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QUANTALE-VALUED WIJSMAN CONVERGENCE

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Abstract. For the hyperspace of non-empty closed sets of a quantale-valued metric space, we define a quantale-valued convergence tower which generalizes the classical Wijsman convergence. We characterize this quantale-valued convergence tower by a quantale-valued neighbourhood tower and show that it is uniformizable. Finally we study compactness and completeness of the quantale-valued Wijsman convergence tower.

1. Introduction

In [26] a convergence notion for closed, non-empty subsets of a metric space (X, d) was introduced, which proved to be fundamental in the study of hyperspaces [3]. A sequence of such sets A_n converges to a set A if the sequence of values of the distance functionals $d(x, A_n)$ converges to the value of the distance functional $d(x, A)$ for each $x \in X$. It is quite obvious, that this is nothing else than convergence in the initial construction with respect to the source $(d(x, \cdot))_{x \in X}$. However, because initial constructions which behave well with respect to the underlying topologies are in general not possible in the category of metric spaces, one is forced right from the start to move to at least the bigger category of topological spaces, i.e. one has to abandon in a certain sense the "metrical information" and has to content oneself with the "topological information". This has led Lowen and Sioen [19] and [18] to consider the category of approach spaces as more suitable supercategory of the category of metric spaces and to study Wijsman convergence there in terms of so-called approach systems and gauges. In [14] we followed these approaches and generalized them for quantale-valued metric spaces.

This paper goes in a similar direction and it has for this reason some overlap in the results with [14]. However, the methods that we use are different. We note that for a restricted choice of quantales, in particular for Lawvere's quantale, the descriptions of approach spaces by L-gauges, L-approach systems, L-approach distances and L-limit

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functions are all equivalent, see [10, 11]. However, e.g. in the probabilistic case with the quantale of distance distribution functions, the transitions between the different descriptions do not always yield isomorphism functors between the corresponding categories [10,11]. It makes therefore sense, to study a theory of Wijsman convergence from a different point of view in the general quantale-valued case.

While much of our inspiration goes back to [18], noting that approach structures can as well be defined by limit functions, we generalize the setting in [18,19] further by considering quantale-valued metric spaces. In this way, the theory developped here applies e.g. also to probabilistic metric spaces [25]. We use the framework of topological quantale-valued convergence tower spaces [12] in order to define Wijsman convergence for the hyperspace of non-empty closed subsets of a quantale-valued metric space as an initial construction. We show that the quantale-valued metrical coreflection of the resulting quantale-valued Wijsman convergence tower is the quantale-valued Hausdorff metric as studied in [2]. Restricting to the "classical" case of metric spaces by using Lawvere's quantale, we note that the topological coreflection of the quantale-valued Wijsman convergence tower is the classical Wijsman topology [3]. We furthermore describe the quantale-valued Wijsman convergence tower by suitable towers of neighbourhood filters and uniform towers, thus showing that, just as in the classical case, also the quantale-valued topological coreflection is uniformizable, i.e. is completely regular. We show that the indices of compactness of a quantale-valued metric space and both the hyperspaces endowed with the quantale-valued Hausdorff-metric and the quantale-valued Wijsman convergence tower, respectively, coincide. Finally we present some results about completeness of the quantale-valued Wijsman convergence tower.

2. Preliminaries

We consider in this paper complete lattices L for which $\top \neq \bot$ for the top element $\top$ and the bottom element $\bot$. In any complete lattice L we can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$ and $\alpha \lhd \bigvee_{j \in J} \beta_j$ iff $\alpha \lhd \beta_i$ for some $i \in J$. A complete lattice is completely distributive if and only if we have $\alpha = \bigvee\{\beta : \beta \lhd \alpha\}$ for any $\alpha \in L$, [22]. For more results on lattices we refer to [7].

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice, is called a *commutative and integral quantale* if $(L, *)$ is a commutative semigroup for which the top element of L acts as the unit, i.e. $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, i.e. $(\bigvee_{i \in J} \alpha_i) * \beta = \bigvee_{i \in J}(\alpha_i * \beta)$, see e.g. [8]. In a commutative and integral quantale we can define an implication $\rightarrow : L \times L \longrightarrow L$ by $\alpha \rightarrow \beta = \bigvee\{\gamma \in L : \alpha * \gamma \leq \beta\}$ for $\alpha, \beta \in L$. Then $\alpha * \beta \leq \gamma$ iff $\alpha \leq \beta \rightarrow \gamma$ for all $\alpha, \beta, \gamma \in L$. We consider in this paper only commutative and integral quantales $\mathsf{L} = (L, \leq, *)$ with completely distributive lattices $(L, \leq)$.

A commutative and integral quantale $\mathsf{L} = (L, \leq, *)$ with underlying completely distributive lattice is called a *value quantale* if $\alpha \vee \beta \lhd \top$ whenever $\alpha \lhd \top$ and $\beta \lhd \top$. This concept goes back to the work of Flagg [6], however he uses the opposite order. Typical examples of such quantales are e.g. the unit interval $[0,1]$ with a left-continuous *t-norm* [25]. Another important example is given by *Lawvere's quantale*, the interval $[0,\infty]$ with the opposite order and addition $\alpha*\beta = \alpha+\beta$ (extended by $\alpha+\infty = \infty+a = \infty$), see e.g. [6]. A further important example is the quantale of distance distribution functions. A *distance distribution function* $\varphi : [0,\infty] \longrightarrow [0,1]$, satisfies $\varphi(x) = \sup_{y<x} \varphi(y)$ for all $x \in [0,\infty]$. The set of all distance distribution functions is denoted by Δ^+. With the pointwise order, the set Δ^+ becomes a completely distributive lattice [6] with top-element ε_0. A quantale operation on Δ^+, $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, is also called a *sup-continuous triangle function* [25]. Note that in this case $(\Delta^+, \leq, *)$ is also a value quantale, [6].

For a set X, we denote its power set by $\mathsf{P}(X)$ and the set of all filters $\mathbb{F}, \mathbb{G}, \ldots$ on X by $\mathsf{F}(X)$. The set $\mathsf{F}(X)$ is ordered by set inclusion and maximal elements of $\mathsf{F}(X)$ in this order are called *ultrafilters*. The set of all ultrafilters on X is denoted by $\mathsf{U}(X)$. In particular, for each $x \in X$, the point filter $[x] = \{A \subseteq X : x \in A\}$ is an ultrafilter. If $\mathbb{F} \in \mathsf{F}(X)$ and $f : X \longrightarrow Y$ is a mapping, then we define $f(\mathbb{F}) \in \mathsf{F}(Y)$ by $f(\mathbb{F}) = \{G \subseteq Y : f(F) \subseteq G \text{ for some } F \in \mathbb{F}\}$. In particular, we have $f([x]) = [f(x)]$ for any $x \in X$. For a filter $\mathbb{G} \in \mathsf{F}(Y)$ the set $\{f^{-1}(G) : G \in \mathbb{G}\}$ is a filter basis whenever none of the $f^{-1}(G)$ is empty. In this case we denote by $f^{-1}(\mathbb{G})$ the filter on X generated by this filter basis and say that $f^{-1}(\mathbb{G})$ exists. We then have $f^{-1}(f(\mathbb{F})) \leq \mathbb{F}$ and $\mathbb{G} \leq f(f^{-1}(\mathbb{G}))$ in case $f^{-1}(\mathbb{G})$ exists. For a family of filters $(\mathbb{F}_i)_{i\in I}$ we define their join, $\bigvee_{i\in I} \mathbb{F}_i$, as the filter generated by the filter basis of finite intersections $F_{i_1} \cap \ldots \cap F_{i_n}$ with $F_{i_k} \in \mathbb{F}_{i_k}$ for $k = 1, \ldots, n$, whenever all these finite intersections are non-empty. For $\mathbb{G} \in \mathsf{F}(J)$ and $\mathbb{F}_j \in \mathsf{F}(X)$ for each $j \in J$, we denote $\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}) = \bigvee_{G\in\mathbb{G}} \bigwedge_{j\in G} \mathbb{F}_j \in \mathsf{F}(X)$ the *diagonal filter* [15]. For filters $\Phi, \Psi \in \mathbb{F}(X \times X)$ we define Φ^{-1} as the filter generated by the filter basis $\{F^{-1} : F \in \Phi\}$ where $F^{-1} = \{(x,y) \in X \times X : (y,x) \in F\}$ and $\Phi \circ \Psi$ as the filter generated by the filter basis $\{F \circ G : F \in \Phi, G \in \Psi\}$, whenever $F \circ G \neq \emptyset$ for all $F \in \Phi, G \in \Psi$, where $F \circ G = \{(x,y) \in X \times X : (x,s) \in F, (s,y) \in G \text{ for some } s \in X\}$.

For notions from category theory we refer to the textbooks [1, 21]. A *construct* is a category $\mathcal{C}$ with a faithful functor $U : \mathcal{C} \longrightarrow SET$, from $\mathcal{C}$ to the category of sets. We always consider a construct as a category whose objects are structured sets (S, ξ) and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e. if for every source $(f_i : S \longrightarrow (S_i, \xi_i))_{i\in I}$ there is a unique structure ξ on S, such that a mapping $g : (T, \eta) \longrightarrow (S, \xi)$ is a morphism if and only if for each $i \in I$ the composition $f_i \circ g : (T, \eta) \longrightarrow (S_i, \xi_i)$ is a morphism.

3. Topological L-convergence tower spaces

Let $\mathsf{L} = (L, \leq, *)$ be a commutative and integral quantale with completely distributive lattice $(L, \leq)$.

Let X be a set. A family of mappings $\overline{c} = (c_\alpha : \mathsf{F}(X) \longrightarrow \mathsf{P}(X))_{\alpha\in L}$ which satisfies the axioms

(LC1) $x \in c_\alpha([x])$ for all $x \in X, \alpha \in L$;

(LC2) $\bigcap_{j\in J} c_\alpha(\mathbb{F}_j) = c_\alpha(\bigwedge_{j\in J} \mathbb{F}_j)$ for all $\{\mathbb{F}_j : j \in J\} \subseteq \mathsf{F}(X)$;

(LC3) $\bigcap_{\alpha\in M} c_\alpha(\mathbb{F}) = c_{\bigvee M}(\mathbb{F})$ for all $M \subseteq L$;

(LC4) For all $\mathbb{G}, \mathbb{F}_y \in \mathsf{F}(X)$ $(y \in X)$ we have that
$x \in c_{\alpha*\beta}(\kappa(\mathbb{G}, (\mathbb{F}_y)_{y\in X}))$ whenever $x \in c_\beta(\mathbb{G})$ and $y \in c_\alpha(\mathbb{F}_y)$ for all $y \in X$;
is called an *topological* L*-convergence tower on* X and the pair $(X, \overline{c})$ is called an *topological* L*-convergence tower space*. A mapping $f : X \longrightarrow X'$ between the topological L-convergence tower spaces $(X, \overline{c})$ and $(X', \overline{c'})$, is called *continuous* if, for all $x \in X$ and all $\mathbb{F} \in \mathsf{F}(X)$, $f(x) \in c'_\alpha(f(\mathbb{F}))$ whenever $x \in c_\alpha(\mathbb{F})$. The category of topological L-convergence tower spaces with continuous mappings as morphisms is denoted by $\mathsf{L\text{-}TCTS}$.

For a topological L-convergence tower $\overline{c}$ on X we can define a *limit operator* $\lambda^{\overline{c}} : \mathsf{F}(X) \longrightarrow L^X$ by $\lambda^{\overline{c}}(\mathbb{F})(x) = \bigvee\{\alpha \in L : x \in c_\alpha(\mathbb{F})\}$. This limit operator satisfies the axioms (L1) $\lambda^{\overline{c}}([x])(x) = \top$; (L2) $\lambda^{\overline{c}}(\bigwedge_{j\in J} \mathbb{F}_j) = \bigwedge_{j\in J} \lambda^{\overline{c}}(\mathbb{F}_j)$ and (L3) $\lambda^{\overline{c}}(\mathbb{G})(x) * \bigwedge_{y\in X} \lambda^{\overline{c}}(\mathbb{F}_y)(y) \le \lambda^{\overline{c}}(\kappa(\mathbb{G}, (\mathbb{F}_y)_{y\in X}))$. Conversely, given a limit operator $\lambda : \mathbb{F}(X) \longrightarrow L^X$ satisfying the axioms (L1), (L2) and (L3), $x \in c^\lambda_\alpha(\mathbb{F}) \iff \lambda(\mathbb{F})(x) \ge \alpha$ defines a topological L-convergence tower on X. In this sense, limit operators and topological L-convergence towers are equivalent concepts. For Lawvere's quantale, Lowen [18] describes approach spaces by such limit operators and it is shown in [4] that a description with topological convergence towers, called *limit towers* in the paper, is equivalent. In a quantale-valued generalization of approach spaces, [16] uses a similar approach based on ultrafilters in the realm of monoidal topology. We prefer to work with topological L-convergence towers because arguments and proofs often become more transparent.

We note some simple consequences of the axioms (LC2) and (LC3). For $\mathbb{F}, \mathbb{G} \in \mathsf{F}(X)$ and $\alpha, \beta \in L$ we have that $\mathbb{F} \le \mathbb{G}$ implies $c_\alpha(\mathbb{F}) \subseteq c_\alpha(\mathbb{G})$; and that $\alpha \le \beta$ implies $c_\beta(\mathbb{F}) \subseteq c_\alpha(\mathbb{F})$; and $c_\perp(\mathbb{F}) = X$ for all $\mathbb{F} \in \mathsf{F}(X)$.

We call a topological L-convergence tower space $(X, \overline{c})$ *symmetric* if $y \in c_\alpha([x])$ implies $y \in c_\alpha([x])$ and it is called *separated* if $x, y \in c_\top(\mathbb{F})$ implies $x = y$.

We will now introduce a strengthening of the axiom (LC4). We say that $(X, \overline{c}) \in |\mathsf{L\text{-}TCTS}|$ satisfies the axiom (LCF) if for all sets J, all $\mathbb{G} \in \mathsf{F}(J)$, all $h : J \longrightarrow X$ and all $\mathbb{F}_j \in \mathsf{F}(X)$ $(j \in J)$ we have $x \in c_{\alpha*\beta}(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}))$ whenever $x \in c_\beta(h(\mathbb{G}))$ and $h(j) \in c_\alpha(\mathbb{F}_j)$ for all $j \in J$.

Lemma 3.1. *Let* $(X, \overline{c}) \in |\mathsf{L\text{-}TCTS}|$. *Then (LCF) is equivalent to (LC2) and (LC4).*

Proof. The choice $J = X$ and $h = id_X$ shows that (LCF) implies (LC4). We now show that (LCF) implies (LC2). Let $\mathbb{F}_i \in \mathsf{F}(X)$ for all $i \in J$. With $h(i) = x$ and $\mathbb{G} = [J]$ we obtain $h(\mathbb{G}) = [x]$ and $\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}) = \bigwedge_{i\in J} \mathbb{F}_i$. If $x \in c_\alpha(\mathbb{F}_i)$ for all $i \in J$, then because $x \in c_\top([x])$ and $h(i) = x \in c_\alpha(\mathbb{F}_i)$ we have by (LCF) $x \in c_{\top*\alpha}(\bigwedge_{i\in J} \mathbb{F}_i) = c_\alpha(\bigwedge_{i\in J} \mathbb{F}_i)$.

Let now (LC2) and (LC4) be true. Let J be a set, $h : J \longrightarrow X$, $\mathbb{G} \in \mathsf{F}(J)$ and, for all $i \in J$, let $\mathbb{F}_i \in \mathsf{F}(X)$. If $x \in c_\beta(h(\mathbb{G}))$ and $h(j) \in c_\alpha(\mathbb{F}_j)$ for all $j \in J$, then by (LC2) $h(\mathbb{G}) \geq \mathbb{U}_x^\beta$ and for all $j \in J$ we have $\mathbb{F}_j \geq \mathbb{U}_{h(j)}^\alpha$. With (LC4) we conclude $x \in c_{\beta*\alpha}(\kappa(\mathbb{U}_x^\beta, (\mathbb{U}_{h(j)}^\alpha)))$ and therefore $\mathbb{U}_x^{\alpha*\beta} \leq \kappa(\mathbb{U}_x^\beta, (\mathbb{U}_{h(j)}^\alpha))$. It is not difficult to show that $\kappa(\mathbb{U}_x^\beta, (\mathbb{U}_{h(j)}^\alpha)) \leq \kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})$ and hence, using (LC2), we obtain $x \in c_{\alpha*\beta}(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}))$ and (LCF) is true. □

Theorem 3.2. *The category* L-TCTS *is a topological construct.*

Proof. Let $(f_i : X \longrightarrow (X_i, \overline{c^i}))_{i\in I}$ be a source. We define the initial L-convergence tower on X by $x \in c_\alpha(\mathbb{F})$ if and only if for all $i \in I$ we have $f_i(x) \in c_\alpha^i(f_i(\mathbb{F}))$. The axiom (LC1) is easy to see. We show (LC3). Let $M \subseteq L$ and let $x \in c_\alpha(\mathbb{F})$ for all $\alpha \in M$. Then $f_i(x) \in c_\alpha^i(f_i(\mathbb{F}))$ for all $\alpha \in M$ and all $i \in I$. Hence, for all $i \in I$, $f_i(x) \in c^i_{\bigvee M}(f_i(\mathbb{F}))$ and this means $x \in c_{\bigvee M}(\mathbb{F})$. To show (LCF), let J be a set, $h : J \longrightarrow X$, $\mathbb{G} \in \mathsf{F}(J)$ and for all $j \in J$ let $\mathbb{F}_j \in \mathsf{F}(X)$. If $x \in c_\alpha(h(\mathbb{G}))$ and for all $j \in J$, $h(j) \in c_\beta(\mathbb{F}_j)$, then for all $i \in J$ we have $f_i(x) \in c_\alpha^i(f_i(h(\mathbb{G})))$ and $f_i(h(j)) \in c_\beta^i(f_i(\mathbb{F}_j))$ for all $j \in J$. We denote $k_i = f_i \circ h : J \longrightarrow X_i$ for all $i \in I$. Then $f_i(x) \in c^i_{\alpha*\beta}(\kappa(\mathbb{G}, (f_i(\mathbb{F}_j))_{j\in J}))$ for all $i \in I$. It is not difficult to show that $\kappa(\mathbb{G}, (f_i(\mathbb{F}_j))_{j\in J}) = f_i(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}))$. Hence $f_i(x) \in c^i_{\alpha*\beta}(f_i(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J})))$ for all $i \in I$, i.e. $p \in c_{\alpha*\beta}(\kappa(\mathbb{G}, (\mathbb{F}_j)_{j\in J}))$. □

It is not difficult to show that if all $(X_i, \overline{c^i})$ are symmetric, then also the initial construction is symmetric. Finally we mention that if the source $(f_i : X \longrightarrow (X_i, \overline{c^i}))_{i\in I}$ is *point-separating*, i.e. if for $x \neq y$ there is an $i \in I$ with $f_i(x) \neq f_i(y)$, and if all $(X_i, \overline{c^i})$ are separated, then the initial construction $(X, \overline{c})$ is also separated. For if $x, y \in c_\alpha(\mathbb{F})$ then $f_i(x), f_i(y) \in c_\alpha^i(f_i(\mathbb{F}))$ for all $i \in I$ and hence $f_i(x) = f_i(y)$ for all $i \in I$, which implies $x = y$.

4. L-metric spaces and topological spaces as topological L-convergence tower spaces

For a quantale $\mathsf{L} = (L, \leq, *)$, an L-*metric space* is a pair (X, d) of a set X and an L-*metric* $d : X \times X \longrightarrow L$ such that
(LM1) $d(x, x) = \top$ for all $x \in X$ (*reflexivity*);

(LM2) $d(x, y) * d(y, z) \leq d(x, z)$ for all $x, y, z \in X$ (*transitivity*).
A mapping between two L-metric spaces, $f : (X, d) \longrightarrow (X', d')$ is called an L-*metric morphism* if $d(x_1, x_2) \leq d'(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. We denote the category of L-metric spaces with L-metric morphisms by L-MET.

If the L-metric satisfies $d(x, y) = d(y, x)$ for all $x, y \in X$, it is called *symmetric*. If $d(x, y) = \top$ implies $x = y$, it is called *separated*.

For a value quantale and using the opposite order, Flagg [6] calls an L-metric space a *continuity space*. Other names for the same concept are L-*categories* [8, 17], or L-*preordered sets* [27].

In case $L = \{0,1\}$, an L-metric space is a preordered set. If $\mathsf{L} = ([0,\infty], \geq, +)$, an L-metric space is a quasimetric space. If $\mathsf{L} = (\Delta^+, \leq, *)$, an L-metric space is a probabilistic quasimetric space, see [6].

THEOREM 4.1 ([12]). *The category* $\mathsf{L\text{-}MET}$ *can be coreflectively embedded into the category* $\mathsf{L\text{-}TCTS}$.

Proof. The embedding of $(X,d) \in |\mathsf{L\text{-}MET}|$ into $(X, \overline{c^d}) \in |\mathsf{L\text{-}TCTS}|$ is given by $x \in c^d_\alpha(\mathbb{F}) \iff \bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y) \geq \alpha$. [12] shows the axioms (LC1), (LC2) and (LC3). We need only show the topological axiom (LC4). Let $\bigvee_{G\in\mathbb{G}} \bigwedge_{y\in G} d(x,y) \geq \beta$ and for each $y \in X$, let $\bigvee_{F^y\in\mathbb{F}_y} \bigwedge z \in F^y d(y,z) \geq \alpha$. Consider $\epsilon \lhd \beta$ and $\delta \lhd \alpha$. Then there is $G \in \mathbb{G}$ such that for all $y \in G$ we have $d(x,y) \geq \epsilon$ and for each $y \in X$ there is $F^y \in \mathbb{F}_y$ such that for all $z \in F^y$ we have $d(y,z) \geq \delta$. The set $H^G = \bigcup_{y\in G} F^y \in \bigwedge_{y\in G} \mathbb{F}_y \leq \kappa(\mathbb{G}, (\mathbb{F}_y)_{y\in Y})$ and for $z \in H^G$ then $z \in F^y$ for some $y \in G$. Hence $d(x,z) \geq d(x,y) * d(y,z) \geq \epsilon * \delta$. We conclude $\bigwedge_{z\in H^G} d(x,z) \geq \epsilon * \delta$ and from this we obtain $\bigvee_{K\in\kappa(\mathbb{G},(\mathbb{F}_y)_{y\in Y})} \bigwedge_{u\in K} d(x,u) \geq \bigwedge_{z\in H^G} d(x,z) \geq \epsilon * \delta$. This is true for all $\epsilon \lhd \beta$ and all $\delta \lhd \alpha$ and L being a quantale and $(L, \leq)$ being completely distributive, the claim follows.

Given $(X, \overline{c}) \in |\mathsf{L\text{-}TCTS}|$ we define $d^{\overline{c}}(x,y) = \bigvee_{x\in c_\alpha([y])} \alpha$. Then $(X, d^{\overline{c}}) \in |\mathsf{L\text{-}MET}|$. It was further shown in [12] that for $(X,d) \in |\mathsf{L\text{-}MET}|$ we have $d^{\overline{c^d}} = d$ and for $(X, \overline{c}) \in |\mathsf{L\text{-}TCTS}|$ we have $c_\alpha^{(d^{\overline{c}})}(\mathbb{F}) \subseteq c_\alpha(\mathbb{F})$ for all $\alpha \in L, \mathbb{F} \in \mathsf{F}(X)$. As morphisms are preserved as well, this completes the proof. □

Note that we have $x \in c^d_\alpha([y])$ if and only if $d(x,y) \geq \alpha$, see [12]. Furthermore, it is easy to prove that if (X,d) is symmetric, then $(X, \overline{c^d})$ is symmetric. If (X,d) is symmetric and separated, then also $(X, \overline{c^d})$ is separated. To see this, let $x, y \in c^d_\top(\mathbb{F})$ and let $\epsilon \lhd \top$. Then there are $F^x, F^y \in \mathbb{F}$ such that for all $u \in F^x$ and all $v \in F^y$ we have $d(x,u) \geq \epsilon$ and $d(y,v) \geq \epsilon$. The set $F = F^x \cap F^y \in \mathbb{F}$ and for all all $w \in F$ we have $d(x,w) \geq \epsilon$ and $d(y,w) \geq \epsilon$. By symmetry, we conclude $d(x,y) \geq d(x,w) * d(w,y) \geq \epsilon * \epsilon$, for all $\epsilon \lhd \top$. L being a quantale and $(L, \leq)$ being completely distributive, we conclude $d(x,y) = \top$. This implies $x = y$.

Following [18], we call, for a topological L-convergence tower space $(X, \overline{c})$, the space $(X, d^{\overline{c}})$ the $\mathsf{L\text{-}MET}$*-coreflection*. It is straightforward to show that if $(X, \overline{c})$ is symmetric, then also the $\mathsf{L\text{-}MET}$-coreflection $(X, d^{\overline{c}})$ is symmetric. Furthermore, if $(X, \overline{c})$ is separated, also the $\mathsf{L\text{-}MET}$-coreflection is separated. To see this, let $\epsilon \lhd \top = d^{\overline{c}}(x,y)$. Then there is $\alpha \geq \epsilon$ such that $x \in c_\alpha([y])$. Hence, by (LC3), $x \in c_\epsilon([y])$. Using again (LC3) yields $x \in c_\top([y])$. From (LC1) we obtain $x = y$.

We can define closed sets for an L-metric space (X,d). For $A \subseteq X$ we define the *d-closure of A* by $x \in \overline{A}^d$ if and only if there is $\mathbb{F} \in \mathsf{F}(X)$ such that $A \in \mathbb{F}$ and $x \in c^d_\top(\mathbb{F})$. If L is a value quantale then $x \in \overline{A}^d$ iff $\bigvee_{a\in A} d(x,a) = \top$, see [14]. For an L-metric space (X,d) we call $A \subseteq X$ *closed* (in (X,d)) if $\overline{A}^d \subseteq A$. Then $A \subseteq X$ is closed iff for all $\mathbb{U} \in \mathsf{U}(X)$ we have $x \in A$ whenever $A \in \mathbb{U}$ and $x \in c^d_\top(\mathbb{U})$. The sets $X, \emptyset$ are closed, $A \cup B$ is closed whenever A, B are closed and $\bigcap_{j \in J} A_j$ is closed whenever A_j is closed for all $j \in J$. We denote the set of all non-empty closed sets of an L-metric space by $CL(X)$. We note that, for a value quantale L, the mapping from $CL(X)$ to L^X, $A \longmapsto d(\cdot, A)$ is an injection. In fact, if $d(x,A) = d(x,B)$ for all $x \in X$, then for $x \in A$ we have $d(x,B) = d(x,A) = \top$ and hence $x \in \overline{B} = B$. Similarly, $x \in B$ implies $x \in A$ and hence $A = B$.

We now turn our attention to topological spaces. We identify a topological space $(X, \mathcal{T})$ with the topological convergence space $(X, c^{\mathcal{T}})$, where $c^{\mathcal{T}} : \mathsf{F}(X) \longrightarrow X$ is defined by $x \in c^{\mathcal{T}}(\mathbb{F})$ if and only if $\mathbb{F} \geq \mathbb{U}_x^{\mathcal{T}}$ with the neighbourhood filter $\mathbb{U}_x^{\mathcal{T}}$ of x in $(X, \mathcal{T})$. For ease of notation, we also write $\mathbb{F} \xrightarrow{\mathcal{T}} x$ for $x \in c^{\mathcal{T}}(\mathbb{F})$.

Theorem 4.2. *The category* TOP *can be coreflectively embedded into the category* L-TCTS.

Proof. Given a topological space $(X, \mathcal{T})$ we define $\overline{c^{\mathcal{T}}} = (c_\alpha^{\mathcal{T}})_{\alpha \in L}$ be $x \in c_\alpha^{\mathcal{T}}(\mathbb{F})$ if and only if $\mathbb{F} \xrightarrow{\mathcal{T}} x$ in case $\alpha \neq \bot$ and for $\alpha = \bot$ we define $c_\bot^{\mathcal{T}}(\mathbb{F}) = X$. Then it is not difficult to show that $(X, \overline{c^{\mathcal{T}}})$ is a topological L-convergence tower space and that for a continous mapping $f : (X, \mathcal{T}) \longrightarrow (X', \mathcal{T}')$ also $f : (X, \overline{c^{\mathcal{T}}}) \longrightarrow (X', \overline{c^{\mathcal{T}'}})$ is continuous.

For a topological L-convergence tower space $(X, \overline{c})$ we define the topological space $(X, \mathcal{T}^{\overline{c}})$ by $\mathbb{F} \xrightarrow{\mathcal{T}^{\overline{c}}} x$ if and only if $x \in c_\top(\mathbb{F})$. Clearly, this defines the convergence of a topology and for a continuous mapping $f : (X, \overline{c}) \longrightarrow (X', \overline{c'})$ also $f : (X, \mathcal{T}^{\overline{c}}) \longrightarrow (X', \mathcal{T}^{\overline{c'}})$ is continous.

For $(X, \mathcal{T}) \in |\mathsf{TOP}|$ we have $c^{\mathcal{T}} = c_\top^{\mathcal{T}}$ and for $(X, \overline{c}) \in |\mathsf{L}\text{-}\mathsf{TCTS}|$ we have $c_\alpha^{(\mathcal{T}^{\overline{c}})}(\mathbb{F}) \subseteq c_\alpha(\mathbb{F})$ for all $\alpha \in L, \mathbb{F} \in \mathsf{F}(X)$. To see the latter we consider $\alpha \neq \bot$. We have $x \in c_\alpha^{(\mathcal{T}^{\overline{c}})}(\mathbb{F})$ if and only if $\mathbb{F} \xrightarrow{\mathcal{T}^{\overline{c}}} x$. This is equivalent to $x \in c_\top(\mathbb{F})$ and the axiom (LC3) implies $x \in c_\alpha(\mathbb{F})$. □

Following [18], for a topological L-convergence tower space $(X, \overline{c})$ we call $(X, \mathcal{T}^{\overline{c}})$ the *topological coreflection*.

5. The L-Wisjman convergence tower of an L-metric space

A commutative and integral quantale $\mathsf{L} = (L, \leq, *)$ becomes a symmetric, separated L-metric space if we define $d_{\mathsf{L}}(\alpha, \beta) = (\alpha \to \beta) \wedge (\beta \to \alpha)$, $(\alpha, \beta \in L)$. In the case of Lawvere's quantale $\mathsf{L} = ([0, \infty], \geq, +)$ we have $\alpha \to \beta = (\beta - \alpha) \vee 0$ and $d_{\mathsf{L}}(\alpha, \beta) = |\alpha - \beta|$ is the standard metric on $[0, \infty]$. We denote the underlying topological L-convergence tower space of (L, d_{L}) by $(L, \overline{c^{\mathsf{L}}})$.

For an L-metric space (X, d) there is an important L-metric on $CL(X)$. For $A, B \in CL(X)$ we define $H_d(A,B) = \bigwedge_{x \in X} d_{\mathsf{L}}(d(x,A), d(x,B))$. Then $H_d : CL(X) \times CL(X) \longrightarrow L$ is called the L-*Hausdorff metric* of the L-metric space (X, d). It is symmetric and for, a value quantale L, separated. Furthermore, $H_d(A,B) = \bigwedge_{a \in A} d(a, B) \wedge \bigwedge_{b \in B} d(A, b)$. For further details see [2].

In the sequel, we mimick the definition of Wijsman convergence [26]. We define an L-convergence tower $\overline{c^W}$ on $CL(X)$ by defining for $A \in CL(X)$ and $\mathfrak{F} \in \mathsf{F}(CL(X))$

$$A \in c^W_\alpha(\mathfrak{F}) \iff d(x,A) \in c^{\mathsf{L}}_\alpha(d(x,\mathfrak{F})) \quad \text{for all } x \in X.$$

Here, we define $d(x,\mathfrak{F})$ as the filter generated by the filter basis $\{d(x,\mathcal{F}) : \mathcal{F} \in \mathfrak{F}\}$, i.e. the image of the filter $\mathfrak{F}$ for the mapping $d(x,\cdot) : CL(X) \longrightarrow L$. It is clear by this definition that $\overline{c^W}$ is the initial topological L-convergence tower on $CL(X)$ with respect to the source

$$\left(d(x,\cdot) : \begin{cases} CL(X) \longrightarrow (L, \overline{c^{\mathsf{L}}}) \\ A \longmapsto d(x,A) \end{cases}\right)_{x \in X}.$$

As (L, d_{L}) is symmetric and separated, also $(L, \overline{c^{\mathsf{L}}})$ is symmetric and separated and, in case of a value quantale L, the family of mappings $(d(x,\cdot))_{x\in X}$ being point-separating, we see that $(CL(X), \overline{c^W})$ is symmetric and separated in this case.

We note that convergence in the topological coreflection of $(CL(X), \overline{c^W})$ means convergence in the $\top$-level. So we have

$$\mathfrak{F} \xrightarrow{\mathcal{T}^{\overline{c^W}}} A \iff d(x,A) \in c^{\mathsf{L}}_\top(d(x,\mathfrak{F})) \quad \text{for all } x \in X.$$

This means that $d(x,\mathfrak{F}) \geq \mathbb{U}^{d_{\mathsf{L}}}_{d(x,A)}$. In case of Lawvere's quantale, i.e. in the metric case, this is convergence in the Wijsman topology, [3]. Hence, as in the case of Lawvere's quantale [18], a natural candidate for the Wijsman topology of an L-metric space is the topological coreflection of $(CL(X), \overline{c^W})$.

Also the following results have been established for the definition of the Wijsman structure by gauges in the case of Lawvere's quantele in [18]. Similar results in terms of L-gauges have been obtained in [14].

Proposition 5.1. *Let (X,d) be an L-metric space. The L-MET-coreflection of the L-Wijsman convergence tower is the L-Hausdorff metric, i.e. we have $d^{\overline{c^W}} = H_d$.*

Proof. We have by definition of the L-MET-coreflection, for $A, B \in CL(X)$,

$$d^{\overline{c^W}}(A,B) = \bigvee\{\alpha \in L : d(x,A) \in c^{\mathsf{L}}_\alpha(d(x,[B])) \text{ for all } x \in X\}.$$

From $d(x,[B]) = d(x,\cdot)([B]) = [d(x,\cdot)(B)] = [d(x,B)]$ we conclude $d(x,A) \in c^{\mathsf{L}}_\alpha(d(x,[B])) = c^{\mathsf{L}}_\alpha([d(x,B)])$ if and only if $d_{\mathsf{L}}(d(x,A), d(x,B)) \geq \alpha$. As a consequence, we get $d^{\overline{c^W}}(A,B) = \bigvee\{\alpha \in L : d_{\mathsf{L}}(d(x,A), d(x,B)) \geq \alpha \text{ for all } x \in X\}$ and hence $d^{\overline{c^W}}(A,B) = \bigwedge_{x\in X} d_{\mathsf{L}}(d(x,A), d(x,B)) = H_d(A,B)$. □

Following [18,19] we call, for an L-metric space (X,d), a topological L-convergence tower $\overline{c}$ on $CL(X)$ *admissible* [20] if $\psi : (X, \overline{c^d}) \longrightarrow (CL(X), \overline{c})$ defined by $\psi(x) = \{x\}$ is an embedding in L-TCTS. Clearly, we need that the one-point subsets $\{x\}$ are closed, i.e. that (X,d) is separated.

Proposition 5.2. *Let (X,d) be a separated and symmetric L-metric space. Then $\overline{c^W}$ is admissible.*

Proof. We first show that $\psi : (X, \overline{c^d}) \longrightarrow (CL(X), \overline{c^W})$ is continuous. To this end, we first note that from transitivity (LM2) we obtain $d(x,y) \leq d(z,x) \to d(z,y)$ and $d(y,x) \leq d(z,y) \to d(z,x)$ for all $x,y,z \in X$. Hence, by symmetry, we have $d(x,y) \leq d_{\mathsf{L}}(d(z,x), d(z,y))$. Let now $x \in c^d_\alpha(\mathbb{F})$. Then, for all $z \in X$, we have

$$\alpha \leq \bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d(x,y) \leq \bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d_{\mathsf{L}}(d(z,x), d(z,y)).$$

Therefore also
$$\bigvee_{H \in d(z,\psi(\mathbb{F}))} \bigwedge_{\beta \in H} d_L(d(z,x),\beta) = \bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d_{\mathsf{L}}(d(z,x), d(z,\{y\}) \geq \alpha,$$

which means $d(z,\{x\}) = d(z,x) \in c^{\mathsf{L}}_\alpha(d(z,\psi(\mathbb{F})))$ for all $z \in X$, i.e. $\psi(x) = \{x\} \in c^W_\alpha(\psi(\mathbb{F}))$.

Next we show that $\psi^{-1} : (\psi(X), \overline{c^W}|_{\psi(X)}) \longrightarrow (X, \overline{c^d})$ is continuous. Here, $\overline{c^W}|_{\psi(X)}$ denotes the subspace structure of $\overline{c^W}$, i.e. the initial structure on $\psi(X) = \{\{x\} : x \in X\}$ with respect to the inclustion mapping $\iota : \psi(X) \longrightarrow (CL(X), \overline{c^W})$. We have $\psi(x) = \{x\} \in c^W_\alpha|_{\psi(X)}(\mathfrak{F}) = c^W_\alpha(\iota(\mathfrak{F}))$ if and only if for all $z \in X$ we have $d(z,x) = d(z,\psi(x)) \in c^{\mathsf{L}}_\alpha(d(z,\iota(\mathfrak{F}))) = c^W_\alpha(\iota(d(z,\mathfrak{F})))$. This is equivalent to $\bigvee_{\mathcal{F} \in \mathfrak{F}} \bigwedge_{\{y\} \in \mathcal{F}} d_{\mathsf{L}}(d(z,x), d(z,y)) \geq \alpha$ for all $z \in X$. Choosing $z = x$ this entails $\bigvee_{H \in \psi^{-1}(\mathfrak{F})} \bigwedge_{y \in H} d(\psi^{-1}(\{x\}), y) = \bigvee_{\mathcal{F} \in \mathfrak{F}} \bigwedge_{\{y\} \in \mathcal{F}} d(x,y) \geq \alpha$, i.e. $\psi^{-1}(\{x\}) \in c^d_\alpha(\psi^{-1}(\mathfrak{F}))$. □

6. Neighbourhoods and uniformization

For a topological L-convergence tower space $(X, \overline{c})$, $x \in X$ and $\alpha \in L$ we define the *α-neighbourhood filter of x* by $\mathbb{U}^{\overline{c},\alpha}_x = \bigwedge_{x \in c_\alpha(\mathbb{F})} \mathbb{F}$. We then have, by (LC2), $x \in c_\alpha(\mathbb{F})$ if and only if $\mathbb{F} \geq \mathbb{U}^{\overline{c},\alpha}_x$.

PROPOSITION 6.1. *Let $(X, \overline{c})$ be a topological L-convergence tower space. The system $\overline{\mathbb{U}^{\overline{c}}} = (\mathbb{U}^{\overline{c},\alpha}_x)_{\alpha \in L, x \in X}$ then satisfies the following properties.*

(U0) $\mathbb{U}^{\overline{c},\bot}_x = \bigwedge \mathsf{F}(X)$;

(U1) $\mathbb{U}^{\overline{c},\alpha}_x \leq [x]$;

(U2) $\mathbb{U}^{\overline{c},\alpha * \beta}_x \leq \kappa(\mathbb{U}^{\overline{c},\beta}_x, (\mathbb{U}^{\overline{c},\alpha}_y)_{y \in X})$;

(U3) $\mathbb{U}^{\overline{c},\bigvee A}_x = \bigvee_{\alpha \in A} \mathbb{U}^{\overline{c},\alpha}_x$ *for* $\emptyset \neq A \subseteq L$.

Furthermore, a mapping $f : X \longrightarrow X'$ between topological L-convergence tower spaces $(X, \overline{c}), (X', \overline{c'})$ is continuous if and only if for all $x \in X$ and all $\alpha \in L$ we have $\mathbb{U}^{\overline{c'},\alpha}_{f(x)} \leq f(\mathbb{U}^{\overline{c},\alpha}_x)$.

Proof. The properties follow easily from the properties of a topological L-convergence tower space. We demonstrate only (U2). By (LC2), we have $x \in c_\beta(\mathbb{U}^{\overline{c},\beta}_x)$ and $y \in c_\alpha(\mathbb{U}^{\overline{c},\alpha}_y)$ for all $y \in X$. Hence by (LC4), $x \in c_{\alpha*\beta}(\kappa(\mathbb{U}^{\overline{c},\beta}_x, (\mathbb{U}^{\overline{c},\alpha}_y)_{y \in X}))$ and again (LC2) implies $\mathbb{U}^{\overline{c},\alpha*\beta}_x \leq \kappa(\mathbb{U}^{\overline{c},\beta}_x, (\mathbb{U}^{\overline{c},\alpha}_y)_{y \in X})$. □

For Lawvere's quantale a description of approach spaces by such systems of neighbourhood filters was explicitely verified in [9].

If we have a system of filters $\overline{\mathbb{U}} = (\mathbb{U}_x^\alpha)_{\alpha\in L, x\in X}$ satisfying the properties (U0) – (U3), then we define a topological L-convergence tower $\overline{c^{\overline{\mathbb{U}}}}$ by $x \in c_\alpha^{\overline{\mathbb{U}}}(\mathbb{F})$ if and only if $\mathbb{F} \geq \mathbb{U}_x^\alpha$.

Furthermore, it is not difficult to show that for a topological L-convergence tower space $(X, \overline{c})$ we have $\overline{c^{(\overline{\mathbb{U}^{\overline{c}}})}} = \overline{c}$ and for a pair $(X, \overline{\mathbb{U}})$ with (U0) – (U3) we have $\overline{\mathbb{U}^{c^{\overline{\mathbb{U}}}}} = \overline{\mathbb{U}}$. Hence we can characterize topological L-convergence tower spaces by their L-neighbourhood systems.

Let now (X, d) be an L-metric space. Flagg [6] defines, for $x \in X$ and $\epsilon \lhd \top$ the ϵ-ball around x by $F^d(x, \epsilon) = \{y \in X : d(x, y) \rhd \epsilon\}$. The collection $\{F^d(x, \epsilon) : \epsilon \lhd \top\}$ is then a filter basis, provided that L is a value quantale. Similarly, we would need to consider also the collections $\{F^d(x, \epsilon) : \epsilon \lhd \alpha\}$ for $\alpha \in L$. Unfortunately, these are in general not filter bases unless we require that $\epsilon, \delta \lhd \alpha$ implies $\epsilon \vee \delta \lhd \alpha$. However, this implication is not true in the important probabilistic case, $\mathsf{L} = (\Delta^+, \leq)$.

Example 6.2. Define $f_{\delta,\epsilon}(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \delta \\ \epsilon & \text{if } x > \delta \end{cases}$.

Then $f_{\delta,\epsilon} \in \Delta^+$ and for $\eta \in \Delta^+$ we have $\eta = \bigvee\{f_{\delta,\epsilon} \lhd \eta\}$ and $f_{\delta,\epsilon} \lhd \eta$ if and only if $\epsilon < \eta(\delta)$. Define now $\eta(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$. Then $f_{1/2,1/4}, f_{3/4,1/2} \lhd \eta$, but $f_{1/2,1/4} \vee f_{3/4,1/2} \not\lhd \eta$. Else there would be $f_{\delta,\epsilon}$ such that $f_{1/2,1/4} \vee f_{3/4,1/2} \leq f_{\delta,\epsilon} \lhd \eta$ and hence $f_{1/2,1/2} \leq f_{\delta,\epsilon}$, but $f_{1/2,1/2} \not\lhd \eta$.

A way out is to replace the well-below relation $\lhd$ by the *way-below relation* [7], $\alpha \prec \beta$, if for all *directed* subsets $D \subseteq L$, $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. This relation can be defined for any complete lattice and has similar properties as the well-below relation but additionally satisfies $\epsilon \vee \delta \prec \alpha$ whenever $\epsilon, \delta \prec \alpha$. Moreover, $\alpha \lhd \beta$ implies $\alpha \prec \beta$. Hence, $\beta = \bigvee\{\alpha \in L : \alpha \prec \beta\}$ for all $\beta \in L$. A complete lattice with the latter property is called *continuous* [7]. Hence, a completely distributive, complete lattice is a continuous lattice.

Proposition 6.3. *Let the complete lattice L satisfy the property that $\alpha, \beta \lhd \gamma$ implies $\alpha \vee \beta \lhd \gamma$ for all $\alpha, \beta \in L$. Then*
(i) $L_\gamma = \{\delta \in L : \delta \lhd \gamma\}$ is directed.

(ii) If $\gamma \neq \bot$, then for all $\alpha \in L$ we have $\alpha \prec \gamma \iff \alpha \lhd \gamma$.

Proof. For (ii): $\alpha \prec \gamma = \bigvee L_\gamma$ implies $\alpha \leq \delta$ for some $\delta \lhd \gamma$ and hence $\alpha \lhd \gamma$. □

Proposition 6.4. *For a complete lattice L, the following are equivalent.*
(i) $\alpha, \beta \lhd \gamma$ implies $\alpha \vee \beta \lhd \gamma$ for all $\gamma \neq \bot$.

(ii) $\alpha \prec \beta \iff \alpha \lhd \beta$ for all $\alpha, \beta \in L, \beta \neq \bot$.

Hence, we can define the *α-neighbourhood filter of* x, $\mathbb{U}_x^{d,\alpha}$, as the filter with the filter basis $\{B^d(x,\epsilon) : \epsilon \prec \alpha\}$ with the *ε-balls* $B^d(x,\epsilon) = \{y \in X : d(x,y) \succ \epsilon\}$.

PROPOSITION 6.5. *Let* (X,d) *be an* L*-metric space. Then* $\mathbb{U}_x^{d,\alpha} = \mathbb{U}_x^{\overline{c^d},\alpha}$.

Proof. We have $\bigvee_{F\in\mathbb{U}_x^{d,\alpha}} \bigwedge_{y\in F} d(x,y) \geq \bigvee_{\epsilon\prec\alpha} \bigwedge_{y\in B(X,\epsilon)} d(x,y) \geq \bigvee_{\epsilon\prec\alpha} \epsilon = \alpha$. Hence $x \in c_\alpha^d(\mathbb{U}_x^{d,\alpha})$ and this implies $\mathbb{U}_x^{d,\alpha} \geq \mathbb{U}_x^{\overline{c^d},\alpha}$. Conversely, for $U \in \mathbb{U}_x^{d,\alpha}$ there is $\epsilon \prec \alpha$ such that $B^d(x,\epsilon) \subseteq U$. Hence $d(x,y) \succ \epsilon$ implies $y \in U$. Let now $x \in c_\alpha^d(\mathbb{F})$. Then $\bigvee_{F\in\mathbb{F}} \bigwedge_{y\in F} d(x,y) \geq \alpha \succ \epsilon$. As the set $\{\bigwedge_{y\in F} d(x,y) : F \in \mathbb{F}\}$ is directed because $\mathbb{F}$ is a filter, there is $F^\epsilon \in \mathbb{F}$ such that $\bigwedge_{y\in F^\epsilon} d(x,y) \succ \epsilon$, i.e. $F^\epsilon \subseteq B^d(x,\epsilon) \subseteq U$. Hence $U \in \bigwedge_{x\in c_\alpha^d(\mathbb{F})} \mathbb{F}$ and we have $\mathbb{U}_x^{d,\alpha} \leq \mathbb{U}_x^{\overline{c^d},\alpha}$. □

Of particular interest for us are initial constructions. The following result can be shown in the classical way.

PROPOSITION 6.6. *Let* $\left(f_i : X \longrightarrow (X_i, \overline{c^i})\right)_{i\in I}$ *and let* $\overline{c}$ *be the initial structure on* X *with respect to this source. We denote the α-neighbourhood filter on* $(X_i,\overline{c^i})$ *by* $\mathbb{U}_{x_i}^{i,\alpha}$ *and* $\mathbb{U}_x^\alpha$ *is the α-neighbourhood filter on* $(X,\overline{c})$. *Then* $\mathbb{U}_x^\alpha = \bigvee_{i\in I} f_i^{-1}(\mathbb{U}_{f_i(x)}^{i,\alpha})$.

We are now in the position to discuss the neighbourhoods for the Wijsman L-convergence tower on $CL(X)$. To this end, we denote the ϵ-balls in (L, d_L) by $B^\mathsf{L}(x,\epsilon)$ and the generated α-neighborhood filter by $\mathbb{U}_x^{\mathsf{L},\alpha}$. The α-neighbourhood filter of $A \in CL(X)$ in $(CL(X),\overline{c^W})$ is denoted by $\mathcal{U}_A^{W,\alpha}$. From Proposition 6.6 we obtain, that, for $F \subseteq X$ finite and $\epsilon \prec \alpha$, the sets $\mathcal{B} = \bigcap_{x\in F} d(x,\cdot)^{-1}(B^\mathsf{L}(d(x,A),\epsilon))$ are a basis for $\mathcal{U}_A^{W,\alpha}$. In fact, as finite intersection of $d(x,\cdot)^{-1}(U)$ with $U \in \mathbb{U}_{d(x,A)}^{\mathsf{L},\alpha}$ also $\mathcal{B} \in \mathbb{U}_A^{W,\alpha}$ and for $\mathcal{U} \in \mathbb{U}_A^{W,\alpha}$ there is a finite set $F \subseteq X$ such that $\mathcal{U} \supseteq \bigcap_{x\in F} d(x,\cdot)^{-1}(U_{d(x,A)}^{\mathsf{L},\alpha})$ with $U_{d(x,A)}^{\mathsf{L},\alpha} \in \mathbb{U}_{d(x,A)}^{\mathsf{L},\alpha}$. Hence, for all $x \in F$ there is $\epsilon_x \prec \alpha$ such that $B^\mathsf{L}(d(x,A),\epsilon_x) \subseteq U_{d(x,A)}^{\mathsf{L},\alpha}$. With $\epsilon = \bigvee_{x\in F} \epsilon_x \prec \alpha$ then also $B^\mathsf{L}(d(x,A),\epsilon) \subseteq U_{d(x,A)}^{\mathsf{L},\alpha}$ and $\mathcal{B}$ is in fact member of a basis of $\mathcal{U}_A^{W,\alpha}$.

Now we note that $B \in \mathcal{B}$ if and only if for all $x \in F$ we have $d(x,B) \in B^\mathsf{L}(d(x,A),\epsilon)$, i.e. if and only if for all $x \in F$, $d_\mathsf{L}(d(x,A),d(x,B)) \succ \epsilon$. If the way-below relation is *multiplicative* [7], i.e. if $\alpha,\beta \succ \gamma$ implies $\alpha \wedge \beta \succ \gamma$, the latter is equivalent to $d_F(A,B) = \bigwedge_{x\in F} d_\mathsf{L}(d(x,A),d(x,B)) \succ \epsilon$. We collect all this in the following theorem.

THEOREM 6.7. *Let* (X,d) *be an* L*-metric space. Then a basis for the α-neighbourhood filter* $\mathcal{U}_A^{W,\alpha}$ *of* $A \in CL(X)$ *in* $(CL(X),\overline{c^W})$ *is given by the sets*

$\mathcal{B}_{F,A,\epsilon} = \{B \in CL(X) : d_\mathsf{L}(d(x,A),d(x,B)) \succ \epsilon$ *for all* $x \in F\}$, $F \subseteq X$ *finite* , $\epsilon \prec \alpha$.

If the way-below relation is multiplicative, then a basis is given by

$$\mathcal{B}_{F,A,\epsilon} = \{B \in CL(X) : \bigwedge_{x\in F} d_\mathsf{L}(d(x,A),d(x,B)) \succ \epsilon\},\ F \subseteq X \text{ finite },\ \epsilon \prec \alpha.$$

For $\alpha = \top$ we obtain a basis of the neighbourhood filter of $A \in CL(X)$ of the topological coreflection of $(CL(X), \overline{c^W})$, $(CL(X), \mathcal{T}^W)$ as

$$\mathcal{B}_{F,A,\epsilon} = \{B \in CL(X) : d_{\mathsf{L}}(d(x,A), d(x,B)) \succ \epsilon \text{ for all } x \in F\},\ F \subseteq X \text{ finite },\epsilon \prec \top.$$

This again shows that for Lawvere's quantale $\mathsf{L} = ([0,\infty], \geq, +)$, the topological coreflection of the Wijsman L-convergence tower is the classical Wijsman topology on $CL(X)$.

For an L-metric space (X,d), we define for a finite set $F \subseteq X$ and $\epsilon \prec \top$ the set

$$\mathcal{B}_{F,\epsilon} = \{(A,B) \in CL(X) \times CL(X) : d_{\mathsf{L}}(d(x,A), d(x,B)) \succ \epsilon \text{ for all } x \in F\}.$$

It is not difficult to show that for $\alpha \in L$ the set $\mathfrak{B}_\alpha = \{\mathcal{B}_{F,\epsilon} : \epsilon \prec \alpha\}$ is a filter basis on $CL(X) \times CL(X)$ and we denote the generated filter by $\mathcal{U}^{W,\alpha}$. The following result shows that the collection of these filters is an L-uniform tower [13].

Proposition 6.8. *Let (X,d) be an L-metric space. Then $\overline{\mathcal{U}^W} = (\mathcal{U}^{W,\alpha})_{\alpha\in L}$ is an L-uniform tower on $CL(X)$, i.e. we have*
(LUT1) $\mathcal{U}^{W,\alpha} \leq [\Delta_{CL(X)}]$ with $\Delta_{CL(X)} = \{(A,A) : A \in CL(X)\}$;
(LUT2) $\mathcal{U}^{W,\alpha} \leq (\mathcal{U}^{W,\alpha})^{-1}$;
(LUT3) $\mathcal{U}^{W,\alpha*\beta} \leq \mathcal{U}^{W,\alpha} \circ \mathcal{U}^{W,\beta}$;
(LUT4) $\mathcal{U}^{W,\alpha} \leq \mathcal{U}^{W,\beta}$ whenever $\alpha \leq \beta$;
(LUT5) $\mathcal{U}^{W,\perp} = \bigwedge \mathbb{F}(CL(X) \times CL(X))$;
(LUT6) $\mathcal{U}^{W,\bigvee A} \leq \bigvee_{\alpha\in A} \mathcal{U}^{W,\alpha}$ whenever $\emptyset \neq A \subseteq L$.

Proof. We only show (LUT3) by proving that for $\epsilon \prec \alpha * \beta$ there are $\delta_1 \prec \alpha$, $\delta_2 \prec \beta$ such that $\mathcal{B}_{F,\delta_1} \circ \mathcal{B}_{F,\delta_2} \subseteq \mathcal{B}_{F,\epsilon}$. First we note that the set $D = \{\delta_1 * \delta_2 : \delta_1 \prec \alpha, \delta_2 \prec \beta\}$ is directed, because $(\delta_1 * \delta_2) * (\epsilon_1 * \epsilon_2) \leq (\delta_1 \vee \epsilon_1) * (\delta_2 \vee \epsilon_2)$ for all $\delta_1, \delta_2, \epsilon_1, \epsilon_2 \in L$ and as $\delta_1 \vee \epsilon_1 \prec \alpha$ and $\delta_2 \vee \epsilon_2 \prec \beta$ if $\delta_1, \epsilon_1 \prec \alpha$ and $\delta_2, \epsilon_2 \prec \beta$. Hence for $\epsilon \prec \alpha * \beta = \bigvee\{\delta_1 * \delta_2 : \delta_1 \prec \alpha, \delta_2 \prec \beta\}$, there are $\delta_1 \prec \alpha$ and $\delta_2 \prec \beta$ such that $\epsilon \prec \delta_1 * \delta_2$. Let now $(A,C) \in \mathcal{B}_{F,\delta_1} \circ \mathcal{B}_{F,\delta_2}$. Then there is $B \in CL(X)$ such that $d_{\mathsf{L}}(d(x,A), d(x,B)) \succ \delta_1$ and $d_{\mathsf{L}}(d(x,B), d(x,C)) \succ \delta_2$ for all $x \in F$. Thus $d_{\mathsf{L}}(d(x,A), d(x,C)) \succ \delta_1 * \delta_2 \succ \epsilon$ and $(A,C) \in \mathcal{B}_{F,\epsilon}$. □

Note that $\mathcal{U}^{W,\top}$ is a uniformity on $CL(X)$. Furthermore, it is clear that $\mathcal{B}_{F,\epsilon}(A) = \{B \in CL(X) : (A,B) \in \mathcal{B}_{F,\epsilon}\} = \mathcal{B}_{F,A,\epsilon}$. Hence, for any $A \in CL(X)$, $\mathcal{U}^{W,\alpha}(A) = \mathcal{U}_A^{W,\alpha}$ is the α-neighbourhood filter in $(CL(X), \overline{c^W})$. In this sense, the L-Wijsman convergence tower space $(CL(X), \overline{c^W})$ is L-*uniformizable*, i.e. there is an L-uniform tower that generates the L-convergence tower. Restricting to $\alpha = \top$ we obtain the following result for the topological coreflection $(CL(X), \mathcal{T}^W)$.

Proposition 6.9. *Let (X,d) be an L-metric space. Then the topological coreflection $(CL(X), \mathcal{T}^W)$ of $(CL(X), \overline{c^W})$ is uniformizable and, hence, is completely regular.*

For Lawvere's quantale this result specializes to the known result that the classical Wijsman topology is completely regular, see [3].

7. Compactness

The results of this section parallel those of [14, Section 8] and [18, Section 10.1], however we need new methods in the proofs. Let $(X,\overline{c})$ be a topological L-convergence tower space. We define the *index of compactness* by

$$\chi_c((X,\overline{c})) = \bigwedge_{\mathbb{U}\in \mathsf{U}(X)} \bigvee_{x\in X} \bigvee_{x\in c_\alpha(\mathbb{U})} \alpha.$$

This definition originates from the corresponding definition for approach spaces [18]. We first motivate the concept.

Proposition 7.1. *Let $(X,\mathcal{T})$ be a topological space. Then $(X,\mathcal{T})$ is compact if and only if $\chi_c((X,\overline{c^{\mathcal{T}}}))=\top$.*

Proof. This is trivial, as $x\in c^{\mathcal{T}}_\alpha(\mathbb{U})$ if and only if $\mathbb{U}$ converges to x in $(X,\mathcal{T})$. □

Proposition 7.2. *Let $(X,\overline{c})$ be a topological L-convergence tower space. If $(X,\mathcal{T}^{\overline{c}})$ is compact, then $\chi_c(X,\overline{c})=\top$.*

Proof. Let $\mathbb{U}\in\mathsf{U}(X)$. By compactness of $(X,\mathcal{T}^{\overline{c}})$, there is $x\in X$ with $x\in c_\top(\mathbb{U})$ and hence $\bigvee_{x\in X}\bigvee_{x\in c_\alpha(\mathbb{U})}\alpha=\top$. We conclude $\bigwedge_{\mathbb{U}\in\mathsf{U}(X)}\bigvee_{x\in X}\bigvee_{x\in c_\alpha(\mathbb{U})}\alpha=\top$. □

Proposition 7.3. *Let (X,d) be an L-metric space. Then*

$$\chi_c((X,\overline{c^d}))=\bigvee\{\alpha\in L:\exists F\subseteq X \text{ finite s.t. } X=\bigcup_{x\in F}B^d(x,\alpha)\}.$$

Proof. Let first $\epsilon\lhd\chi_c((X,\overline{c^d}))$. Then for all $\mathbb{U}\in\mathsf{U}(X)$ there is $x\in X$ and $\alpha\succ\epsilon$ such that $x\in c^d_\alpha(\mathbb{U})$. From $\alpha\rhd\epsilon$ we conclude $\alpha\succ\epsilon$ and hence $\bigvee_{U\in\mathbb{U}}\bigwedge_{y\in U}d(x,y)\geq\alpha\succ\epsilon$. As $\mathbb{U}$ is a filter, the set $\{\bigwedge_{y\in U}d(x,y):U\in\mathbb{U}\}$ is directed. Thus, for all $\mathbb{U}\in\mathsf{U}(X)$ there is $x\in X$ and $U\in\mathbb{U}$ such that $U\subseteq B^d(x,\epsilon)$ and we conclude for all $\mathbb{U}\in\mathsf{U}(X)$ we have $B^d(x,\epsilon)\in\mathbb{U}$ for some $x\in X$. Assume now that for all finite subsets $F\subseteq X$ we had $X\neq\bigcup_{x\in F}B^d(x,\epsilon)$. For convenience, we denote the complement of an ϵ-ball around x by $C(x,\epsilon)=(B^d(x,\epsilon))^c$. Then $\mathbb{B}=\{\bigcap_{x\in F}C(x,\epsilon):F\subseteq X \text{ finite }\}$ is a filter basis. Let $\mathbb{U}$ be an ultrafilter finer than the filter generated by this filter basis. As seen above then $B^d(x,\epsilon)\in\mathbb{U}$ for some $x\in X$ in contradiction to $C(x,\epsilon)\in\mathbb{U}$. Therefore there is a finite set $F\subseteq X$ such that $X=\bigcup_{x\in F}B^d(x,\epsilon)$ and we conclude $\bigvee\{\alpha\in L:\exists F\subseteq X \text{ finite s.t. } X=\bigcup_{x\in F}B^d(x,\alpha)\}\geq\epsilon$. This is true for all $\epsilon\lhd\chi_c((X,\overline{c^d}))$ and by complete distributivity then $\chi_c((X,\overline{c^d}))\leq\bigvee\{\alpha\in L:\exists F\subseteq X \text{ finite s.t. } X=\bigcup_{x\in F}B^d(x,\alpha)\}$.

Let now $\epsilon\lhd\bigvee\{\alpha\in L:\exists F\subseteq X \text{ finite s.t. } X=\bigcup_{x\in F}B^d(x,\alpha)\}$. Then there is $\alpha\in L$ such that $\epsilon\leq\alpha$ and $X=\bigcup_{x\in F}B^d(x,\alpha)$ for some finite set $F\subseteq X$. Let $\mathbb{U}\in\mathsf{U}(X)$. As $X\in\mathbb{U}$, there is $x\in F$ such that $B^d(x,\alpha)\in\mathbb{U}$ and, as $\epsilon\leq\alpha$ also $B^d(x,\epsilon)\in\mathbb{U}$. Also, for $\epsilon'\prec\epsilon$ we have $B^d(x,\epsilon')\in\mathbb{U}$, from which we conclude $\mathbb{U}^{d,\epsilon}_x\leq\mathbb{U}$, i.e. $x\in c^d_\epsilon(\mathbb{U})$. As $\mathbb{U}$ was an arbitrary ultrafilter on X we obtain $\chi_c((X,\overline{c^d}))\geq\epsilon$ and again the complete distributivity of L leads to $\chi_c((X,\overline{c^d}))\geq\bigvee\{\alpha\in L:\exists F\subseteq X \text{ finite s.t. } X=\bigcup_{x\in F}B^d(x,\alpha)\}$. □

For Lawvere's quantale $\mathsf{L} = ([0,\infty], \geq, +)$ we see that a metric space (X, d) is totally bounded if and only if $\chi_c((X, \overline{c^d})) = 0$. Hence, we will call an L-metric space (X, d) *totally bounded* if $\chi_c((X, \overline{c^d})) = \top$.

THEOREM 7.4. *Let $\prec$ be multiplicative and let (X, d) be a separated and symmetric L-metric space. Then $\chi_c((X,d)) = \chi_c((CL(X), H_d)) = \chi_c((CL(X), \overline{c^W}))$.*

Proof. We first show $\chi_c((CL(X), H_d)) \leq \chi_c((CL(X), \overline{c^W}))$. For $B \in CL(X)$ and an ultrafilter $\mathcal{U} \in \mathsf{U}(CL(X))$ we have

$$\begin{aligned}\bigvee\{\alpha \in L : B \in c_\alpha^{H_d}(\mathcal{U})\} &= \bigvee_{\mathcal{U}\in\mathcal{U}} \bigwedge_{A\in\mathcal{U}} \bigwedge_{x\in X} d_{\mathsf{L}}(d(x,A), d(x,B)) \\ &\leq \bigwedge_{x\in X} \bigvee_{\mathcal{U}\in\mathcal{U}} \bigwedge_{A\in\mathcal{U}} d_{\mathsf{L}}(d(x,A), d(x,B)).\end{aligned}$$

Hence,

$$\begin{aligned}\chi_c((CL(X), H_d)) &\leq \bigwedge_{\mathcal{U}\in\mathsf{U}(CL(X))} \bigvee_{B\in CL(X)} \bigwedge_{x\in X} \bigvee_{\mathcal{U}\in\mathcal{U}} \bigwedge_{A\in\mathcal{U}} d_{\mathsf{L}}(d(x,A), d(x,B)) \\ &= \bigwedge_{\mathcal{U}\in\mathsf{U}(CL(X))} \bigvee_{B\in CL(X)} \bigvee_{A\in c_\alpha^W(\mathcal{U})} \alpha \;=\; \chi_c((CL(X), \overline{c^W})).\end{aligned}$$

Next we show $\chi_c((CL(X), \overline{c^W})) \leq \chi_c((X,d))$. As (X, d) is separated, for $x \in X$ the one-point sets $\{x\}$ are in $CL(X)$. Hence, for any function $\varphi : X \longrightarrow L$ we have $\bigvee_{B\in CL(X)} \bigwedge_{x\in B} \varphi(x) = \bigvee_{x\in X} \varphi(x)$. Let now $\mathbb{U} \in \mathsf{U}(X)$ be an ultrafilter on X. We define $\mathfrak{B}_{\mathbb{U}} = \{\mathcal{Y}_U : U \in \mathbb{U}\}$ with $\mathcal{Y}_U = \{\{x\} : x \in U\}$. As the sets $\mathcal{Y}_U$ are non-empty and $\mathcal{Y}_{U_1} \cap \mathcal{Y}_{U_2} = \mathcal{Y}_{U_1\cap U_2}$, $\mathfrak{B}_{\mathbb{U}}$ is a filter basis on $CL(X)$. Let $\mathcal{U}_{\mathbb{U}}$ be an ultrafilter on $CL(X)$ which is finer than the filter generated by $\mathfrak{B}_{\mathbb{U}}$. We conclude

$$\begin{aligned}\chi_c((X, \overline{c^W})) &= \bigwedge_{\mathcal{U}\in\mathsf{U}(CL(X))} \bigvee_{B\in CL(X)} \bigwedge_{x\in X} \bigvee_{\mathcal{U}\in\mathcal{U}} \bigwedge_{A\in\mathcal{U}} d_{\mathsf{L}}(d(x,B), d(x,A)) \\ &\leq \bigwedge_{\mathcal{U}\in\mathsf{U}(CL(X))} \bigvee_{B\in CL(X)} \bigwedge_{x\in X B} \bigvee_{\mathcal{U}\in\mathcal{U}} \bigwedge_{A\in\mathcal{U}} d(x,A),\end{aligned}$$

as for $x \in B$ we have $d_{\mathsf{L}}(d(x,B), d(x,A)) = d_{\mathsf{L}}(\top, d(x,A)) = d(x,A)$. Hence

$$\begin{aligned}\chi_c((X, \overline{c^W})) &\leq \bigvee_{x\in X} \bigvee_{\mathcal{U}\in\mathcal{U}_{\mathbb{U}}} \bigwedge_{A\in\mathcal{U}} d(x,A) \\ &= \bigvee_{x\in X} \bigvee_{\mathcal{Y}_U : U\in\mathbb{U}} \bigwedge_{\{y\}\in\mathcal{Y}_U} d(x,\{y\}) \;=\; \bigvee_{x\in X} \bigvee_{U\in\mathbb{U}} \bigwedge_{y\in U} d(x,y).\end{aligned}$$

This is true for all $\mathbb{U} \in \mathsf{U}(X)$ and hence $\chi_c((X, \overline{c^W})) \leq \bigwedge_{\mathbb{U}\in\mathsf{U}(X)} \bigvee_{x\in X} \bigvee_{U\in\mathbb{U}} \bigwedge_{y\in U} d(x,y) = \chi_c((X,d))$.

The proof that $\chi_c((X,d)) \leq \chi_c((CL(X), H_d))$ follows along the same lines of the corresponding part of the proof of [14, Theorem 8.5]. □

COROLLARY 7.5. *Let $\prec$ be multiplicative and let (X, d) be a separated and symmetric L-metric space. Then (X, d) is totally bounded if and only if $(CL(X), H_d)$ is totally bounded.*

8. Some results on completeness

Let $(X, \overline{c})$ be a topological L-convergence tower space and let $\mathbb{F} \in \mathsf{F}(X)$. We call $\mathbb{F}$ a *Cauchy filter* if $\bigvee_{x \in X} \bigvee_{x \in c_\alpha(\mathbb{F})} \alpha = \top$. As usual, this generalizes the corresponding concept in the theory of approach spaces [18]. Furthermore, we call $\mathbb{F}$ *convergent to* $x_0 \in X$ if $x_0 \in c_\top(\mathbb{F})$. The space $(X, \overline{c})$ is called *complete* if every Cauchy filter converges to some $x_0 \in X$.

We first characterize Cauchy filters for an L-metric space, justifying the definition.

Proposition 8.1. *Let (X, d) be an L-metric space and let $\mathbb{F} \in \mathsf{F}(X)$. Then $\mathbb{F}$ is a Cauchy filter in $(X, \overline{c^d})$ if and only if for all $\epsilon \lhd \top$ there is $x \in X$ such that $B^d(x, \epsilon) \in \mathbb{F}$.*

Proof. Let first $\mathbb{F}$ be a Cauchy filter in $(X, \overline{c^d})$ and let $\epsilon \lhd \top$. Then there is $x \in X$ and $\alpha \rhd \epsilon$ such that $x \in c^d_\alpha(\mathbb{F})$, i.e. $\bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d(x, y) \geq \alpha \rhd \epsilon$. Thus there is $x \in X$ and $F \in \mathbb{F}$ such that for all $y \in F$ we have $d(x, y) \rhd \epsilon$. The latter implies that $F \subseteq B^d(x, \epsilon)$ and consequently, $B^d(x, \epsilon) \in \mathbb{F}$.

Conversely, let $\epsilon \lhd \top$. Then there is $x \in X$ and $F \in \mathbb{F}$ such that $F \subseteq B^d(x, \epsilon)$. Hence, there is $x \in X$ and $F \in \mathbb{F}$ such that $\bigwedge_{y \in F} d(x, y) \geq \epsilon$. We conclude $\bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d(x, y) \geq \epsilon$, i.e. $x \in c^d_\epsilon(\mathbb{F})$. It follows $\bigvee_{x \in X} \bigvee_{x \in c^d_\alpha(\mathbb{F})} \alpha \geq \epsilon$ and the complete distributivity then yields that $\mathbb{F}$ is a Cauchy filter. □

It is not difficult to show with this result that the definition of Cauchy filter in [5] and our definition coincides for an L-metric space.

Proposition 8.2. *Let (X, d) be an L-metric space and let $\mathbb{F} \in \mathsf{F}(X)$.*

If $\bigvee_{F \in \mathbb{F}} \bigwedge_{x, y \in F} d(x, y) = \top$, then $\mathbb{F}$ is a Cauchy filter in $(X, \overline{c^d})$. If L is a value quantale and (X, d) is symmetric, then also the converse is true.

Proof. Let first $\bigvee_{F \in \mathbb{F}} \bigwedge_{x, y \in F} d(x, y) = \top \rhd \epsilon$. Then there is $F_\epsilon \in \mathbb{F}$ such that for all $x, y \in F_\epsilon$ we have $d(x, y) \geq \epsilon$, i.e. for all $x \in F_\epsilon$ we have $\bigwedge_{y \in F_\epsilon} d(x, y) \geq \epsilon$. Hence for all $x \in F_\epsilon$ we have $x \in c^d_\epsilon(\mathbb{F})$ and therefore $\bigvee_{x \in X} \bigvee_{x \in c^d_\alpha(\mathbb{F})} \alpha \geq \epsilon$. As $\epsilon \lhd \top$ was arbitrary, the complete distributivity yields that $\mathbb{F}$ is a Cauchy filter.

Let now $\bigvee_{x \in X} \bigvee_{x \in c_\alpha(\mathbb{F})} \alpha = \top \rhd \epsilon$. Choose $\delta \lhd \top$ such that $\delta * \delta \rhd \epsilon$. Then there is $x_\epsilon \in X$ and $\alpha \rhd \delta$ such that $x_\epsilon \in c^d_\alpha(\mathbb{F})$, i.e. $\bigvee_{F \in \mathbb{F}} \bigwedge_{y \in F} d(x_\epsilon, y) \geq \alpha \rhd \delta$. Thus there is $F_\delta \in \mathbb{F}$ such that for all $y \in F_\delta$ we have $d(x_\epsilon, y) \geq \delta$. Let now $y, z \in F_\delta$. Then $d(x_\epsilon, y) \geq \delta$ and $d(z, x_\epsilon) = d(x_\epsilon, z) \geq \delta$ and hence by (LM2) $d(z, y) \geq \delta * \delta \rhd \epsilon$. We conclude $\bigvee_{F \in \mathbb{F}} \bigwedge_{z, y \in F} d(z, y) \geq \epsilon$ and the complete distributivity implies $\bigvee_{F \in \mathbb{F}} \bigwedge_{z, y \in F} d(z, y) = \top$. □

This characterization is the definition of a Cauchy filter in [23, 24].

Proposition 8.3. *Let $(X, \overline{c})$ be a topological L-convergence tower space. If $\mathbb{F} \in \mathsf{F}(X)$ is a Cauchy filter in the L-MET-coreflection $(X, d^{\overline{c}})$, then $\mathbb{F}$ is a Cauchy filter.*

Proof. Let $\epsilon \lhd \top$. Then there is $x \in X$ such that $B^{d^{\overline{c}}}(x, \epsilon) \in \mathbb{F}$. Hence, there is $F_\epsilon \in \mathbb{F}$ such that for all $y \in F_\epsilon$ we have $\bigvee\{\alpha \in L : x \in c_\alpha([y])\} \succ \epsilon$. As the set

$\{\alpha \in L : x \in c_\alpha([y])\}$ is directed by (LC3), we conclude that for each $y \in F_\epsilon$ there is $\alpha_y \succ \epsilon$ such that $x \in c_{\alpha_y}([y]) \subseteq c_\epsilon([y])$. We conclude that there is $F_\epsilon \in \mathbb{F}$ such that for all $y \in F_\epsilon$ we have $[y] \geq \mathbb{U}^x_\epsilon$ and therefore $\mathbb{F} \geq [F_\epsilon] = \bigwedge_{y\in F_\epsilon}[y] \geq \mathbb{U}^x_\epsilon$, i.e. $x \in c_\epsilon(\mathbb{F})$. Consequently, $\bigvee_{x\in X}\bigvee_{x\in c_\alpha(\mathbb{F})} \alpha \geq \epsilon$. Taking the join over all $\epsilon \lhd \top$ and using the complete distributivity, the claim follows. □

In the sequel, we shall often assume that the topological L-convergence tower is induced by an L-uniform tower $\overline{\mathcal{U}}$ in the sense that for the neighbourhood filters we have $\mathbb{U}^\alpha_x = \mathcal{U}^\alpha(x) = \{U(x) : U \in \mathcal{U}^\alpha\}$ with $U(x) = \{y \in X : (x, y) \in U\}$. We write $(X, \overline{c}) = (X, \overline{c^{\overline{\mathcal{U}}}})$ in this case. Note that $x \in c^{\overline{\mathcal{U}}}_\alpha(\mathbb{F})$ if and only if $\mathbb{F} \times [x] \geq \mathcal{U}^\alpha$. From this we conclude that $(X, \overline{c^{\overline{\mathcal{U}}}})$ is symmetric.

Proposition 8.4. *Let L be a value quantale and let $(X, \overline{c^{\overline{\mathcal{U}}}})$ be a topological L-convergence tower space, where $\overline{c^{\overline{\mathcal{U}}}}$ is induced by an L-uniform tower $\overline{\mathcal{U}}$. If $\mathbb{F} \in \mathsf{F}(X)$ is a Cauchy filter in $(X, \overline{c^{\overline{\mathcal{U}}}})$, then it is a Cauchy filter in the uniform space $(X, \mathcal{U}^\top)$.*

Proof. If $\bigvee_{x\in X}\bigvee_{\mathbb{F}\times[x]\geq\mathcal{U}^\alpha} \alpha = \top \rhd \epsilon$, we choose $\delta \lhd \top$ such that $\epsilon \lhd \delta * \delta$. Then there is $x_\delta \in X$ and $\alpha_\delta \geq \delta$ such that $\mathbb{F} \times [x_\delta] \geq \mathcal{U}^{\alpha_\delta} \geq \mathcal{U}^\delta$. It follows $\mathbb{F} \times \mathbb{F} = (\mathbb{F} \times [x_\delta]) \circ ([x_\delta] \times \mathbb{F}) \geq \mathcal{U}^\delta \circ \mathcal{U}^\delta \geq \mathcal{U}^{\delta*\delta} \geq \mathcal{U}^\epsilon$. Hence $\mathbb{F} \times \mathbb{F} \geq \mathcal{U}^{\bigvee\{\epsilon:\epsilon\lhd\top\}} = \mathcal{U}^\top$. □

Proposition 8.5. *Let $(X, \overline{c})$ be a topological L-convergence tower space. If $\chi_c(X, \overline{c}) = \top$ and $(X, \overline{c})$ is complete, then the topological coreflection $(X, \mathcal{T}^{\overline{c}})$ is compact.*

Proof. Let $\mathbb{U} \in \mathsf{U}(X)$. As $\top = \chi_c(X, \overline{c})$, $\mathbb{U}$ is a Cauchy filter and as $(X, \overline{c})$ is complete, $x_0 \in c_\top(\mathbb{U})$ for some $x_0 \in X$, i.e. $\mathbb{U}$ converges to x_0 in $(X, \mathcal{T}^{\overline{c}})$. □

We now note that for a filter $\mathbb{F}$, convergence to x in $(X, \overline{c^{\overline{\mathcal{U}}}})$ is equivalent to $x \in c^{\overline{\mathcal{U}}}_\top(\mathbb{F})$, i.e. to $\mathbb{F} \geq \mathcal{U}^\top(x)$. But the latter means that $\mathbb{F}$ convergens to x in the topological space $(X, \mathcal{T}^{\mathcal{U}^\top})$ induced by the uniform space $(X, \mathcal{U}^\top)$. Hence the topological coreflection $(X, \mathcal{T}^{\overline{c^{\overline{\mathcal{U}}}}})$ is uniformizable by $\mathcal{U}^\top$.

Proposition 8.6. *Let L be a value quantale and let $(X, \overline{c^{\overline{\mathcal{U}}}})$ be a topological L-convergence tower space induced by the uniform tower $\overline{\mathcal{U}}$. If $(X, \mathcal{T}^{\overline{c^{\overline{\mathcal{U}}}}})$ is compact, then $(X, \overline{c^{\overline{\mathcal{U}}}})$ is complete.*

Proof. We have just seen that $(X, \mathcal{T}^{\overline{c^{\overline{\mathcal{U}}}}})$ is compact and uniformizable by $\mathcal{U}^\top$ and hence $(X, \mathcal{U}^\top)$ is complete. Let now $\mathbb{F}$ be a Cauchy filter in $(X, \overline{c^{\overline{\mathcal{U}}}})$. Then $\mathbb{F}$ is a Cauchy filter in $(X, \mathcal{U}^\top)$ and hence there is $x \in X$ such that $\mathbb{F} \geq \mathcal{U}^\top(x)$. This means $x \in c^{\overline{\mathcal{U}}}_\top(\mathbb{F})$ and $(X, \overline{c^{\overline{\mathcal{U}}}})$ is complete. □

We collect all this in the following theorem.

Theorem 8.7. *Let L be a value quantale and $(X, \overline{c^{\overline{\mathcal{U}}}})$ be a topological L-convergence tower space induced by the uniform tower $\overline{\mathcal{U}}$. The following are equivalent.*
(i) $\chi_c(X, \overline{c^{\overline{\mathcal{U}}}}) = \top$ and $(X, \overline{c^{\overline{\mathcal{U}}}})$ is complete. *(ii) $(X, \mathcal{T}^{\overline{c^{\overline{\mathcal{U}}}}})$ is compact.*

We now apply this theorem to quantale-valued Wijsman convergence.

Corollary 8.8. *Let $\prec$ be multiplicative and let (X, d) be a separated and symmetric L-metric space. Then $(CL(X), \mathcal{T}^W)$ is compact if and only if (X, d) is totally bounded and $(CL(X), \overline{c^W})$ is complete.*

Proof. $(CL(X), \overline{c^W})$ is L-uniformizable and $\chi_c((X, \overline{c^d})) = \chi_c((CL(X), \overline{c^W}))$. □

We finally turn our attention to the completenes of the L-MET-coreflection.

Proposition 8.9. *Let L be a value quantale and let $(X, \overline{c^{\overline{\mathcal{U}}}}) \in |\mathsf{L}\text{-}\mathsf{TCTS}|$ with an L-uniform tower $\overline{\mathcal{U}}$. If $(X, \overline{c^{\overline{\mathcal{U}}}})$ is complete then $(X, d^{\overline{c^{\overline{\mathcal{U}}}}})$ is complete.*

Proof. Let $\mathbb{F}$ be a Cauchy filter in $(X, d^{\overline{c^{\overline{\mathcal{U}}}}})$. Then $\mathbb{F}$ is a Cauchy filter in $(X, \overline{c^{\overline{\mathcal{U}}}})$ and hence there is $x_0 \in X$ such that $\mathbb{F} \times [x_0] \geq \mathcal{U}^\top$. As $\mathbb{F}$ is a Cauchy filter $(X, d^{\overline{c^{\overline{\mathcal{U}}}}})$ we have $\bigvee_{F \in \mathbb{F}} \bigwedge_{x,y \in F} d^{\overline{c^{\overline{\mathcal{U}}}}}(x, y) = \top$. Let $\epsilon \lhd \top$. Then there is $F \in \mathbb{F}$ such that for all $x, y \in F$ we have $\bigvee_{x \in c^{\mathcal{U}}_\alpha([y])} \alpha \rhd \epsilon$. Hence, there is $F \in \mathbb{F}$ such that for all $x, y \in F$ there is $\alpha \geq \epsilon$ such that $x \in c^{\mathcal{U}}_\alpha([y]) \subseteq c^{\mathcal{U}}_\epsilon([y])$. This means $[y] \times [x] \geq \mathcal{U}^\epsilon$ for all $x, y \in F$ and hence $\mathbb{F} \times [x] \geq [F] \times [x] = \bigwedge_{y \in F} [y] \times [x] \geq \mathcal{U}^\epsilon$ for all $x \in F$. We conclude $([x_0] \times \mathbb{F}) \circ (\mathbb{F} \circ [x]) = [x_0] \times [x] \geq \mathcal{U}^{\epsilon * \top} = \mathcal{U}^\epsilon$, i.e. $x_0 \in c^{\mathcal{U}}_\epsilon([x])$ for all $x \in F$. We conclude $\bigvee_{F \in \mathbb{F}} \bigwedge_{x \in F} d^{\overline{c^{\overline{\mathcal{U}}}}}(x_0, x) \geq \epsilon$ and the complete distributivity again yields $\bigvee_{F \in \mathbb{F}} \bigwedge_{x \in F} d^{\overline{c^{\overline{\mathcal{U}}}}}(x_0, x) = \top$. This is the same as $x_0 \in c_\top^{d^{\overline{c^{\overline{\mathcal{U}}}}}}(\mathbb{F})$. □

Corollary 8.10. *Let L be a value quantale and let $(X, d) \in |\mathsf{L}\text{-}\mathsf{MET}|$. If $(CL(X), \overline{c^W})$ is complete then also $(CL(X), H_d)$ is complete.*

Remark 8.11. For Lawvere's quantale, we even have: (X, d) complete $\Longleftrightarrow (CL(X), H_d)$ complete $\Longleftrightarrow (CL(X), \overline{c^W})$ complete, see [18]. If these equivalences remain true for an arbitrary quantale is still an open question.

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