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A NEW COUPLED FIXED POINT THEOREM VIA SIMULATION FUNCTION WITH APPLICATION

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Abstract. In this paper, we prove a coupled fixed point theorem, using the concept of simulation function, which generalizes the works of Bhaskar et al., Sintunavarat et al. and Zlatanov. The validity of main results is verified through interesting examples. As sequel we also prove that the theorem has a vital application in solving a system of nonlinear impulsive fractional stochastic differential equations.

1. Introduction

The concept of simulation function and the notion of $\mathbb{Z}$-contraction were introduced by Khojasteh et al. [5] in 2015; Roldan Lopez de Hierro et al. [11] slightly modified the definition of Khojasteh et al. and proved some fixed point theorems. Ran and Reurings [10] introduced the concept of partially ordered metric space and proved an analogue of Banach's fixed point theorem. Bhaskar et al. [2] proved a coupled fixed point theorem in a partially ordered metric space, using a mixed monotone property, which was later extended by Sintunavarat et al. [15] in 2012. Later Sabetghadam et al. [12] extended the theory of Bhaskar et al. in the context of partially ordered cone metric space. The theory was further extended by Luong et al. [7] and Sedghi et al. [14]

It is well known that ordinary differential equations and fractional stochastic differential equations play a significant role in constructing SIR epidemic models. Likewise, a system of impulsive fractional stochastic differential equations has its indispensable part, in constructing dynamic models, which are used in the treatment of cancer. In recent past, wherever we cross a problem of solving a system of differential equations, whatever the type it may be, the arrival of fixed point theorems is an unavoidable one. Several fixed point theorems were proved to show the existence of a unique solution for various types of systems of equations like ordinary differential equations,

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functional equations, matrix equations, fractional differential equations and impulsive fractional differential equations (see [2, 6, 8, 10, 14, 16, 17]).

In Section 2, we give some basic definitions and results which are useful in subsequent sections. In Section 3, we prove an interesting coupled fixed point theorem, by framing a contractive condition, using the concept of simulation function; which generalizes the works of Bhaskar et al., Sintunavarat et al. and Zlatanov. In what follows, we justify our work by appropriate examples. In Section 4, we prove that the theory developed in Section 3 has a significant application in solving a system of impulsive fractional stochastic differential equations using coupled fixed points.

2. Preliminaries

Let X be a nonempty set equipped with a partial order. A function F from X^2 to X is said to satisfy mixed monotone property, if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y. A point $(x, y) \in X^2$ is said to be a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.1 ([11]). A function $\zeta : [0,\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:

(ζ1) $\zeta(0, 0) = 0$;

(ζ2) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ3) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Remark 2.2. A simulation function need not be continuous. Indeed, let $\zeta : [0,\infty)^2 \to \mathbb{R}$ be a mapping defined as $\zeta(t, s) = 3(s-t)$ if $s < t$ and $\zeta(t, s) = \frac{s}{4} - t$ if $s \geq t$, then clearly ζ is a simulation function. Now by letting $t = 1$, it follows that $\zeta(1, s) = 3(s-1)$ if $s < 1$ and $\zeta(1, s) = \frac{s}{4} - 1$ if $s \geq 1$. Thus ζ is discontinuous at $(1, 1)$.

Now let us see some theorems which are already known and described in the literature. To avoid repeated use of symbols and terminologies, we fix the notations to be used throughout the paper.

Let $(X, \preceq, d)$ be a complete partially ordered metric space and F be a function from X^2 to X, that satisfies mixed monotone property. As stated in [2], we endow the product space X^2 with the partial order: for $(x, y), (u, v) \in X^2$, "$(u, v) \preceq (x, y)$ if $x \succeq u$ and $y \preceq v$".

Theorem 2.3 ([2]). *Let $F : X^2 \to X$ be a mapping that satisfies*

$$d\left(F(x, y), F(u, v)\right) \leq \frac{k}{2}\left(d(x, u) + d(y, v)\right) \tag{1}$$

for some $k \in [0, 1)$ and for all $x \succeq u, y \preceq v$. Suppose that F is continuous or X has the following properties:

(i) If $\{x_n\}$ is a non-decreasing sequence with $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,

(ii) If $\{y_n\}$ is a non-increasing sequence with $y_n \to y$, then $y_n \succeq y$ for all $n \in \mathbb{N}$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point. If in addition for any two $(x, y), (z, t) \in X^2$, there exists $(u, v) \in X^2$, which is comparable with (x, y) and (z, t), then the coupled fixed point is unique.

THEOREM 2.4 ([15]). *Let X and F be as in Theorem 2.3. Instead of (1) assume that*

$$d\left(F(x,y), F(u,v)\right) \leq \phi\left(\frac{d(x,u)+d(y,v)}{2}\right) \text{ for all } x \succeq u,\ y \preceq v, \tag{2}$$

where $\phi : [0, \infty) \to [0, \infty)$ is an increasing function with $0 = \phi(0) < \phi(t) < t$ and $\lim_{r\to t^+} \phi(r) < t$ for all $t > 0$. Then F has a coupled fixed point. If in addition for any two $(x, y), (z, t) \in X^2$, there exists $(u, v) \in X^2$, which is comparable with (x, y) and (z, t), then the coupled fixed point is unique.

THEOREM 2.5 ([18]). *Let X and F be as in Theorem 2.3. Instead of (1) assume that*

$$d(F(x,y), F(u,v)) \leq kd(x,u) + ld(y,v) \text{ for all } x \succeq u,\ y \preceq v, \tag{3}$$

where $k, l \in [0, 1)$, $k+l < 1$. Then F has a coupled fixed point. If in addition every pair of elements in X^2 has a lower or an upper bound, then the coupled fixed point is unique.

3. A theorem on coupled fixed point

In this section we prove a theorem on coupled fixed points using a new contractive condition that involves a simulation function. Further we justify the significance of the theory with suitable examples.

THEOREM 3.1. *Assume one of the following*
(CF1) F is continuous

(CF2) X has the following properties:

(i) If $\{x_n\}$ is a non-decreasing sequence with $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$,

(ii) If $\{y_n\}$ is a non-increasing sequence with $y_n \to y$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.
Let

$$\zeta\left(d\left(F(x,y), F(u,v)\right), \max\{d(x,u), d(y,v)\}\right) \geq 0, \text{ for all } x \succeq u,\ y \preceq v \tag{4}$$

where $\zeta(t, s)$ is a simulation function such that $\zeta(t, s_1) \leq \zeta(t, s_2)$ whenever $t \leq s_1 \leq s_2$. If there exists $x_0, y_0 \in X$ such that $x_0 \succeq F(x_0, y_0)$, $y_0 \preceq F(y_0, x_0)$, then F has a coupled fixed point.

Proof. Firstly, we wish to show the existence of two sequences $\{x_n\}$ and $\{y_n\}$ so that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. By using the mixed monotone property of F, construct two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$, where $\{x_n\}$ is increasing and $\{y_n\}$ is decreasing. By (4) and (ζ2) we get $d(x_n, x_{n+1}) < \max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n)\}$.

Suppose $\max\{d(x_{n-1},x_n),d(y_{n-1},y_n)\}=d(x_{n-1},x_n)$ except for finitely many n. Then we have $d(x_n,x_{n+1})<d(x_{n-1},x_n)$, which in turn implies that $\{d(x_n,x_{n+1})\}$ is a monotonically decreasing sequence that converges to its glb (say) s. If $s>0$, then we have $\lim_{n\to\infty} d(x_n,x_{n+1})=\lim_{n\to\infty} d(x_{n-1},x_n)=s$. Combining ($\zeta 3$) and (4) it clear to see that $0\leq \limsup_{n\to\infty}\zeta(d(x_n,x_{n+1}),d(x_{n-1},x_n))<0$, which is not possible and hence it follows that $\lim_{n\to\infty} d(x_n,x_{n+1})=0$. Now since $d(y_{n-1},y_n)\leq d(x_{n-1},x_n)$, we see that $\lim_{n\to\infty} d(y_n,y_{n+1})=0$.

The proof is analogous if we let $\max\{d(x_{n-1},x_n),d(y_{n-1},y_n)\}=d(y_{n-1},y_n)$ except for finitely many n. Let $\mathscr{C}$ be an infinite collection of all $n_k\in\mathbb{N}$ for which $\max\{d(x_{n_k-1},x_{n_k}),d(y_{n_k-1},y_{n_k})\}=d(x_{n_k-1},x_{n_k})$ and $\mathscr{D}$ be an infinite collection of all $m_k\in\mathbb{N}$ for which $\max\{d(x_{m_k-1},x_{m_k}),d(y_{m_k-1},y_{m_k})\}=d(y_{n_k-1},y_{m_k})$. Then it can be seen that the sequences $\{d(x_{n_k-1},x_{n_k})\}$ and $\{d(x_{m_k-1},x_{m_k})\}$ are distinct subsequences of $d(x_{n-1},x_n)$. Let $r_{n_k}=d(x_{n_k-1},x_{n_k})$; then by (4), we have

$$\begin{aligned}r_{n_k}=d(x_{n_k},x_{n_k-1})<&d(y_{n_k-1},y_{n_k-2})<d(y_{n_k-2},y_{n_k-3})\\ &<\cdots<d(y_{n_{k-1}+1,y_{n_{k-1}}})<d(x_{n_{k-1}},x_{n_{k-1}-1})=r_{n_{k-1}}.\end{aligned}$$

Thus $\{r_{n_k}\}$ is a decreasing subsequence which converges to its glb (say) l. If $l>0$, then $\lim_{n_k\to\infty} r_{n_k}=\lim_{n_k\to\infty} r_{n_{k-1}}=l$. Now since $\zeta(t,s_1)\leq\zeta(t,s_2)$ whenever $t\leq s_1\leq s_2$, we have $0\leq\zeta(r_{n_k},d(y_{n_k-1},y_{n_k-2}))<\zeta(r_{n_k},r_{n_{k-1}})$. On the other hand, by ($\zeta 3$), we obtain $\limsup_{n_k\to\infty}\zeta(r_{n_k},r_{n_{k-1}})<0$, which is not possible and therefore $\lim_{n_k\to\infty} d(x_{n_k},x_{n_k-1})=0$. Similarly, we prove $\lim_{m_k\to\infty} d(x_{m_k-1},x_{m_k})=\lim_{n_k\to\infty} d(y_{n_k-1},y_{n_k})=\lim_{m_k\to\infty}=d(y_{m_k-1},y_{m_k})=0$. Thus it follows that, in all the cases discussed above, $\lim_{n\to\infty} d(x_n,x_{n+1})=\lim_{n\to\infty} d(y_n,y_{n+1})=0$.

Secondly, we wish to show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose exactly one of the sequence is not Cauchy, (say) $\{y_n\}$ is not Cauchy. Then there exists $\epsilon>0$ such that for $p(n)>q(n)>n$, $d(y_{p(n)},y_{q(n)})\geq\epsilon$ and $d(y_{p(n)-1},y_{q(n)})<\epsilon$. Then we have $d(y_{p(n)},y_{q(n)})\leq d(y_{p(n)},y_{p(n)-1})+\epsilon$ and hence it follows that $\lim_{n\to\infty} d(y_{p(n)},y_{q(n)})=\epsilon$, as $d(y_{p(n)},y_{q(n)})\geq\epsilon$. Since $\{x_n\}$ is Cauchy, it can be seen that

$$\max\{d(x_{p(n)-1},x_{q(n)-1}),d(y_{p(n)-1},y_{q(n)-1})\}=d(y_{p(n)-1},y_{q(n)-1})$$

for all $p(n)-1,q(n)-1\geq N$. Now by using (4) and ($\zeta 2$) we have

$$\begin{aligned}d(y_{p(n)},y_{q(n)})&<d(y_{p(n)-1},y_{q(n)-1})\\ &\leq d(y_{p(n)},y_{p(n)-1})+d(y_{p(n)},y_{q(n)})+d(y_{q(n)},y_{q(n)-1}).\end{aligned}$$

Now by letting $n\to\infty$ on both sides of the above inequality we get that, $\lim_{n\to\infty} d(y_{p(n)-1},y_{q(n)-1})=\epsilon$. Combining ($\zeta 3$) and (4) we have

$$0\leq\limsup_{n\to\infty}\zeta(d(y_{p(n)},y_{q(n)}),\max\{d(x_{p(n)-1},x_{q(n)-1}),d(y_{p(n)-1},y_{q(n)-1})\})<0,\quad (5)$$

which is not possible and hence $\{y_n\}$ must be a Cauchy sequence as desired. The other cases can be proved analogously.

Now suppose $\{x_n\}$ and $\{y_n\}$ are not Cauchy sequences; then there exists $\epsilon>0$ such that for $p(n)>q(n)>n$, $d(x_{p(n)},x_{q(n)})\geq\epsilon$, $d(y_{p(n)},y_{q(n)})\geq\epsilon$, $d(x_{p(n)-1},x_{q(n)})<\epsilon$ and $d(y_{p(n)-1},y_{q(n)})<\epsilon$. Proceeding in a way similar to the above we can prove that $\lim_{n\to\infty} d(x_{p(n)-1},x_{q(n)-1})=\lim_{n\to\infty} d(y_{p(n)-1},y_{q(n)-1})=\epsilon$. Hence it follows that, $\lim_{n\to\infty} \max\{d(x_{p(n)-1}, x_{q(n)-1}), d(y_{p(n)-1}, y_{q(n)-1})\} = \epsilon$. Thus we arrive at a contradictions, by the argument similar to the one in (5) as desired. Thus $\{x_n\}$ and $\{y_n\}$ converge to some $x, y \in X$, respectively.

Now suppose (CF1) is true; it follows that

$$x = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = F(x, y) \text{ and } y = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = F(y, x).$$

This in turn implies that (x, y) is the required coupled fixed point of F. Suppose (CF2) is true. Then by our assumptions we have $x_n \preceq x$ and $y_n \succeq y$ for all n. Using the contractive condition it follows that

$$\begin{aligned} 0 &\le \zeta\left(d(x_n, F(x, y)), \max\{d(x_n, x), d(y_n, y)\}\right) \\ &\le \limsup_{n\to\infty} \zeta\left(d(x_n, F(x, y)), \max\{d(x_n, x), d(y_n, y)\}\right) \\ &\le \limsup_{n\to\infty} \max\{d(x_n, x), d(y_n, y)\} - d(x_n, F(x, y)) \quad (\text{by } (\zeta 2)) \\ &\le -d(x, F(x, y)). \end{aligned}$$

But by $(\zeta 1)$ we have, $d(x, F(x, y)) = 0$ and hence $F(x, y) = x$. Similarly, we can prove $F(y, x) = y$. This completes the proof. □

In the forthcoming theorem, in order to show the existence of a unique coupled fixed point, we impose an additional condition to the hypotheses of Theorem 3.1. Note that this additional condition is equivalent to the conditions stated in the works of Bhaskar et al. [2, Theorem 2.4] and Berinde [1, Remark 3].

Theorem 3.2. *In addition to the assumptions in Theorem 3.1, if for any given* (x, y) *and* (z, t) *in* X^2, *there exists* $(u, v) \in X^2$ *such that* $x, z \succeq u$ *and* $y, t \preceq v$, *then* F *has a unique coupled fixed point.*

Proof. By Theorem 3.1, F has a coupled fixed point (x, y). We wish to show that $x = y$. Suppose x and y are comparable; then by using (4), it is easy to see that $\zeta(d(F(x, y), F(y, x)), \max\{d(x, y), d(x, y)\}) \ge 0$ and therefore $\zeta(d(x, y), d(x, y)) \ge 0$. Using $(\zeta 2)$ and $(\zeta 1)$ it follows that $d(x, y) = 0$ which implies $x = y$.

On the other hand, suppose x and y are not comparable. Then by our assumption there exist $z \in X$ such that either $x, y \preceq z$ or $z \preceq x, y$. Without loss of generality, $x, y \preceq z$, then by mixed monotone property we have,

$$\begin{aligned} F(x, y) \preceq F(z, y) &\text{ and } F(y, x) \succeq F(y, z), \\ F(y, x) \preceq F(z, x) &\text{ and } F(x, y) \succeq F(x, z), \\ F(x, x) \preceq F(z, x) &\text{ and } F(x, x) \succeq F(x, z). \end{aligned}$$

Thus it follows that

$$\begin{aligned} F^n(x, z) \preceq F^n(x, y) \preceq F^n(z, y), \\ F^n(y, z) \preceq F^n(y, x) \preceq F^n(z, x), \\ F^n(x, z) \preceq F^n(x, x) \preceq F^n(z, x). \end{aligned}$$

Now by applying the above discussed inequalities to (4), we get

$$\zeta(d(x_n, F^n(x, z)), \max\{d(x_{n-1}, F^{n-1}(x, z)), d(y_{n-1}, F^{n-1}(z, x))\}) \ge 0, \tag{6}$$

$$\zeta(d(y_n, F^n(z,x)), \max\{d(x_{n-1}, F^{n-1}(x,z)), d(y_{n-1}, F^{n-1}(z,x))\}) \geq 0, \tag{7}$$

$$\zeta(d(F^n(x,x), F^n(x,z)), \max\{d(F^{n-1}(x,x), F^{n-1}(x,z)), d(F^{n-1}(x,x), F^{n-1}(z,x))\}) \geq 0, \tag{8}$$

$$\zeta(d(F^n(x,x), F^n(z,x)), \max\{d(F^{n-1}(x,x), F^{n-1}(x,z)), d(F^{n-1}(x,x), F^{n-1}(z,x))\}) \geq 0. \tag{9}$$

Here, we claim that

$$\lim_{n\to\infty} F^n(x,z) = x \text{ and } \lim_{n\to\infty} F^n(z,x) = y. \tag{10}$$

Suppose that

$$\max\{d(x_{n-1}, F^{n-1}(x,z)), d(y_{n-1}, F^{n-1}(z,x))\} = d(x_{n-1}, F^{n-1}(x,z)) \tag{11}$$

holds for all but finitely many n. Then by ($\zeta 2$), ($\zeta 3$), (6) and (7) it follows that, $\lim_{n\to\infty} d(x_n, F^n(x,z)) = 0$ and therefore $\lim_{n\to\infty} F^n(x,z) = x$. But by (11) we have, $d(y_{n-1}, F^{n-1}(z,x)) < d(x_{n-1}, F^{n-1}(x,z))$ and therefore $\lim_{n\to\infty} d(y_n, F^n(z,x)) = 0$, which implies $\lim_{n\to\infty} F^n(z,x) = y$, thus the claim holds in this case.

If $\max\{d(x_{n-1}, F^{n-1}(x,z)), d(y_{n-1}, F^{n-1}(z,x))\} = d(y_{n-1}, F^{n-1}(z,x))$ (except for finitely many n), the proof is similar. Let $\mathscr{C}$ be an infinite collection of all $n_k \in \mathbb{N}$ for which $\max\{d(x_{n_k-1}, F^{n_k-1}(x,z)), d(y_{n_k-1}, F^{n_k-1}(z,x))\} = d(x_{n_k-1}, F^{n_k-1}(x,z))$, and $\mathscr{D}$ be an infinite collection of all $m_k \in \mathbb{N}$ for which $\max\{d(x_{m_k-1}, F^{m_k-1}(x,z)), d(y_{m_k-1}, F^{m_k-1}(z,x))\} = d(y_{m_k-1}, F^{m_k-1}(x,z))$. Then, it can be seen that the sequences $\{d(x_{n_k}, F^{n_k}(x,z))\}$ and $\{d(x_{m_k}, F^{m_k}(x,z))\}$ are nothing but distinct subsequences of $d(x_n, F^n(x,z))$. Now by using (6), (7), and ($\zeta 2$) repeatedly, we have

$$d(x_{n_k}, F^{n_k}(x,z)) < d(x_{n_k-1}, F^{n_k-1}(x,z)) < d(y_{n_k-2}, F^{n_k-2}(z,x)) < \cdots < d(d(y_{n_{k-1}-1}, F^{n_{k-1}-1}(z,x))) < d(x_{n_{k-1}}, F^{n_{k-1}}(x,z)),$$

which implies $\{d(x_{n_k}, F^{n_k}(x,z))\}$ is a decreasing sequence that converges to its glb (say) s. Suppose $s > 0$, then $\lim_{n\to\infty} d(x_{n_k}, F^{n_k}(x,z)) = \lim_{n\to\infty} d(x_{n_{k-1}}, F^{n_{k-1}}(x,z)) = s$, and since $\zeta(t, s_1) \leq \zeta(t, s_2)$ whenever $t \leq s_1 \leq s_2$, we have

$$\begin{aligned} 0 &\leq \zeta(d(x_{n_k}, F^{n_k}(x,z)), d(x_{n_k-1}, F^{n_k-1}(x,z))) \\ &\leq \zeta(d(x_{n_k}, F^{n_k}(x,z)), d(x_{n_{k-1}}, F^{n_{k-1}}(x,z))). \end{aligned}$$

However, by ($\zeta 3$), it follows that $\limsup_{n\to\infty} \zeta(d(x_{n_k}, F^{n_k}(x,z)), d(x_{n_{k-1}}, F^{n_{k-1}}(x,z))) < 0$, which is not possible. Therefore $\lim_{n_k\to\infty} d(x_{n_k}, F^{n_k}(x,z)) = 0$.

Similarly, we prove that $\lim_{n_k\to\infty} d(y_{n_k}, F^{n_k}(z,x)) = \lim_{m_k\to\infty} d(x_{m_k}, F^{m_k}(x,z)) = \lim_{m_k\to\infty} d(y_{m_k}, F^{m_k}(z,x)) = 0$, which in turn implies $\lim_{n\to\infty} d(x_n, F^n(x,z)) = 0$ and $\lim_{n\to\infty} d(x_n, F^n(x,z)) = 0$.

Now by using (8) and (9), instead of (6) and (7), it can be easily derived that $\lim_{n\to\infty} F^n(z,x) = x = \lim_{n\to\infty} F^n(x,z)$. By triangular inequality, we have $d(x,y) \leq d(x, F^n(x,z)) + d(F^n(x,z), F^n(z,x)) + d(F^n(z,x), y)$, then by letting $n \to \infty$ in the above inequality we have $d(x,y) = 0$ and therefore $x = y$. Thus (x,x) is a coupled fixed point of F.

Suppose (u,u) is another coupled fixed point such that x and u are comparable; then by using (4), we have $\zeta(d(F(x,x), F(u,u)), \max\{d(x,u), d(x,u)\}) \geq 0$, which implies $\zeta(d(x,u), d(x,u)) \geq 0$. Using ($\zeta 1$) and ($\zeta 2$) it follows that $d(x,u) = 0$ and hence $x = u$. Suppose u, x are not comparable. Then by our assumption there exists $z \in X$ such that either $x, u \preceq z$ or $x, u \succeq z$. Without loss of generality, let $x, u \preceq z$; by mixed monotone property we have,

$$\begin{aligned} F(x,z) \preceq F(z,z) &\text{ and } F(z,u) \succeq F(z,z), \\ F(u,z) \preceq F(z,z) &\text{ and } F(z,x) \succeq F(z,z), \\ F(x,x) \preceq F(z,x) &\text{ and } F(x,x) \succeq F(x,z), \\ F(u,u) \preceq F(z,u) &\text{ and } F(u,u) \succeq F(u,z). \end{aligned}$$

Thus it follows that

$$\begin{aligned} F^n(x,z) \preceq F^n(z,z) \preceq F^n(z,u), \\ F^n(u,z) \preceq F^n(z,z) \preceq F^n(z,x), \\ F^n(x,z) \preceq F^n(x,x) \preceq F^n(z,x), \\ F^n(u,z) \preceq F^n(u,u) \preceq F^n(u,x). \end{aligned}$$

Now by applying the above discussed inequalities to (4), we get

$$\zeta(d(F^n(x,z), F^n(z,u)), \max\{d(F^{n-1}(x,z), F^{n-1}(z,u)), d(F^{n-1}(z,x), F^{n-1}(u,z))\}) \geq 0, \tag{12}$$

$$\zeta(d(F^n(z,x), F^n(u,z)), \max\{d(F^{n-1}(x,z), F^{n-1}(z,u)), d(F^{n-1}(z,x), F^{n-1}(u,z))\}) \geq 0, \tag{13}$$

$$\zeta(d(F^n(x,x), F^n(x,z)), \max\{d(F^{n-1}(x,x), F^{n-1}(x,z)), d(F^{n-1}(x,x), F^{n-1}(z,x))\}) \geq 0, \tag{14}$$

$$\zeta(d(F^n(x,x), F^n(z,x)), \max\{d(F^{n-1}(x,x), F^{n-1}(x,z)), d(F^{n-1}(x,x), F^{n-1}(z,x))\}) \geq 0, \tag{15}$$

$$\zeta(d(F^n(u,u), F^n(u,z)), \max\{d(F^{n-1}(u,u), F^{n-1}(u,z)), d(F^{n-1}(u,u), F^{n-1}(z,u))\}) \geq 0, \tag{16}$$

$$\zeta(d(F^n(u,u), F^n(z,u)), \max\{d(F^{n-1}(u,u), F^{n-1}(u,z)), d(F^{n-1}(u,u), F^{n-1}(z,u))\}) \geq 0. \tag{17}$$

From (12) and (13), we get $\lim_{n\to\infty} d(F^n(x,z), F^n(z,u)) = 0$. Using (14), (15), (16) and (17), we can see that

$$\lim_{n\to\infty} F^n(z,x) = x \text{ and } \lim_{n\to\infty} F^n(x,z) = x \quad \lim_{n\to\infty} F^n(z,u) = u \text{ and } \lim_{n\to\infty} F^n(u,z) = u.$$

By triangular inequality, we have $d(u,x) \leq d(u, F^n(z,u)) + d(F^n(z,u), F^n(x,z)) + d(F^n(x,z), x)$, then by letting $n \to \infty$ in the above inequality we have $d(x,u) = 0$ and therefore $x = u$. Thus (x,x) is a unique coupled fixed point as desired. □

Corollary 3.3. *Let X and F be as in Theorem 3.2. Instead of (4) assume that*

$$d(F(x,y), F(u,v)) \leq k(\max\{d(x,u), d(y,v)\}) \text{ for all } x \succeq u,\ y \preceq v, \tag{18}$$

where $k \in [0,1)$. Then F has a unique coupled fixed point.

Proof. If we let $\zeta(t,s) = ks - t$ in Theorem 3.2, then we are done. □

Corollary 3.4. *Let X and F be as in Theorem 3.2. Instead of* (4) *assume that*

$$d\left(F(x,y), F(u,v)\right) \leq \phi\left(\max\{d(x,u), d(y,v)\}\right) \text{ for all } x \succeq u,\ y \preceq v, \qquad (19)$$

where $\phi : [0,\infty) \to [0,\infty)$ is a increasing function with $0 = \phi(0) < \phi(t) < t$ and $\lim_{r\to t^+} \phi(r) < t$ for each $t > 0$. Then F has a unique coupled fixed point.

Proof. If we let $\zeta(t,s) = \phi(s) - t$ in Theorem 3.2, then the proof follows. □

Corollary 3.5. *Let (X,d) be a complete metric space and $F : X^2 \to X$ be a mapping satisfying the contractive condition* (4) *or* (18) *or* (19) *for all $x,y,u,v \in X$. Then F has a unique coupled fixed point.*

Note that the additional condition stated in Theorem 3.2 is only a sufficient one. That is, F may possess a unique coupled fixed point, even in the state where the additional assumption fails. We prove our claim through the following example.

Example 3.6. Let $X = [0,1]$ be a space equipped with usual metric and let us define a relation on X denoted by '$\preceq$' as follows:

"$x \preceq y$ if there exists $q \geq 1$ in $\mathbb{R}$ such that $x = qy$".

Further, we write $x \prec y$ if there exists $q > 1$ in $\mathbb{R}$ such that $x = qy$ and write $x = y$ if for $q = 1$ it is $x = qy$. We claim this defines a partial order. As the reflexivity and antisymmetry of the relation follows trivially it ren=mais to show that the relation is transitive. For if $x \preceq y$ and $y \preceq z$, then there exist $q_1, q_2 \geq 1$ such that $x = q_1 y$ and $y = q_2 z$, which in turn implies $x = q_1(q_2 z) = (q_1 q_2)z$ as desired.

Here note that the condition: "for any given (x,y) and (z,t) in X^2, there exists (u,v) in X^2 such that $x,z \succeq u$ and $y,t \preceq v$" fails. For if we let $(x,y) = (0,0)$ and $(z,t) = (1,1)$, then it is not possible to find a (u,v) in X^2 such that $0,1 \succeq u$ and $1,0 \preceq v$.

Now, let $F : X^2 \to X$ be the continuous function defined by $F(x,y) = \frac{x}{x+2y+2}$ and let ζ be the simulation function defined by $\zeta(t,s) = \frac{9s}{10} - t$. Then it is easy to verify that both F and ζ satisfy the hypothesis of Theorem 3.1 and $(0,0)$ is a coupled fixed point of F. In Particular, $(0,0)$ is the only coupled fixed point of F. For, if there exists a coupled fixed point $(x,y) \neq (0,0)$ of F, then the equations $\frac{x}{x+2y+2} = x$ and $\frac{y}{y+2x+2} = y$ must possess a solution in $[0,1]$, which is not possible.

Subsequently, we give some nontrivial examples to verify the validity of Theorem 3.2. We start with an example in which both (CF1) and (CF2) are true, followed by the one in which (CF1) fails.

Example 3.7. Let $X = [0,1]$ with usual order and usual metric. Let $F : X^2 \to X$ be a continuous function defined by $F(x,y) = \frac{x}{x+2y+1}$ and let ζ be a simulation function defined by $\zeta(t,s) = \frac{s}{s+1} - t$. Then it is easy to verify that F and ζ satisfy the hypotheses of Theorem 3.2 and $(0,0)$ is the unique coupled fixed point of F.

Example 3.8. Let $X = [0,1]$ be equipped with usual order and usual metric. Let $F : X^2 \to X$ be a function defined by

$$F(x,y) = \begin{cases} \frac{x}{2} & \text{if } 0 \le x \le \frac{1}{2},\ 0 \le y \le 1 \\ \frac{x}{3} & \text{if } \frac{1}{2} < x \le 1,\ 0 \le y \le 1. \end{cases}.$$

Here note that F is not continuous at $(\frac{1}{2}, \frac{1}{2})$. Let ζ be a simulation function defined by $\zeta(t,s) = \frac{s}{7} - t$. Then it is easy to see that F and ζ satisfy the hypothesis of Theorem 3.2 and $(0,0)$ is the unique coupled fixed point of F.

We skip the example, the one in which (CF2) fails, since Example 3.6 shows the same. For in Example 3.6 if we let $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, then it is clear that $\{x_n\}$ is an increasing sequence that converges to 0 and $\frac{1}{n} \not\preceq 0$ for all n. Next we give some remarks to assert that, the Corollaries 3.3 and 3.4 of Theorem 3.2 are generalizations of the results in [2, 15, 18].

Remark 3.9. 1. If F satisfies (1), then it satisfies (18). But the converse is not true. For example, let $k = \frac{1}{2}$ and let $F : [0,1]^2 \to [0,1]$ be a function defined by $F(x,y) = \frac{x}{2}$. Then F satisfies (18). But by letting $(x,y) = (1,0)$ and $(u,v) = (0,0)$, it is easy to see that there is no $k \in [0,1)$ such that (1) is true.

2. If F satisfies (2), then it satisfies (19). But the converse is not true. For example, in Example 3.7 take $\phi(s) = \frac{s}{s+1}$. Then (19) holds. But if we let $(x,y) = (1,0)$ and $(u,v) = (0,0)$, then (2) fails.

3. If F satisfies (3), then it satisfies (18). But the converse is not true. For example, let $k = 0.98$ and let $F : [0,1]^2 \to [0,1]$ be a function defined by $F(x,y) = \frac{2x}{\sin^2(x+y)+\cos(x+y)+3}$. Then F satisfies (18). It can be seen clearly that there exist no $k, l \in (0,1)$ with $k + l < 1$ so that (3) is true.

4. Applications

In this section we prove the existence of unique solution for a system of impulsive fractional stochastic differential equation with finite delay. First we give some definitions and results which are already available in the literature.

Definition 4.1 ([9]). The Caputo derivative of at least n-times differentiable function $f : [0,\infty) \to \mathbb{R}$ is defined by ${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_0^t \frac{f^{(n)}(s)ds}{(t-s)^{\alpha+1-n}}, n-1 < \alpha < n$.

Lemma 4.2 ([4]). *Let $N(t)$ be a Poisson process with intensity λ and f be continuous. Then* $E\left[\int_0^t f(s)dN(s)\right] = \lambda \int_0^t f(s)ds$.

Lemma 4.3. *[4, Itȯ Isometry] Let $W(t)$ be a Wiener process. Then*

$$E\|\int_0^t f(s)dW(s)\|^2 = E\int_0^t \|f(s)\|^2 ds.$$

$\mathscr{L}^2(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}, \mathbb{H},) = \mathscr{L}^2(\Omega, \mathbb{H})$ is a Banach space of all measurable, square integrable, $\mathbb{H}$-valued random variables with norm $\|x(t)\|^2_{\mathscr{L}^2} = E\|x(t,w)\|^2_{\mathbb{H}}$, where $E(h) = \int_\Omega h(w)d\mathbb{P}$. $\mathscr{B} = \mathcal{C}([-a,0], \mathscr{L}^2(\Omega,\mathbb{H}))$ is a Banach space of all continuous maps from $[-a,0]$ into $\mathscr{L}^2(\Omega,\mathbb{H})$ which satisfy $\sup E\|\phi(t)\|^2 < \infty$ with norm $\|\phi\|_{\mathscr{B}} = \sup_{t\in[-d,0]} E\|\phi(t)\|^2_{\mathbb{H}}$. To consider the impulsive effects in the system; consider a Banach space $\mathscr{B}_I = B^1([-a,T], \mathscr{L}^2(\Omega,\mathbb{H}))$ of all continuous functions from $[-a,T]$ to $\mathscr{L}^2(\Omega,\mathbb{H})$, which are continuously differentiable on $[0,T]$ except for finite number of points $t_i \in (0,T)$, $i = 1,2,\dots,m$, at which $x'(t_i^+)$ and $x'(t_i^-)$ exist so that $x'(t_i^-) = x'(t_i)$, such that $\|x\|_{\mathscr{B}_I} = \sup_{t\in(0,T]} \sum_{j=0}^{1} \{E\|x^j(t)\|^2_{\mathbb{H}}\}, x \in \mathscr{B}_I$. Now consider the following coupled system of impulsive fractional stochastic differential equations with finite delay.

$$\begin{aligned}
{}^C D_t^\alpha x(t) =& f(t,x(t),y(t),G(x(t),y(t))) + g(t,x(t),y(t),G(x(t),y(t)))\frac{dW(t)}{dt} \\
&+ h(t,x(t),y(t),G(x(t),y(t)))\frac{dN(t)}{dt},\ t\in(0,T],\ t\neq t_i, \qquad (20)\\
{}^C D_t^\alpha y(t) =& f(t,y(t),x(t),G(y(t),x(t))) + g(t,y(t),x(t),G(y(t),x(t)))\frac{dW(t)}{dt} \\
&+ h(t,y(t),x(t),G(y(t),x(t)))\frac{dN(t)}{dt},\ t\in(0,T],\ t\neq t_i, \qquad (21)\\
x(t) =& y(t) = \phi(t),\ x'(t) = y'(t) = \psi(t),\ t\in[-a,0], \qquad (22)\\
\Delta(x(t_i),y(t_i)) =& I_i(x(t_i^-),y(t_i^-)),\ \Delta(x'(t_i),y'(t_i)) = Q_i(x(t_i^-),y(t_i^-)), i = 1,\dots,m.
\end{aligned}$$

where ${}^C D$ denote Caputo derivative, $\alpha \in (1,2)$, $W(t)$ is a wiener process, $N(t)$ is a poisson process with intensity λ, $G^* = \sup_{t\in[0,t]} \int_0^t k(s)ds \leq \infty$ and $G(x(t),y(t)) = \int_0^t k(s)\left(\frac{x(t)+y(t)}{2}\right)ds$. In addition consider the following assumptions.

(A1) $x(0) = y(0) = \phi(0)$ and $x'(0) = y'(0) = \psi(0)$ on $[-a,0]$.

(A2) Let $a_1, a_2, b_1, b_2, c_1, c_2$ be positive constants and let $f,g,h : (0,T]\times\mathbb{B}\times\mathbb{B}\times\mathbb{H} \to \mathbb{H}$ be nonlinear continuous functions.

$$E\|f(t,\phi_1,\phi_2,x) - f(t,\psi_1,\psi_2,x)\|^2_{\mathbb{H}} \leq a_1 \max\{\|\phi_1-\psi_1\|^2_{\mathscr{B}_I}, \|\phi_2-\psi_2\|^2_{\mathscr{B}_I}\} + a_2\|x-y\|^2_{\mathbb{H}}$$
$$E\|g(t,\phi_1,\phi_2,x) - g(t,\psi_1,\psi_2,x)\|^2_{\mathbb{H}} \leq b_1 \max\{\|\phi_1-\psi_1\|^2_{\mathscr{B}_I}, \|\phi_2-\psi_2\|^2_{\mathscr{B}_I}\} + b_2\|x-y\|^2_{\mathbb{H}}$$
$$E\|h(t,\phi_1,\phi_2,x) - h(t,\psi_1,\psi_2,x)\|^2_{\mathbb{H}} \leq c_1 \max\{\|\phi_1-\psi_1\|^2_{\mathscr{B}_I}, \|\phi_2-\psi_2\|^2_{\mathscr{B}_I}\} + c_2\|x-y\|^2_{\mathbb{H}}$$

for every $x,y \in \mathbb{H}$, $t\in(0,T]$ and $\phi_1,\phi_2,\psi_1,\psi_2 \in \mathbb{B}$.

(A3) Let $I_k, Q_k : \mathbb{H}\times\mathbb{H} \to \mathbb{H}$ be continuous functions and there exist positive constants L_I and L_Q such that

$$E\|I_k(x,y) - I_k(u,v)\|^2_{\mathbb{H}} \leq L_I \max\{E\|x-u\|^2_{\mathbb{H}}, E\|y-v\|^2_{\mathbb{H}}\}$$
$$E\|Q_k(x,y) - Q_k(u,v)\|^2_{\mathbb{H}} \leq L_Q \max\{E\|x-u\|^2_{\mathbb{H}}, E\|y-v\|^2_{\mathbb{H}}\}.$$

THEOREM 4.4. *Let*

$$k = 5(mL_I + mT^2L_Q) + \frac{5T^{2\alpha}}{(\Gamma(a))^2}\left(\frac{a_1+a_2G^*}{\alpha^2} + \frac{(b_1+b_2G^*) + \lambda^2(c_1+c_2G^*)}{T(2\alpha-1)}\right)$$

If the system of impulsive fractional stochastic differential equations satisfies (A1), (A2), (A3) and if $k<1$, then the system has a unique solution.

Proof. We know that the solving of the system of impulsive fractional stochastic differential equations (20)–(22) is equivalent to the solving of following integral equations. If $t \in (0, t_1]$,

$$x(t) = \begin{cases} \phi(0) + \psi(0)t + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} f(s, x(s), y(s), G(x(s), y(s)))ds \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} g(s, x(s), y(s), G(x(s), y(s)))dW(s) \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} h(s, x(s), y(s), G(x(s), y(s)))dN(s) \end{cases}$$

$$y(t) = \begin{cases} \phi(0) + \psi(0)t + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} f(s, y(s), x(s), G(y(s), x(s)))ds \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} g(s, y(s), x(s), G(y(s), x(s)))dW(s) \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} h(s, y(s), x(s), G(y_s, x(s)))dN(s) \end{cases}$$

If $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, m$

$$x(t) = \begin{cases} \phi(0) + \psi(0)t + \sum\limits_{j=1}^{i} [I_j(x(t_j^-), y(t_j^-)) + Q_j(x(t_j^-), y(t_j^-))(t-t_j)] \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} f(s, x(s), y(s), G(x(s), y(s)))ds \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} g(s, x(s), y(s), G(x(s), y(s)))dW(s) \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} h(s, x(s), y(s), G(x(s), y(s)))dN(s) \end{cases}$$

$$y(t) = \begin{cases} \phi(0) + \psi(0)t + \sum\limits_{j=1}^{i} [I_j(y(t_j^-), x(t_j^-)) + Q_j(y(t_j^-), x(t_j^-))(t-t_j)] \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} f(s, y(s), x(s), G(y(s), x(s)))ds \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} g(s, y(s), x(s), G(y(s), x(s)))dW(s) \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} h(s, y(s), x(s), G(y_s, x(s)))dN(s). \end{cases}$$

Define a function $F : \mathscr{B}_I^2 \to \mathscr{B}_I$ as follows.

For $t \in (0, t_1]$,

$$F(x(t), y(t)) = \begin{cases} \phi(0) + \psi(0)t + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} f(s, x(s), y(s), G(x(s), y(s)))ds \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} g(s, x(s), y(s), G(x(s), y(s)))dW(s) \\ \qquad + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{(\alpha-1)} h(s, x(s), y(s), G(x(s), y(s)))dN(s); \end{cases}$$

for $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, m$

$$F(x(t), y(t)) = \begin{cases} \phi(0) + \psi(0)t + \sum_{j=1}^{i} [I_j(x(t_j^-), y(t_j^-)) + Q_j(x(t_j^-), y(t_j^-))(t - t_j)] \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} f(s, x(s), y(s), G(x(s), y(s)))ds \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} g(s, x(s), y(s), G(x(s), y(s)))dW(s) \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} h(s, x(s), y(s), G(x(s), y(s)))dN(s). \end{cases}$$

We have to prove that the above integral equations have unique solution. It is enough to prove F has unique coupled fixed point. Let $(x(t), y(t))$ and $(u(t), v(t))$ be two different elements in $\mathscr{B}_I \times \mathscr{B}_I$. Let

$$E_1 = E\|\int_0^t (t-s)^{\alpha-1}[f(s, x(s), y(s), G(x(s), y(s))) - f(s, u(s), v(s), G(u(s), v(s)))]ds\|_{\mathbb{H}}^2$$

$$E_2 = E\|\int_0^t (t-s)^{\alpha-1}[g(s, x(s), y(s), G(x(s), y(s))) - g(s, u(s), v(s), G(u(s), v(s)))]dW(s)\|_{\mathbb{H}}^2$$

$$E_3 = E\|\int_0^t (t-s)^{\alpha-1}[h(s, x(s), y(s), G(x(s), y(s))) - h(s, u(s), v(s), G(u(s), v(s)))]dN(s)\|^2$$

$$A_i = E\|I_i(x(t_i^-), y(t_i^-)) - I_i(u(t_i^-), v(t_i^-))\|_{\mathbb{H}}^2$$

$$B_i = E\|Q_i(x(t_i^-), y(t_i^-)) - Q_i(u(t_i^-), v(t_i^-))\|_{\mathbb{H}}^2, \quad i=1, 2 \ldots, m$$

$$M = \max\{\|x(t) - u(t)\|_{\mathscr{B}_I}^2, \|y(t) - v(t)\|_{\mathscr{B}_I}^2\}.$$

Suppose $t \in (0, t_1]$, then we have $E\|F(x(t), y(t)) - F(u(t), v(t))\|_{\mathbb{H}}^2 \le \frac{3}{(\Gamma(\alpha))^2}(E_1 + E_2 + E_3)$. Now we have to calculate E_1,

$$\begin{aligned} E_1 &\le E\|\int_0^t (t-s)^{\alpha-1}ds\|^2 E\|f(s, x(s), y(s), G(x(s), y(s))) - f(s, u(s), v(s), G(u(s), v(s)))\|^2 \\ &\le \frac{T^{2\alpha}}{\alpha^2}\{a_1 \max\{\|x(s) - u(s)\|_{\mathscr{B}}^2, \|y(s) - v(s)\|_{\mathscr{B}}^2\} + a_2\|G(x(s), y(s)) - G(u(s), v(s))\|_{\mathbb{H}}^2\} \\ &\le \frac{T^{2\alpha}}{\alpha^2}\Big\{a_1 \max\{\sup E\|x(s) - u(s)\|_{\mathbb{H}}^2, \sup E\|y(s) - v(s)\|_{\mathbb{H}}^2\} \\ &\quad + a_2\|\int_0^t k(t,s)\frac{x(s) - u(s) + y(s) - v(s)}{2}ds\|_{\mathbb{H}}^2\Big\} \\ &\le \frac{T^{2\alpha}}{\alpha^2}(a_1 M +_2 G^* M) = \frac{T^{2\alpha}M}{\alpha^2}(a_1 + a_2 G^*). \end{aligned}$$

For calculating E_2, using Lemma 4.3 we have

$$\begin{aligned} E_2 &\le E\int_0^t \|(t-s)^{\alpha-1}[g(s, x(s), y(s), G(x(s), y(s))) - g(s, u(s), v(s), G(u(s), v(s)))]\|_{\mathbb{H}}^2 ds \\ &\le \frac{T^{2\alpha-1}M}{2\alpha - 1}(b_1 + b_2 G^*). \end{aligned}$$

For calculating E_2, using Lemma 4.2 we have

$$E_3 \leq \lambda^2 E \int_0^t \|(t-s)^{\alpha-1}[h(s,x(s),y(s),G(x(s),y(s))) - h(s,u(s),v(s),G(u(s),v(s)))]\|_{\mathbb{H}}^2 ds$$
$$\leq \lambda^2 \frac{T^{2\alpha-1}M}{2\alpha-1}(c_1 + c_2G^*).$$

Thus we have

$$E\|F(x(t),y(t)) - F(u(t),v(t))\|_{\mathbb{H}}^2 \leq \frac{3T^{2\alpha}M}{(\Gamma(a))^2}\left(\frac{a_1+a_2G^*}{\alpha^2} + \frac{(b_1+b_2G^*)+\lambda^2(c_1+c_2G^*)}{T(2\alpha-1)}\right).$$

Suppose $t \in (t_1, t_2]$.

$$E\|F(x(t),y(t)) - F(u(t),v(t))\|_{\mathbb{H}}^2 \leq 5A_1 + 5B_1E\|(t-t_1)\|^2 + \frac{5}{(\Gamma(a))^2}(E_1+E_2+E_3)$$
$$\leq 5(L_I + T^2L_Q)M + \frac{5T^{2\alpha}M}{(\Gamma(a))^2}\left(\frac{a_1+a_2G^*}{\alpha^2} + \frac{(b_1+b_2G^*)+\lambda^2(c_1+c_2G^*)}{T(2\alpha-1)}\right).$$

Suppose $t \in (t_m, t_{m+1}]$.

$$E\|F(x(t),y(t)) - F(u(t),v(t))\|_{\mathbb{H}}^2 \leq 5E\sum_{i=1}^m A_i + 5E\sum_{i=0}^m B_iE\|t-t_i\|_{\mathbb{H}}^2 + \frac{5}{(\Gamma(a))^2}(E_1+E_2+E_3)$$
$$\leq 5(mL_I + mT^2L_Q)M + \frac{5T^{2\alpha}M}{(\Gamma(a))^2}\left(\frac{a_1+a_2G^*}{\alpha^2} + \frac{(b_1+b_2G^*)+\lambda^2(c_1+c_2G^*)}{T(2\alpha-1)}\right).$$

Thus we get $\|F(x(t),y(t)) - F(u(t),v(t))\|_{\mathscr{B}_I}^2 \leq k\max\{\|x(t)-u(t)\|_{\mathscr{B}_I}^2, \|y(t)-v(t)\|_{\mathscr{B}_I}^2\}$, for all $x(t)$, $y(t)$, $u(t)$ and $v(t) \in \mathscr{B}_I$, where

$$k = 5(mL_I + mT^2L_Q) + \frac{5T^{2\alpha}}{(\Gamma(a))^2}\left(\frac{a_1+a_2G^*}{\alpha^2} + \frac{(b_1+b_2G^*)+\lambda^2(c_1+c_2G^*)}{T(2\alpha-1)}\right) < 1.$$

By Corollary 3.5 with contractive condition (18), F has a unique coupled fixed point in $\mathscr{B}_I$. Thus the given system of impulsive fractional stochastic differential equation has a unique solution. □

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