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ON WEAKLY STRETCH RANDERS METRICS

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Abstract. The class of weakly stretch Finsler metrics contains the class of stretch metric. Randers metrics are important Finsler metrics which are defined as the sum of a Riemann metric and a 1-form. In this paper, we prove that every Randers metric with closed and conformal one-form is a weakly stretch metric if and only if it is a Berwald metric.

1. Introduction

In [5], L. Berwald introduced the non-Riemannian curvature, so-called stretch curvature as a natural extension of Landsberg curvature. He denoted it by $\mathbf{\Sigma}$ and proved that $\mathbf{\Sigma} = 0$ if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. A Finsler metric is said to be stretch metric if $\mathbf{\Sigma} = 0$. Then Matsumoto investigated the class of stretch metrics with scalar flag curvature [8]. In [12], Shibata showed that every stretch metric of non-zero scalar flag curvature is a Riemannian metric of constant sectional curvature. In [13], he proved that a Kropina metric $F = \alpha^2/\beta$ has vanishing stretch curvature if and only if it is a Berwald metric. Bácsó-Matsumoto showed that a Douglas metric on a manifold of dimension $n \geq 3$ is R-quadratic if and only if it is a stretch metric with horizontally constant mean Berwald curvature [3]. In [9], Najafi-Bidabad-Tayebi proved that every R-quadratic Finsler metric is a stretch metric.

In 1941, Randers published a paper concerned with an asymmetric metric in the four-space of general relativity. His metric is in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is gravitation field and $\beta = b_i(x)y^i$ is the electromagnetic field. He regarded these metrics not as Finsler metrics but as "affinely connected Riemannian metrics". This metric was first recognized as a kind of Finsler metric in 1957 by Ingarden, who first named them Randers metrics. In [22], Tayebi-Tabatabeifar showed that a Randers metric $F = \alpha + \beta$ with closed one-form β is a stretch metric if and only if it is a Berwald metric. In [21], Tayebi-Sadeghi characterized the stretch (α, β)-metrics of non-Randers type with vanishing S-curvature. In [4], Bácsó-Szilasi showed

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that if the stretch tensor of a Finsler metric depends only on the position, then it vanishes identically. Recently, Tayebi-Najafi proved that a homogeneous (α, β)-metric is a stretch metric if and only if it is a Berwald metric [18].

Taking the trace with respect to $\mathbf{g}_y$ in first and second variables of $\mathbf{\Sigma}_y$ gives rise to the mean stretch curvature $\bar{\mathbf{\Sigma}}_y$. A Finsler metric is said to be weakly stretch metric if $\bar{\mathbf{\Sigma}} = 0$. In [10], Najafi-Tayebi showed that every compact weakly stretch manifold is a weakly Landsberg manifold. Then, they proved a rigidity theorem stating that every compact weakly stretch manifold with negative flag curvature reduces to a Riemannian manifold. In [16], Tayebi-Izadian studied a square metrics $F = \alpha + 2\beta + \beta^2/\alpha$ with vanishing Douglas curvature. They showed that F is a weakly stretch metric if and only if it reduces to a Berwald metric.

By definition, we have the following

$$\begin{aligned}\{\text{Berwald metrics}\} &\subseteq \{\text{R-quadratic metrics}\}\\ &\subseteq \{\text{Stretch metrics}\} \subseteq \{\text{Weakly stretch metrics}\}.\end{aligned}$$

In [7], Li-Shen found the necessary and sufficient condition under which a Randers metric is R-quadratic. It follows that a weakly stretch Randers metric is not a Berwald metric, in general. Najafi-Tayebi showed that every generalized Berwald Randers metric is a weakly stretch metric if and only if it is a Berwald metric [10]. A Finsler manifold (M, F) is called a generalized Berwald manifold if there exists a covariant derivative ∇ on M such that the parallel translations induced by ∇ preserve the Finsler function F. If the covariant derivative ∇ is also torsion-free, then (M, F) is called a Berwald manifold. It is interesting to find some curvature properties conditions under which a weakly stretch Randers metric reduces to a Berwald metric. Then, we prove the following.

Theorem 1.1. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M and β a closed and conformal 1-form with respect to α. Then F is a weakly stretch metric if and only if it is a Berwald metric.*

Theorem 1.1 is a generalization of [22, Theorem 1.1] which indicated that a Randers metric with closed 1-form and vanishing stretch curvature is a Berwald metric.

In Theorem 1.1, the condition on 1-form is necessary. See the following examples.

Example 1.2 ([7]). Let $|.|$ and $\langle , \rangle$ denote the Euclidean norm and the inner product in $\mathbb{R}^n$, respectively. Consider the following Randers metric defined nearby the origin in $\mathbb{R}^n$

$$F := \frac{\sqrt{|y|^2 - (|xQ|^2|y|^2 - \langle y, xQ\rangle^2)}}{1 - |xQ|^2} - \frac{\langle y, xQ\rangle}{1 - |xQ|^2},$$

where $Q = (q^i_j)$ is an anti-symmetric matrix. F is a weakly stretch metric but it is not a Berwald metric when $Q \neq 0$.

Example 1.3. Let us consider the well-known Shen's fish tank metric as follows. Let $X = (x, y, z) \in \mathbb{B}^3(1) \subset \mathbb{R}^3$ and $Y = (u, v, w) \in T_x\mathbb{B}^3(1)$. Put

$$F = \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2 + w^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2} + \frac{xv - yu}{1 - x^2 - y^2}.$$

The Shen's fish tank metric F is a weakly stretch metric which is not a Berwald metric [11].

There are many Riemannian metrics with nontrivial closed and conformal 1-forms. See the following.

EXAMPLE 1.4 ([23]). The Riemmannian metric $\alpha = e^{\rho}\sqrt{|y|^2 - \kappa\langle x, y\rangle^2}$ has closed and conformal 1-form expressed as $\beta = c\sqrt{1 - \kappa|x|^2}e^{2\rho}\langle x, y\rangle$, where $\kappa = \kappa(|x|^2)$ and $\rho = \rho(|x|^2)$ are two arbitrary functions such that $1 - \kappa|x|^2 > 0$.

Every Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if the 1-form β is a closed 1-form [2]. Then by Theorem 1.1, we get the following.

COROLLARY 1.5. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M and β conformal 1-form with respect to α. Then F is a weakly stretch metric with vanishing Douglas curvature if and only if it is a Berwald metric.*

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on $TM_0 := TM \setminus \{0\}$;

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;

(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian. For $y \in T_xM_0$, define the mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Deicke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where $L_{ijk} := C_{ijk|s}y^s$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L}=\mathbf{0}$.

The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := g^{jk}L_{ijk} = I_{i|s}y^s$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$ [15].

Define the stretch curvature $\mathbf{\Sigma}_y : T_xM\times T_xM\times T_xM\times T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(u,v,w,z) := \Sigma_{ijkl}(y)u^iv^jw^kz^l$, where $\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k})$.

Here " | " denotes the horizontal derivation with respect to the Berwald connection of F. A Finsler metric is said to be stretch metric if $\mathbf{\Sigma} = 0$ [5].

Taking an average on two first indices of the stretch curvature, we get a new non-Riemannian curvature, namely, mean stretch curvature. For $y \in T_xM_0$, define $\bar{\mathbf{\Sigma}}_y : T_xM \times T_xM \to \mathbb{R}$ by $\bar{\mathbf{\Sigma}}_y(u,v) := \bar{\Sigma}_{ij}(y)u^iv^j$, where $\bar{\Sigma}_{ij} := g^{kl}\Sigma_{klij}$.

A Finsler metric is said to be weakly stretch metric if $\bar{\mathbf{\Sigma}} = 0$. Every Landsberg metric or stretch metric is a weakly stretch metric.

Given a Finsler manifold (M,F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i(x,y) := \frac{1}{4}g^{il}\Big\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\Big\}.$$

The vector field $\mathbf{G}$ is called the associated spray to (M,F).

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ and $\mathbf{E}_y : T_xM\times T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u,v) := E_{jk}(y)u^jv^k$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j\partial y^k\partial y^l}, \quad \text{and} \quad E_{jk} := \frac{1}{2}B^m{}_{jkm}.$$

The non-Riemannian quantities $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature of F, respectively. F is a Berwald (resp. weakly Berwald) metric if it satisfies $\mathbf{B} = \mathbf{0}$ (resp. $\mathbf{E} = \mathbf{0}$).

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric, where $\phi = \phi(s)$ is a C^∞ on $(-b_0,b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. For an (α,β)-metric, let us define $b_{i|j}$ by $b_{i;j}\theta^j := db_i - b_j\theta^j_i$, where $\theta^i := dx^i$ and $\theta^j_i := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}),\ s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),\ r_{i0} := r_{ij}y^j,\ r_{00} := r_{ij}y^iy^j,\ r_j := b^ir_{ij},$$
$$s_{i0} := s_{ij}y^j,\ s_j := b^is_{ij},\ s^i{}_j = a^{im}s_{mj},\ s^i{}_0 = s^i{}_jy^j,\ r_0 := r_jy^j,\ s_0 := s_jy^j.$$

Put
$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q', \quad \Theta := \frac{Q - sQ'}{2\Delta},$$
$$\Psi := \frac{Q'}{2\Delta} = \frac{\phi''}{2\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]},$$

where $b^2 := a^{ij}b_ib_j$. Let $G^i = G^i(x,y)$ and $G^i_\alpha = G^i_\alpha(x,y)$ denote the coefficients of F and α, respectively, in the same coordinate system. By definition, we have

$$G^i = G^i_\alpha + Py^i + Q^i, \tag{1}$$

where $P := \alpha^{-1}\Theta[r_{00} - 2\alpha Qs_0]$, $Q^i := \alpha Qs^i{}_0 + \Psi[r_{00} - 2\alpha Qs_0]b^i$. Simplifying (1) yields the following

$$G^i = G^i_\alpha + \alpha Qs^i_0 + (r_{00} - 2\alpha Qs_0)(\alpha^{-1}\Theta y^i + \Psi b^i). \tag{2}$$

Clearly, if β is parallel with respect to α, that is $r_{ij} = 0$ and $s_{ij} = 0$, then $P = 0$ and

$Q^i = 0$. In this case, $G^i = G^i_\alpha$ are quadratic in y and F reduces to a Berwald metric. β is conformal with respect to α if $r_{ij} = ca_{ij}$, where $c = c(x)$ is a scalar function on M. Also, β is a closed 1-form if $s_{ij} = 0$. For more details, see [17, 19].

For an (α, β)-metric $F = \alpha\phi(s)$, the mean Landsberg curvature is given by

$$\begin{aligned} J_j = &- \frac{1}{\Delta\alpha^2(b^2-s^2)}\left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\right](r_0 + s_0)h_i \\ &- \frac{h_i}{2\alpha^3\Delta(b^2-s^2)}\left(\psi_1 + s\frac{\Phi}{\Delta}\right)\left(r_{00} - 2\alpha Q s_0\right) \\ &- \frac{\Phi}{2\alpha^3\Delta^2}\big[\alpha Q(\alpha^2 s_i - s_0 y_i) - \alpha Q' s_0 h_i + \alpha^2\Delta s_{i0} \\ &- \alpha^2(r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0)y_i\big], \end{aligned} \tag{3}$$

where $\Phi := -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''$, $\Psi_1 := \sqrt{b^2 - s^2}\,\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2-s^2}\,\Phi}{\Delta^{\frac{3}{2}}}\right]'$, $h_i := \alpha b_i - sy_i$.

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. For this aim, we remark that the mean stretch curvature is given by following

$$\bar{\Sigma}_{ij} := g^{kl}\Sigma_{klij} = 2g^{kl}(L_{kli|j} - L_{klj|i}). \tag{4}$$

For the Berwald connection, we have $g_{ij|k} = -2L_{ijk}$. Then, we get

$$(g^{ij})_{|k} = 2L^{ij}_{\ \ k}. \tag{5}$$

$$\overset{(5)}{\Longrightarrow} \quad g^{kl}L_{kli|j} = J_{i|j} - 2L^{kl}_{\ \ j}L_{kli}. \tag{6}$$

$$\overset{(4),(6)}{\Longrightarrow} \quad \bar{\Sigma}_{ij} = 2(J_{i|j} - J_{j|i}). \tag{7}$$

By (7), F is a weakly stretch metric if and only if $J_{i|j} = J_{j|i}$.

LEMMA 3.1. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M and β a closed and conformal 1-form with respect to α. Suppose that F is a weakly stretch metric. Then the following hold*

$$A_4\alpha^4 + A_2\alpha^2 + A_0 = 0, \tag{8}$$

$$A_5\alpha^4 + A_3\alpha^2 + A_1 = 0, \tag{9}$$

where

$A_0 := -2(n+1)r_{00;0}\beta^4 + (2n+1)r_{00}^2\beta^3$,

$A_1 := -8(n+1)r_{00;0}\beta^3 + (8n+5)r_{00}^2\beta^2$,

$A_2 := 4(n+1)r_{0;0}\beta^3 - 2(n+1)(5+b^2)r_{00;0}\beta^2 - (6n+4)r_0r_{00}\beta^2 + \big[6n+4+(4n+3)b^2\big]r_{00}^2\beta$,

$A_3 := 12(n+1)r_{0;0}\beta^2 - 4(n+1)(1+b^2)r_{00;0}\beta - 4(3n+2)r_0r_{00}\beta + (4n+3)b^2r_{00}^2$,

$A_4 := 12(n+1)r_{0;0}\beta - 2(n+1)b^2r_{00;0} - 2(3n+2)r_0r_{00}$,

$A_5 := 4(n{+}1)r_{0;0}$ *and* $b^2 := a_{ij}b^ib^j$.

Proof. For a Randers metric $\phi = 1+s$, we have

$$Q = Q - sQ' = 1, \qquad \Delta = 1+s,$$
$$\Phi = -(n+1)(1+s), \quad \Psi_1 = \frac{n+1}{2(1+s)}(s^2+2s+b^2). \tag{10}$$

By assumption, β is a closed and conformal 1-form with respect to α. Thus it satisfies $b_{i;j} = ca_{ij}$, where $c = c(x)$ is a scalar function on M, which implies that

$$s_{ij} = 0, \quad s_i = 0, \quad r_{ij} = ca_{ij}, \quad r_i = cb_i. \tag{11}$$

By putting (11) in (2), we get $G^i = \bar{G}^i + \frac{r_{00}}{2F}\, y^i$. Also, putting (10) and (11) in (3) imply that

$$J_i = -\frac{n+1}{4\alpha^3(1+s)^2}r_{00}h_i + \frac{n+1}{2\alpha^3(1+s)}\Big[\alpha^2 r_{0i} - r_{00}y_i\Big]. \tag{12}$$

Taking a horizontal derivation of (12) with respect to the Berwald connection of F yields the following

$$\begin{aligned} J_{i|j} =& -\frac{(n{+}1)r_{00}}{4\alpha^4(1{+}s)^3}\big[\alpha(1{+}s)(\alpha_{|j}b_i{+}\alpha b_{i|j}{-}s_{|j}y_i{-}sy_{i|j}){-}(\alpha b_i{-}sy_i)\big(3(1{+}s)\alpha_{|j}{+}2\alpha s_{|j}\big)\big]\\ &+\frac{n{+}1}{4\alpha^6(1{+}s)^2}\big[2(1{+}s)(2\alpha r_{i0}\alpha_{|j}{+}\alpha^2 r_{i0|j}{-}y_i r_{00|j}{-}r_{00}y_{i|j})\alpha^3\\ &-2\alpha^2(\alpha^2 r_{i0}{-}r_{00}y_i)(3\alpha_{|j}{+}3s\alpha_{|j}{+}\alpha s_{|j})\big]{-}\frac{(n{+}1)(\alpha b_i{-}sy_i)}{4\alpha^3(1{+}s)^2}r_{00|j}, \end{aligned}$$

where $y_i = \alpha\alpha_{y^i}$. The following holds

$$\begin{aligned} \alpha_{|j} =& \frac{\partial\alpha}{\partial x^j} - G^m_j\frac{\partial\alpha}{\partial y^m} = \frac{\partial\alpha}{\partial x^j} - \bar{G}^m_j\frac{\partial\alpha}{\partial y^m} - \frac{\partial\alpha}{\partial y^m}\Big(\frac{2r_{j0}y^m + r_{00}\delta^m_j}{2F} - \frac{F_j r_{00}y^m}{2F^2}\Big)\\ =& -\frac{\alpha r_{j0}}{F} - \frac{r_{00}y_j}{2\alpha F} + \frac{\alpha F_j r_{00}}{2F^2}. \end{aligned} \tag{13}$$

Also, we have

$$\begin{aligned} b_{i|j} =& \frac{\partial b_i}{\partial x^j} - G^m_j\frac{\partial b_i}{\partial y^m} - b_m\Gamma^m_{ij} = r_{ij} - \frac{r_{ij}\beta + r_{i0}b_j + r_{0j}b_i}{F}\\ &+\frac{(2F_j r_{i0} + 2F_i r_{0j} + F_{ij}r_{00})\beta + (F_i b_j + F_j b_i)r_{00}}{2F^2} - \frac{F_jF_i r_{00}\beta}{F^3}. \end{aligned} \tag{14}$$

By (13) and (14), we get

$$\begin{aligned} \beta_{|j} &= r_{0j} - \frac{1}{2F^2}\big[2Fr_{j0}\beta + Fr_{00}b_j - F_j r_{00}\beta\big],\\ s_{|j} &= \frac{1}{\alpha}r_{j0} - \frac{1}{2\alpha^2 F}\big[\alpha r_{00}b_j - sr_{00}y_j\big]. \end{aligned} \tag{15}$$

The following holds

$$\begin{aligned} y_{i|j} =& \frac{r_{00}y_iF_j + (2\alpha^2 r_{0j} + r_{00}y_j)F_i + (2\alpha^2 r_{i0} + r_{00}y_i)F_j + \alpha^2 r_{00}F_{ij}}{2F^2}\\ &-\frac{\alpha^2 r_{00}F_iF_j}{F^3} - \frac{4r_{j0}y_i + 2r_{i0}y_j + r_{00}a_{ij} + 2\alpha^2 r_{ij}}{2F}. \end{aligned} \tag{16}$$

Since F is a weakly stretch metric, we get $J_{i|j}y^j = 0$. Thus $J_{i|j}y^jb^i = 0$. By putting (13), (14), (15) and (16) in $J_{i|j}y^jb^i = 0$, one can get the following

$$\frac{r_{00}}{A}\big[2(s+b^2)r_{00}\alpha^2\beta-4r_{00}\alpha^2b^2F-2(2r_0\beta+r_{00}b^2)F\alpha^2+6\beta Fr_{00}\beta+4\beta Fr_0\alpha^2+\alpha r_0$$
$$-2\beta(s+b^2)r_{00}\alpha^2-sr_{00}+4(1+s)(\alpha b^2-s\beta)r_{00}\alpha^2\big]+\frac{n+1}{BF}\big[2F^2(\alpha^2r_{i0|0}b^i-r_{00|0}\beta)$$
$$-4Fr_0r_{00}\alpha^2+r_{00}[3Fr_{00}\beta+2Fr_0\alpha^2-(s+b^2)r_{00}\alpha^2]+4Fr_{00}(\alpha^2r_0-r_{00}\beta)\big]\alpha^2$$
$$-\frac{n+1}{4\alpha^3(1+s)^2}(\alpha b^2-s\beta)r_{00|0}=0, \tag{17}$$

where $A := 16\alpha^6(1+s)^4$, $B := 4\alpha^6(1+s)^2$. By a simple calculation, we get

$$r_{ij|k} = r_{ij;k}-\frac{2}{F}\big[r_{ij}r_{0k}+r_{ik}r_{j0}+r_{kj}r_{i0}\big]+\frac{1}{2F^2}\big[F_i(2r_{0k}r_{0j}+r_{00}r_{kj})$$
$$+F_j(2r_{0k}r_{i0}+r_{00}r_{ik})+2F_k(2r_{i0}r_{0j}+r_{00}r_{ij})+(F_{jk}r_{i0}+F_{ik}r_{0j})r_{00}\big]$$
$$-\frac{F_k}{F^3}(r_{0j}F_i+r_{i0}F_j)r_{00}. \tag{18}$$

Multiplying (18) with y^jy^k implies that

$$r_{i0|0} = r_{i0;0}-\frac{6}{F}r_{00}r_{i0}+\frac{1}{2F^2}(5r_{00}F_i+11Fr_{i0})r_{00}. \tag{19}$$

Contracting (19) with y^i implies that

$$r_{00|0} = r_{00;0}-\frac{2}{F}r_{00}^2. \tag{20}$$

By putting (19) and (20) in (17), we get

$$A_5\alpha^5+A_4\alpha^4+A_3\alpha^3+A_2\alpha^2+A_1\alpha+A_0=0. \tag{21}$$

By (21), we get (8) and (9). □

Proof (of Theorem 1.1). Since $r_{ij} = ca_{ij}$ then we have $r_{00} = c\alpha^2$, $r_{i0;j} = c_{x^j}y_i$ and $r_{00;0} = c_0\alpha^2$, where $y_i = a_{im}y^m$ and $c_0 := c_{x^i}y^i$. Putting these relations in (8) and (9) imply that

$$b^2\big[(4n+3)\beta c^2-2(n+1)c_0\big]\alpha^4+2\big[(n+1)(1-b^2)c_0\beta^2-(2n+1)c^2\beta^3\big]\alpha^2+2(n+1)c_0\beta^4=0,$$
$$\big[(4n+3)b^2c^2\big]\alpha^4+\big[-(4n+3)\beta^2c^2-4(n+1)b^2\beta c_0\big]\alpha^2+4(n+1)c_0\beta^3=0. \tag{22}$$

From here, we have $\theta\alpha^2 = c_0\beta^4$, where $\theta := \delta\alpha^2+\eta\beta^2$, and δ and η are two 1-forms on M. This means that $\alpha^2|c_0\beta$, which contradicts with the positive-definiteness of α. Then $c_0 = 0$. In this case, (22) reduces to following $(4n+3)c^2(b^2\alpha^2-\beta^2) = 0$. It is easy to see that $b^2\alpha^2-\beta^2 = 0$ contradicts with the positive-definiteness of α. Hence, $c = 0$. In this case, (11) implies that β is parallel with respect to α and then F reduces to a Berwald metric. □

Corollary 3.2. *Let $F = \alpha+\beta$ be a non-Riemannian Randers metric on a manifold M and β conformal 1-form with respect to α. Then F is a weakly stretch metric with vanishing mean Berwald curvature if and only if it is a Berwald metric.*

Proof. In [6], it is proved that a Randers metric has vanishing mean Berwald curvature if and only if the following holds

$$r_{ij} = -b_i s_j - b_j s_i. \tag{23}$$

Contracting (23) with $y^i y^j$ implies that

$$r_{00} = -2\beta s_0. \tag{24}$$

Let β be a conformal 1-form with respect to α, i.e., $r_{ij} = ca_{ij}$, where $c = c(x)$ is a scalar function on M. Multiplying it with $y^i y^j$ yields

$$r_{00} = c\alpha^2. \tag{25}$$

By (24) and (25), we have

$$c\alpha^2 = -2\beta s_0. \tag{26}$$

Since F is a non-Riemannian metric, then (26) implies that $c = 0$. In this case, we get $r_{ij} = 0$ and (24) implies that $s_i = 0$. By the same method used in the proof of [10, Theorem 1.3], we get $s_{ij} = 0$. Thus F reduces to a Berwald metric. □

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