$ with $t_1 \leq t_2$. Then, we have

$$h(t_2,s) - h(t_1,s) = \begin{cases} t_2 \ln\left(\frac{t_2+s}{t_2}\right), & \text{for } t_1 = 0, \\ t_2 \ln\left(\frac{t_2+s}{t_2}\right) - t_1 \ln\left(\frac{t_1+s}{t_1}\right), & \text{for } t_1 > 0. \end{cases}$$

It is easy to check that the function $s \mapsto (h(t_2,s) - h(t_1,s))$ is nondecreasing on I and the functions $h(t,0)$ and $h(t,1)$ are continuous on I.

Now, we give an example to illustrate Theorem 3.1.

Example 3.2. Consider the following cubic Uryshon-Stieltjes integral equation

$$x(t) = \frac{\sqrt{t}}{5} + \frac{t}{t^2+9}x(t) + \frac{tx^2(t)}{10}\int_0^1 (t+s+x(s)+x(\lambda s))\,ds,\ t \in I = [0,1]. \tag{10}$$

Note that (10) can be written in the form of a cubic integral equation of Uryshon-Stieltjes type, namely

$$x(t) = \frac{\sqrt{t}}{5} + \frac{t}{t^2+9}x(t) + \frac{x^2(t)}{10}\int_0^1 (t+s+x(s)+x(\lambda s))\,d_s h(t,s), \tag{11}$$

where $h(t,s) = ts$. Note that (11) is a particular case of (1) with $f(t) = \frac{\sqrt{t}}{5}$, $g(t,x) = \frac{t}{t^2+1}x$, $u(t,s,x,y) = \frac{1}{10}(t+s+x+y)$ and $h(t,s) = t\,s$.

The function f satisfies assumption (a1) with $\|f\| = \frac{1}{5}$. The function $g(t,x)$ satisfies assumption (a2) with $c = 0.1$, since

$$|g(t,x) - g(t,y)| = \left|\frac{tx}{t^2+9} - \frac{ty}{t^2+9}\right| \leq \frac{1}{10}|x-y|\ \ \forall t \in I,\ (x,y) \in \mathbb{R}^2.$$

Moreover, for an arbitrary nonnegative function $x \in C(I)$ and $t_1, t_2 \in I$ with $t_1 \leq t_2$, we have

$$\begin{aligned} i(Gx) =& |(Gx)(t_2) - (Gx)(t_1)| - ((Gx)(t_2) - (Gx)(t_1)) \\ =& |g(t_2,x(t_2)) - g(t_1,x(t_1))| - (g(t_2,x(t_2)) - g(t_1,x(t_1))) \\ =& \left|\frac{t_2}{t_2^2+9}x(t_2) - \frac{t_1}{t_1^2+9}x(t_1)\right| - \left(\frac{t_2}{t_2^2+9}x(t_2) - \frac{t_1}{t_1^2+9}x(t_1)\right) \\ \leq& \frac{t_2}{t_2^2+9}|x(t_2) - x(t_1)| + \left|\frac{t_2}{t_2^2+9} - \frac{t_1}{t_1^2+9}\right| x(t_1) \\ &- \frac{t_2}{t_2^2+9}(x(t_2) - x(t_1)) - \left(\frac{t_2}{t_2^2+9} - \frac{t_1}{t_1^2+9}\right) x(t_1) \\ =& \frac{t_2}{t_2^2+9}[|x(t_2) - x(t_1)| - (x(t_2) - x(t_1))] \leq \frac{1}{10}\, i(x). \end{aligned}$$

Next, assumptions (a3) and (a4) are satisfied with $u(t,s,x,y) = \frac{1}{10}(t+s+x+y)$, $\psi(x,y) = \frac{1}{5} + \frac{1}{10}(x+y)$ and $\varphi_\nu(t) = \frac{t}{10}$.

Note that the function h satisfies assumption (a5). The inequality appearing in assumption (a6) takes the form $0.2 + 0.1r + 0.2r^2(1 + r) \leq r$, where,

$$\mathrm{T} = \sup\{\bigvee_{s=0}^{1} h(t,s) : t \in I\} = \sup\{(h(t,1) - h(t,0)) : t \in I\} = \sup\{t : t \in I\} = 1,$$

and $r_0 = 1$ is its a positive solution. Also, $c + 2r_0\psi(r_0, r_0)\mathrm{T} = 0.1 + 2 \times 0.2 < 1$. Therefore, Theorem 3.1 guarantees that the equation (10) has a nondecreasing solution.

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