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73, 4 (2021), 223–242
December 2021

research paper
оригинални научни рад

AN EXTENSION OF THE CONCEPT OF γ-CONTINUITY FOR MULTIFUNCTIONS

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Abstract. A function $f : (X, \tau) \to (Y, \tau^*)$ between topological spaces is called γ-continuous if $f^{-1}(W) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(W))) \cup \mathrm{Int}(\mathrm{Cl}(f^{-1}(W)))$ for each open $W \subset Y$, where Cl (resp. Int) denotes the closure (resp. interior) operator on X. When we use the other possible operators obtained by multiple composing Cl and Int, then this condition boils down to the definitions of known types of generalized continuity. The case of multifunctions is quite different. The appropriate condition have two forms: $F^+(W) \subset \mathrm{Cl}(\mathrm{Int}(F^+(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^+(W)))$ called *u.γ.c.* or, $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(F^-(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ called *l.γ.c.*, where $\mathrm{F}^+(\mathrm{W}) = \{x \in X : F(x) \subset W\}$ and $\mathrm{F}^-(\mathrm{W}) = \{x \in X : F(x) \cap W \neq \emptyset\}$. So, one can consider the simultaneous use of the two different inverse images namely, $F^+(W)$ and $F^-(W)$. We will show that in this case the usage of all possible multiple compositions of Cl and Int leads to the new different types of continuity for multifunctions, which together with the previous defined types of continuity forms a collection which is complete in a certain topological sense.

1. Introduction and preliminaries

Throughout the present paper, (X, τ) and (Y, τ^*) will denote topological spaces with no separation properties assumed. Given a nonempty subset A $\subset$ X, we denote by $\mathcal{P}(A)$ the family of all subsets of A. The closure and the interior of a subset A of a topological space (X, τ) are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. We will regard Cl and Int as operators acting on $\mathcal{P}(X)$.

Remark 1.1. If we denote by $O_1 \vee O_2$ the sum of two operators O_1, O_2 on $\mathcal{P}(X)$ defined by $O_1 \vee O_2(A) = O_1(A) \cup O_2(A)$ for any $A \in \mathcal{P}(X)$, then it is easy to check (see [13]), that by using the sum operation and composing the members of $\{\mathrm{Cl}, \mathrm{Int}\}$ we may construct at most the following new operators: $\mathrm{Int} \circ \mathrm{Cl} \circ \mathrm{Int}$, $\mathrm{Cl} \circ \mathrm{Int}$, $\mathrm{Int} \circ \mathrm{Cl}$, $\mathrm{Cl} \circ \mathrm{Int} \vee \mathrm{Int} \circ \mathrm{Cl}$, $\mathrm{Cl} \circ \mathrm{Int} \circ \mathrm{Cl}$. Of course, the set of all such operators including the operator Int, which will be denoted by $\mathcal{I}_0$, is closed with respect to the operation $\vee$.

2020 Mathematics Subject Classification: 54C05, 54C08, 54C60, 54A05, 58C07.
Keywords and phrases: Multifunction; upper semi continuity; quasi-continuity; γ-continuity.

The usage of these operators leads to the following definitions:
A subset $A \subset X$ is said to be α-open [24] (resp. semi-open [8], pre-open [18], b-open [2] (or γ-open [3], or sp-open [9]), β-open [20] (or ps-open [1]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$). The family of all α-open (resp. semi-open, pre-open, γ-open, β-open) sets in (X, τ) is denoted by $\alpha O(X, \tau)$ (resp. $SO(X, \tau)$, $PO(X, \tau)$, $\gamma O(X, \tau)$, $\beta O(X, \tau)$). The union of all α-open (resp. semi-open, pre-open, γ-open, β-open) sets of X contained in A is called α-interior (resp. semi-interior, pre-interior, γ-interior, β-interior) of A and is denoted by $\alpha\,\mathrm{Int}(A)$ (resp. $s\,\mathrm{Int}(A)$, $p\,\mathrm{Int}(A)$, $\gamma\,\mathrm{Int}(A)$, $\beta\,\mathrm{Int}(A)$).

By a multifunction $F : (X, \tau) \to (Y, \tau^*)$ we mean a map defined on X with values being nonempty subsets of Y. We will denote the upper and lower inverse images of a subset B of Y by $\mathrm{F}^+(\mathrm{B})$ and $\mathrm{F}^-(\mathrm{B})$ respectively, that is, $\mathrm{F}^+(\mathrm{B}) = \{x \in X : F(x) \subset B\}$ and $\mathrm{F}^-(\mathrm{B}) = \{x \in X : F(x) \cap B \neq \emptyset\}$, see [4].

Let us recall some classical notions – a multifunction $F : (X, \tau) \to (Y, \tau^*)$ is said to by upper semi continuous (briefly u.s.c.) (resp. lower semi continuous (briefly l.s.c.)) at a point $x \in X$, [12, 14, 19, 25], if $x \in \mathrm{Int}(F^+(W))$ (resp. $x \in \mathrm{Int}(F^-(W))$) for each open subset W of Y such that $x \in F^+(W)$ (resp. $x \in F^-(W)$). We will say that a multifunction $F : (X, \tau) \to (Y, \tau^*)$ is *u.s.c.* (resp. *l.s.c.*) if it has this property at each point which means that $F^+(W) \subset \mathrm{Int}(F^+(W))$ (resp. $F^-(W) \subset \mathrm{Int}(F^-(W))$) for each $W \in \tau^*$. One can also consider the other two possible relations, namely $F^+(W) \subset \mathrm{Int}(F^-(W))$ and $F^-(W) \subset \mathrm{Int}(F^+(W))$. We denote these types of continuity as *u.l.s.c.* and *l.u.s.c.*, respectively. It is easy to show that the first one is equivalent to $x \in \mathrm{Int}(F^-(W))$ for any open subset $W \subset Y$ such that $x \in F^+(W)$ for any $x \in X$, which defines the type of continuity introduced in [10]. The second property means that $x \in \mathrm{Int}(F^+(W))$ for any open subset $W \subset Y$ such that $x \in F^-(W)$ or equivalently, F is *u.s.c.* at x and $F(x)$ is a singleton.

The usage of the operators $\mathrm{Int} \circ \mathrm{Cl} \circ \mathrm{Int}$, $\mathrm{Cl} \circ \mathrm{Int}$, $\mathrm{Int} \circ \mathrm{Cl}$, $\mathrm{Cl} \circ \mathrm{Int} \vee \mathrm{Int} \circ \mathrm{Cl}$ or $\mathrm{Cl} \circ \mathrm{Int} \circ \mathrm{Cl}$ instead of Int leads to the following types of generalized continuity.

Definition 1.2. A multifunction $F : (X, \tau) \to (Y, \tau^*)$ is called (α) *u.α.c.* (or *l.α.c.*) [23] (resp. (q) *u.q.c.* (or *l.q.c.*) [27], (p) *u.p.c.* (or *l.p.c.*) [26], (γ) *u.γ.c.* (or *l.γ.c.*) [21]), (β) *u.β.c.* (or *l.β.c.*) [28], [7] at a point x ∈ X if,

- $x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(W))))$ (or $x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W)))))$ (resp.
- $x \in \mathrm{Cl}(\mathrm{Int}(F^+(W)))$ (or $x \in \mathrm{Cl}(\mathrm{Int}(F^-(W))))$,
- $x \in \mathrm{Int}(\mathrm{Cl}(F^+(W)))$ (or $x \in \mathrm{Int}(\mathrm{Cl}(F^-(W))))$,
- $x \in \mathrm{Cl}(\mathrm{Int}(F^+(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^+(W)))$ (or $x \in \mathrm{Cl}(\mathrm{Int}(F^-(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W))))$),
- $x \in \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))$ (or $x \in \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(W)))))$ for each $W \in \tau^*$ such that $x \in F^+(W)$ (or $x \in F^-(W)$).

A multifunction $F : (X, \tau) \to (Y, \tau^*)$ is called *u.α.c.* (or *l.α.c.*) (resp. *u.q.c.* (or *l.q.c.*), *u.p.c.* (or *l.p.c.*), *u.γ.c.* (or *l.γ.c.*), *u.β.c.* (or *l.β.c.*)) if it has this property at each point $x \in X$.

Remark 1.3. It is evident that a multifunction $F : (X, \tau) \to (Y, \tau^*)$ is $u.\alpha.c.$ (or $l.\alpha.c.$) (resp. $u.q.c.$ (or $l.q.c.$), $u.p.c.$ (or $l.p.c.$), $u.\gamma.c.$ (or $l.\gamma.c.$), $u.\beta.c.$ (or $l.\beta.c.$)) if and only if
- $F^+(W) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(W))))$ (or $F^-(W) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W)))))$ (resp.
- $F^+(W) \subset \mathrm{Cl}(\mathrm{Int}(F^+(W)))$ (or $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(F^-(W))))$,
- $F^+(W) \subset \mathrm{Int}(\mathrm{Cl}(F^+(W)))$ (or $F^-(W) \subset \mathrm{Int}(\mathrm{Cl}(F^-(W))))$,
- $F^+(W) \subset \mathrm{Cl}(\mathrm{Int}(F^+(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^+(W)))$ (or $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(F^-(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W))))$,
- $F^+(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))$ (or $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(W))))))$ for each $W \in \tau^*$.

The lemma stated below guarantees, that the property $u.\alpha.c.$ (or $l.\alpha.c.$) (resp. $u.q.c.$ (or $l.q.c.$), $u.p.c.$ (or $l.p.c.$), $u.\gamma.c.$ (or $l.\gamma.c.$), $u.\beta.c.$ (or $l.\beta.c.$)) is in fact equivalent to
- $F^+(W) = \alpha\,\mathrm{Int}(F^+(W))$ (or $F^-(W) = \alpha\,\mathrm{Int}(F^-(W)))$ (resp.
- $F^+(W) = s\,\mathrm{Int}(F^+(W))$ (or $F^-(W) = s\,\mathrm{Int}(F^-(W)))$,
- $F^+(W) = p\,\mathrm{Int}(F^+(W))$ (or $F^-(W) = p\,\mathrm{Int}(F^-(W)))$,
- $F^+(W) = \gamma\,\mathrm{Int}(F^+(W))$ (or $F^-(W) = \gamma\,\mathrm{Int}(F^-(W)))$,
- $F^+(W) = \beta\,\mathrm{Int}(F^+(W))$ (or $F^-(W) = \beta\,\mathrm{Int}(F^-(W))))$ for each $W \in \tau^*$.

Lemma 1.4 ([2]). *The following hold for any subset A of a space (X, τ):*

(a) $\alpha\,\mathrm{Int}(A) = A \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$*;* *(d)* $\gamma\,\mathrm{Int}(A) = s\,\mathrm{Int}(A) \cup p\,\mathrm{Int}(A)$*,*
(b) $s\,\mathrm{Int}(A) = A \cap \mathrm{Cl}(\mathrm{Int}(A))$*;*
(c) $p\,\mathrm{Int}(A) = A \cap \mathrm{Int}(\mathrm{Cl}(A))$*;* *(e)* $\beta\,\mathrm{Int}(A) = A \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$*.*

Remark 1.5. For the same reasons as in Remark 1.1, the following set of operators $\mathcal{I} = \{\mathrm{Int}, \alpha\,\mathrm{Int}, s\,\mathrm{Int}, p\,\mathrm{Int}, \gamma\,\mathrm{Int}, \beta\,\mathrm{Int}\}$ is closed with respect to the operation $\vee$.

The requirements stated in the above forms of generalized continuity of a multifunction $F : (X, \tau) \to (Y, \tau^*)$ apply to only one of the two types of the inverse images of open subsets $W \subset Y$, namely to upper inverse image F^+(W) for the upper types of generalized continuity, and lower inverse image F^-(W) for the lower types. It is obvious, however, that a multifunction F designates unequivocally the pairs (F^+(W), F^-(W)) which are identified by $W \in \tau^*$. So, it is justified to establish the requirements for the pairs (F^+(W), F^-(W)), where $W \in \tau^*$. The simultaneous use of these two types of inverse images leads to the following types of generalized continuity.

Definition 1.6. A multifunction $F : (X, \tau) \to (Y, \tau^*)$ is said to be
(α) $u.l.\alpha.c.$ (or $l.u.\alpha.c.$) [30](resp.
(p) $u.l.p.c.$ (or $l.u.p.c.$) [30],
(q) $u.l.q.c.$ [30](or $l.u.q.c.$ [5–7, 15, 22]),

(γ) *u.l.γ.c.* (or *l.u.γ.c.*)

(β) *u.l.β.c.* (or *l.u.β.c.*) [30])) at a point $x \in X$ if,

- $x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ (or $x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(W)))))$ (resp.
- $x \in \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ (or $x \in \mathrm{Int}(\mathrm{Cl}(F^+(W))))$,
- $x \in \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ (or $x \in \mathrm{Cl}(\mathrm{Int}(F^+(W))))$,
- $x \in \mathrm{Cl}(\mathrm{Int}(F^-(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$

(or $x \in \mathrm{Cl}(\mathrm{Int}(F^+(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^+(W))))$,

- $x \in \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(W))))$ (or $x \in \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))))$, for each $W \in \tau^*$ such that $x \in F^+(W)$ (or $x \in F^-(W)$).

A multifunction $F : (X, \tau) \to (Y, \tau^*)$ is called *u.l.α.c.* (or *l.u.α.c.*) (resp. *u.l.q.c.* (or *l.u.q.c.*), *u.l.p.c.* (or *l.u.p.c.*), *u.l.γ.c.* (or *l.u.γ.c.*), *u.l.β.c.* (or *l.u.β.c.*)) if it has this property at each point $x \in X$.

In [5,7] the property *l.u.q.c.* was used under the name of the minimality or *u.s.c.o.* (*u.s.c.* with compact values) multifunction. In [7, Theorem 5.2] and [15, 22] this property was used independently of the *u.s.c.*

Of course, if a single-valued function $f : (X, \pi) \to (Y, \tau)$ is treated as a multifunction $F : (X, \tau) \to (Y, \tau^*)$ given by F(x) = $\{f(x)\}$ for all x $\in X$, we have $F^+(B) = F^-(B) = f^{-1}(B)$ for any $B \subset Y$ and consequently, the properties marked by (α), (p), (q), (γ) or (β) are equivalent to the α-continuity [17], pre-continuity [18], semi-continuity (quasi-continuity) [11,16], γ-continuity3 (b-continuity) or β-continuity20). respectively.

Remark 1.7. It is, of course, clear that a multifunction $F : (X, \tau) \to (Y, \tau^*)$ is *u.l.s.c* (or *l.u.s.c.*) (resp. *u.l.α.c.* (or *l.u.α.c.*) *u.l.q.c.* (or *l.u.q.c.*), *u.l.p.c.* (or *l.u.p.c.*), *u.l.γ.c.* (or *l.u.γ.c.*) *u.l.β.c.* (or *l.u.β.c.*)) if and only if

- $F^+(W) \subset \mathrm{Int}(F^-(W))$ (or $F^-(W) \subset \mathrm{Int}(F^+(W)))$ (resp.
- $F^+(W) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ (or $F^-(W) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(W)))))$
- $F^+(W) \subset \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ (or $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(F^+(W))))$,
- $F^+(W) \subset \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ (or $F^-(W) \subset \mathrm{Int}(\mathrm{Cl}(F^+(W))))$,
- $F^+(W) \subset \mathrm{Cl}(\mathrm{Int}(F^-(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ (or $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(F^+(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^+(W))))$,
- $F^+(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(W))))$ (or $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))))$ for each $W \in \tau^*$.

Analogously as in Remark 1.3, these types of continuity can be characterized by means of equality. For this purpose, we will use the following operators.

Definition 1.8 ([29]). Given a topological space (X,τ) we define the following operators $\mathcal{P}(X) \vec{\times} \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathcal{P}(X) \vec{\times} \mathcal{P}(X) = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : A \subset B\}$:

(a_u) $\mathrm{Int}_u(A,B){=}B\cap \mathrm{Int}(A)$,

(a_l) $\mathrm{Int}_l(A,B){=}A\cap \mathrm{Int}(B)$,

(b_u) $\alpha\,\mathrm{Int}_u(A,B){=}B\cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,

(b_l) $\alpha\,\mathrm{Int}_l(A,B){=}A\cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))$,

(c_u) $s\,\mathrm{Int}_u(A,B){=}B\cap \mathrm{Cl}(\mathrm{Int}(A))$,

(c_l) $s\,\mathrm{Int}_l(A,B){=}A\cap \mathrm{Cl}(\mathrm{Int}(B))$,

(d_u) $p\,\mathrm{Int}_u(A,B){=}B\cap \mathrm{Int}(\mathrm{Cl}(A))$,

(d_l) $p\,\mathrm{Int}_l(A,B){=}A\cap \mathrm{Int}(\mathrm{Cl}(B))$,

(e_u) $\gamma\,\mathrm{Int}_u(A,B){=}s\,\mathrm{Int}_u(A,B)\cup p\,\mathrm{Int}_u(A,B)$,

(e_l) $\gamma\,\mathrm{Int}_l(A,B){=}s.\,\mathrm{Int}_l(A,B)\cup p.\,\mathrm{Int}_l(A,B)$,

(f_u) $\beta\,\mathrm{Int}_u(A,B){=}B\cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,

(f_l) $\beta\,\mathrm{Int}_l(A,B){=}A\cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B)))$.

Directly from this definition we get the following characterizations.

LEMMA 1.9. *A multifunction $F:(X,\tau)\to(Y,\tau^*)$ is u.l.s.c (or l.u.s.c.) (resp. u.l.α.c. (or l.u.α.c.) u.l.q.c. (or l.u.q.c.), u.l.p.c. (or l.u.p.c.), u.l.β.c. (or l.u.β.c.), u.l.γ.c. (or l.u.γ.c.)) if and only if*
- $F^+(W)=\mathrm{Int}_l(F^+(W),F^-(W))$ *(or* $F^-(W)=\mathrm{Int}_u(F^+(W),F^-(W))$*) (resp.*
- $F^+(W)=\alpha\,\mathrm{Int}_l(F^+(W),F^-(W))$ *(or* $F^-(W)=\alpha\,\mathrm{Int}_u(F^+(W),F^-(W))$*),*
- $F^+(W)=q\,\mathrm{Int}_l(F^+(W),F^-(W))$ *(or* $F^-(W)=q\,\mathrm{Int}_u(F^+(W),F^-(W))$*),*
- $F^+(W)=p\,\mathrm{Int}_l(F^+(W),F^-(W))$ *(or* $F^-(W)=p\,\mathrm{Int}_u(F^+(W),F^-(W))$*),*
- $F^+(W)=\gamma\,\mathrm{Int}_l(F^+(W),F^-(W))$ *(or* $F^-(W)=\gamma\,\mathrm{Int}_u(F^+(W),F^-(W))$*),*
- $F^+(W)=\beta\,\mathrm{Int}_l(F^+(W),F^-(W))$ *(or* $F^-(W)=\beta\,\mathrm{Int}_u(F^+(W),F^-(W))$*)) for each* $W\in\tau^*$.

It is easy to see that the classical topological operators presented in Lemma 1.4 can be expressed as compositions of the above operators and the diagonal operator $\Delta:\mathcal{P}(X)\to\mathcal{P}(X)\times\mathcal{P}(X)$ given by $\Delta(A)=(A,A)$ for any $A\in\mathcal{P}(X)$. More precisely:

REMARK 1.10. The following hold for a subset A of a topological space (X,τ):

(a) $\mathrm{Int}_u(A,A){=}\,\mathrm{Int}_l(A,A){=}\,\mathrm{Int}(A)$;

(b) $\alpha\,\mathrm{Int}_u(A,A){=}\alpha\,\mathrm{Int}_l(A,A){=}\alpha\,\mathrm{Int}(A)$;

(c) $s\,\mathrm{Int}_u(A,A){=}s\,\mathrm{Int}_l(A,A){=}s\,\mathrm{Int}(A)$;

(d) $p\,\mathrm{Int}_u(A,A){=}s\,\mathrm{Int}_l(A,A){=}s\,\mathrm{Int}(A)$;

(e) $\gamma\,\mathrm{Int}_u(A,A){=}\gamma\,\mathrm{Int}_l(A,A){=}\gamma\,\mathrm{Int}(A)$;

(f) $\beta\,\mathrm{Int}_u(A,A){=}\beta\,\mathrm{Int}_l(A,A){=}\beta\,\mathrm{Int}(A)$.

Let us note that the equivalence conditions presented in Remark 1.7 and Lemma 1.9 show that the definitions of considered types of continuity actually determine relations $\mathcal{R}\subset\mathcal{P}(X)\times\mathcal{P}(X)$ described in terms of the operators Cl and Int such that $(\mathrm{F}^+(\mathrm{W}),\ \mathrm{F}^-(\mathrm{W}))\in\mathcal{R}$ for each $W\in\tau^*$. So, it is convenient to use the following general concept.

DEFINITION 1.11 ([29]). Let $\mathcal{R}$ be a binary relation on $\mathcal{P}(X)$. Then we say that a multifunction $F:(X,\tau)\to(Y,\tau^*)$ is $\mathcal{R}$-continuous if for any $W{\in}\tau^*$, $(F^+(W),F^-(W)){\in}\mathcal{R}$.

A direct application of Remark 1.7 and Lemma 1.9 shows that the considered types of continuity can be presented in terms of relations as follows.

LEMMA 1.12. *The following hold for any multifunction $F : (X, \tau) \to (Y, \tau^*)$:*
(i) The property of being u.s.c. (or l.s.c.) (resp. u.α.c. (or l.α.c.), u.q.c. (or l.q.c.), u.p.c. (or l.p.c.), u.γ.c. (or l.γ.c.), u.β.c. (or l.β.c.)) is equivalent to

- $\mathcal{C}(\pi_u, \operatorname{Int} \circ \pi_u)$*-continuity (or* $\mathcal{C}(\pi_l, \operatorname{Int} \circ \pi_l)$*-continuity (resp.*
- $\mathcal{C}(\pi_u, \alpha \operatorname{Int} \circ \pi_u)$*-continuity (or* $\mathcal{C}(\pi_l, \alpha \operatorname{Int} \circ \pi_l)$*-continuity,*
- $\mathcal{C}(\pi_u, s \operatorname{Int} \circ \pi_u)$*-continuity (or* $\mathcal{C}(\pi_l, s \operatorname{Int} \circ \pi_l)$*-continuity,*
- $\mathcal{C}(\pi_u, p \operatorname{Int} \circ \pi_u)$*-continuity (or* $\mathcal{C}(\pi_l, p \operatorname{Int} \circ \pi_l)$*-continuity,*
- $\mathcal{C}(\pi_u, \gamma \operatorname{Int} \circ \pi_u)$*-continuity (or* $\mathcal{C}(\pi_l, \gamma \operatorname{Int} \circ \pi_l)$*-continuity,*
- $\mathcal{C}(\pi_u, \beta \operatorname{Int} \circ \pi_u)$*-continuity (or* $\mathcal{C}(\pi_l, \beta \operatorname{Int} \circ \pi_l)$*-continuity).*

(ii) The property of being u.l.s.c. (or l.u.s.c.) (resp. u.l.α.c. (or l.u.α.c.), u.l.q.c. (or l.u.q.c.), u.l.p.c. (or l.u.p.c.), u.l.γ.c. (or l.u.γ.c.), u.l.β.c. (or l.u.β.c.)) is equivalent to

- $\mathcal{C}(\pi_u, \operatorname{Int}_l)$*-continuity (or.* $\mathcal{C}(\pi_l, \operatorname{Int}_u)$*-continuity) (resp.*
- $\mathcal{C}(\pi_u, \alpha \operatorname{Int}_l)$*-continuity (or.* $\mathcal{C}(\pi_l, \alpha \operatorname{Int}_u)$*-continuity)*
- $\mathcal{C}(\pi_u, s \operatorname{Int}_l)$*-continuity (or* $\mathcal{C}(\pi_l, s \operatorname{Int}_u)$*-continuity),*
- $\mathcal{C}(\pi_u, p \operatorname{Int}_l)$*-continuity (or* $\mathcal{C}(\pi_l, p \operatorname{Int}_u)$*-continuity),*
- $\mathcal{C}(\pi_u, \gamma \operatorname{Int}_l)$*-continuity (or* $\mathcal{C}(\pi_l, \gamma \operatorname{Int}_u)$*-continuity),*
- $\mathcal{C}(\pi_u, \beta \operatorname{Int}_l)$*-continuity (or* $\mathcal{C}(\pi_l, \beta \operatorname{Int}_u)$*-continuity)).*

It is easy to see that in the case of *u.γ.c.*, *l.γ.c.*, *u.l.γ.c.* and *l.u.γ.c.*, the corresponding set of coincidence points can be presented as follows:
- $\mathcal{C}(\pi_u, \gamma \operatorname{Int} \circ \pi_u) = \mathcal{C}(\pi_u, s \operatorname{Int} \circ \pi_u \oplus p \operatorname{Int} \circ \pi_u)$,
- $\mathcal{C}(\pi_l, \gamma \operatorname{Int} \circ \pi_l) = \mathcal{C}(\pi_l, s \operatorname{Int} \circ \pi_l \oplus p \operatorname{Int} \circ \pi_l)$,
- $\mathcal{C}(\pi_u, \gamma \operatorname{Int}_l) = \mathcal{C}(\pi_u, s \operatorname{Int}_l \oplus p \operatorname{Int}_l)$ and
- $\mathcal{C}(\pi_l, \gamma \operatorname{Int}_u) = \mathcal{C}(\pi_l, s \operatorname{Int}_u \oplus p \operatorname{Int}_u)$, where $\Psi_1 \oplus \Psi_2$ means the sum of the operators $\Psi_1, \Psi_2 : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $(\Psi_1 \oplus \Psi_2)(A, B) = \Psi_1(A, B) \cup \Psi_2(A, B)$ for any $(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X)$.

We will determine all possible types of continuity understood as $\mathcal{C}(\pi_u, \Psi_1 \oplus \Psi_2)$-continuity or $\mathcal{C}(\pi_l, \Psi_1 \oplus \Psi_2)$-continuity, where

$$\begin{aligned}\Psi_1, \Psi_2 \in &\{\operatorname{Int} \circ \pi_u, \alpha \operatorname{Int} \circ \pi_u, s \operatorname{Int} \circ \pi_u, p \operatorname{Int} \circ \pi_u, \gamma \operatorname{Int} \circ \pi_u, \beta \operatorname{Int} \circ \pi_u\} \\ &\cup \{\operatorname{Int}_l, \alpha \operatorname{Int}_l, s \operatorname{Int}_l, p \operatorname{Int}_l, \gamma \operatorname{Int}_l, \beta \operatorname{Int}_l\}, \qquad \text{or} \\ \Psi_1, \Psi_2 \in &\{\operatorname{Int} \circ \pi_l, \alpha \operatorname{Int} \circ \pi_l, s \operatorname{Int} \circ \pi_l, p \operatorname{Int} \circ \pi_l, \gamma \operatorname{Int} \circ \pi_l, \beta \operatorname{Int} \circ \pi_l\} \\ &\cup \{\operatorname{Int}_u, \alpha \operatorname{Int}_u, s \operatorname{Int}_u, p \operatorname{Int}_u, \gamma \operatorname{Int}_u, \beta \operatorname{Int}_u\}, \quad \text{respectively.}\end{aligned}$$

2. New types of generalized continuity

The following theorem identifies all possible types of $\mathcal{C}(\pi_u, \Psi_1 \oplus \Psi_2)$-continuity, in the case if $\Psi_1, \Psi_2 \in \mathcal{I}_{\pi_u} \cup \mathcal{I}_l$, where

$$\mathcal{I}_{\pi_u} = \{\operatorname{Int} \circ \pi_u, \alpha \operatorname{Int} \circ \pi_u, s \operatorname{Int} \circ \pi_u, p \operatorname{Int} \circ \pi_u, \gamma \operatorname{Int} \circ \pi_u, \beta \operatorname{Int} \circ \pi_u\}, \quad \text{and}$$

$\mathcal{I}_l = \{\mathrm{Int}_l, \alpha\,\mathrm{Int}_l, s\,\mathrm{Int}_l, p\,\mathrm{Int}_l, \gamma\,\mathrm{Int}_l, \beta\,\mathrm{Int}_l\}$.

In the parts (1)–(14) we set certain new types of continuity and we introduce their abbreviations.

Theorem 2.1. *Let $F : (X, \tau) \to (Y, \tau^*)$ be a multifunction and suppose that Ψ_1 and Ψ_2 are operators belonging to $\mathcal{I}_{\pi_u} \cup \mathcal{I}_l$. Then $\mathcal{C}(\pi_u, \Psi_1 \oplus \Psi_2)$-continuity of F is equivalent to one of the following types of continuity:*

(1) $\mathcal{C}(\pi_u, \alpha\,\mathrm{Int} \circ \pi_u \oplus \mathrm{Int}_l)$-continuity (briefly $u.[\alpha, c].c.$),

(2) $\mathcal{C}(\pi_u, s\,\mathrm{Int} \circ \pi_u \oplus \mathrm{Int}_l)$-continuity (briefly $u.[q, c].c.$),

(3) $\mathcal{C}(\pi_u, s\,\mathrm{Int} \circ \pi_u \oplus \alpha\,\mathrm{Int}_l)$-continuity (briefly $u.[q, \alpha].c.$),

(4) $\mathcal{C}(\pi_u, s\,\mathrm{Int} \circ \pi_u \oplus p\,\mathrm{Int}_l)$-continuity (briefly $u.[q, p].c.$),

(5) $\mathcal{C}(\pi_u, p\,\mathrm{Int} \circ \pi_u \oplus \mathrm{Int}_l)$-continuity (briefly $u.[p, c].c.$),

(6) $\mathcal{C}(\pi_u, p\,\mathrm{Int} \circ \pi_u \oplus \alpha\,\mathrm{Int}_l)$-continuity (briefly $u.[p, \alpha].c$),

(7) $\mathcal{C}(\pi_u, p\,\mathrm{Int} \circ \pi_u \oplus s\,\mathrm{Int}_l)$-continuity (briefly $u.[p, q].c.$),

(8) $\mathcal{C}(\pi_u, \gamma\,\mathrm{Int} \circ \pi_u \oplus \mathrm{Int}_l)$-continuity (briefly $u.[\gamma, c].c.$),

(9) $\mathcal{C}(\pi_u, \gamma\,\mathrm{Int} \circ \pi_u \oplus \alpha\,\mathrm{Int}_l)$-continuity (briefly $u.[\gamma, \alpha].c.$),

(10) $\mathcal{C}(\pi_u, \beta\,\mathrm{Int} \circ \pi_u \oplus \mathrm{Int}_l)$-continuity (briefly $u.[\beta, c].c.$),

(11) $\mathcal{C}(\pi_u, \beta\,\mathrm{Int} \circ \pi_u \oplus \alpha\,\mathrm{Int}_l)$-continuity (briefly $u.[\beta, \alpha].c.$),

(12) $\mathcal{C}(\pi_u, \beta\,\mathrm{Int} \circ \pi_u \oplus s\,\mathrm{Int}_l)$-continuity (briefly $u.[\beta, q].c.$),

(13) $\mathcal{C}(\pi_u, \beta\,\mathrm{Int} \circ \pi_u \oplus p\,\mathrm{Int}_l)$-continuity (briefly $u.[\beta, p].c.$),

(14) $\mathcal{C}(\pi_u, \beta\,\mathrm{Int} \circ \pi_u \oplus \gamma\,\mathrm{Int}_l)$-continuity (briefly $u.[\beta, \gamma].c$),

(15) $u.s.c.$, $u.\alpha.c.$, $u.q.c.$, $u.p.c.$, $u.\gamma.c.$ or $u.\beta.c.$,

(16) $u.l.s.c.$, $u.l.\alpha.c.$, $u.l.q.c.$, $u.l.p.c.$, $u.l.\gamma.c.$ or $u.l.\beta.c.$

Proof. The set $\mathcal{I}_{\pi_u}$ is closed with respect to the operation $\oplus$. Indeed, if $\Psi_1, \Psi_2 \in \mathcal{I}_{\pi_u}$, then by definition, there exist operators $O_1, O_1 \in \mathcal{I}$ (Remark 1.5), such that $\Psi_1 = O_1 \circ \pi_u$ and $\Psi_2 = O_2 \circ \pi_u$. So, $\Psi_1 \oplus \Psi_2(A, B) = O_1 \circ \pi_u(A, B) \cup O_2 \circ \pi_u(A, B) = O_1(A) \cup O_2(A) = O_1 \vee O_2(A) = O_1 \vee O_2 \circ \pi_u(A, B)$ for any $(A, B) \in \mathcal{P}(X) \vec{\times} \mathcal{P}(X)$. Since, according to Remark 1.5, $O_1 \vee O_2 \in \mathcal{I}$, this proves that $\Psi_1 \oplus \Psi_2 \in \mathcal{I}_{\pi_u}$. Consequently, in the case when $\Psi_1, \Psi_2 \in \mathcal{I}_{\pi_u}$, according to Lemma 1.12 (i), $\mathcal{C}(\pi_u, \Psi_1 \oplus \Psi_2)$-continuity means one of the following: *u.s.c.*, *u.α.c.*, *u.q.c.*, *u.p.c.*, *u.γ.c.* or *u.β.c.*

Analogously, if $\Psi_1, \Psi_2 \in \mathcal{I}_l$ then, since for any $\Psi \in \mathcal{I}_l$ there exists $O \in \mathcal{I}_0$ (see Remark 1.1), such that $\Psi(A, B) = A \cap O(B)$ for all $(A, B) \in \mathcal{P}(X) \vec{\times} \mathcal{P}(X)$, we have $\Psi_1 \oplus \Psi_2(A, B) = (A \cap O_1(B)) \cup (A \cap O_2(B)) = A \cap (O_1(B) \cup O_2(B)) = A \cap O_1 \vee O_2(B)$ for some $O_1, O_1 \in \mathcal{I}_0$. According to Remark 1.1, $O_1 \vee O_2 \in \mathcal{I}_0$, this proves that $\Psi_1 \oplus \Psi_2 \in \mathcal{I}_l$, consequently, the set $\mathcal{I}_l$ is closed with respect to the operation $\oplus$. Therefore, in the case when $\Psi_1, \Psi_2 \in \mathcal{I}_l$, Lemma 1.12 1.12 - implies that $\mathcal{C}(\pi_u, \Psi_1 \oplus \Psi_2)$-continuity means one of the following: *u.l.s.c.*, *u.l.α.c.*, *u.l.q.c.*, *u.l.p.c.*, *u.l.γ.c.* or *u.l.β.c.*

Now we will consider all the cases when $\Psi_1 \in \mathcal{I}_{\pi_u}$ and $\Psi_2 \in \mathcal{I}_l$.

At first, let us assume that $\Psi_1 = \operatorname{Int}\circ\pi_u$. Then, for every $\Psi_2 \in \mathcal{I}_l$ we have $\Psi_1 \oplus \Psi_2 = \Psi_2$. Indeed, since $\Psi_2 \in \mathcal{I}_l$, then there exists $O \in \mathcal{I}_0$ such that $\Psi_2(A,B) = A \cap O(B)$ for all $(A,B) \in \mathcal{P}(X)\vec{\times}\mathcal{P}(X)$. So, $\Psi_1 \oplus \Psi_2(A,B) = \operatorname{Int}(A) \cup (A \cap O(B)) = A \cap O(B) = \Psi_2(A,B)$ because $\operatorname{Int}(A) \subset A \cap O(B)$.

Now suppose that $\Psi_1 = \alpha\operatorname{Int}\circ\pi_u$. Then, in the case when $\Psi_2 = \operatorname{Int}_l$, the operator $\Psi_1 \oplus \Psi_2$ determines the relation noted in item (1), while in the case if $\Psi_2 \in \mathcal{I}_l \setminus \{\operatorname{Int}_l\}$, for every $(A,B) \in \mathcal{P}(X)\vec{\times}\mathcal{P}(X)$ we have $\Psi_2(A,B) = A \cap O(B)$, where $O \in \mathcal{I}_0 \setminus \{\operatorname{Int}\}$. Consequently, $\Psi_1 \oplus \Psi_2(A,B) = \alpha\operatorname{Int}(A) \cup (A \cap O(B)) = A \cap O(B) = \Psi_2(A,B)$ because $\alpha\operatorname{Int}(A) \subset A \cap O(B)$.

Next, we assume that $\Psi_1 = s\operatorname{Int}\circ\pi_u$. Then, in the case when Ψ_2 is equal to Int_l (resp. $\alpha\operatorname{Int}_l$, $p\operatorname{Int}_l$), the operator $\Psi_1 \oplus \Psi_2$ determines the relation noted in item (2) (resp. (3), (4)). In the other cases i.e., if $\Psi_2 \in \mathcal{I}_l \setminus \{\operatorname{Int}_l, \alpha\operatorname{Int}_l, p\operatorname{Int}_l\}$, we have $\Psi_1 \oplus \Psi_2 = \Psi_2$ for the same reason as in the previous parts of the proof.

Analogously to the above, if $\Psi_1 = p\operatorname{Int}\circ\pi_u$ and $\Psi_2 \in \{\operatorname{Int}_l, \alpha\operatorname{Int}_l, q\operatorname{Int}_l\}$, then the operator $\Psi_1 \oplus \Psi_2$ determines the relation noted in item (5), (6) or (7). And, $\Psi_1 \oplus \Psi_2 \in \mathcal{I}_l$ when $\Psi_2 \in \mathcal{I}_l \setminus \{\operatorname{Int}_l, \alpha\operatorname{Int}_l, q\operatorname{Int}_l\}$.

Suppose now that $\Psi_1 = \gamma\operatorname{Int}\circ\pi_u$. If Ψ_2 is equal to Int_l or $\alpha\operatorname{Int}_l$, then the operator $\Psi_1 \oplus \Psi_2$ determines the relation noted in item (8) or (9), respectively.

If $\Psi_2 = s\operatorname{Int}_l$, then we have

$$\begin{aligned}
\Psi_1 \oplus \Psi_2(A,B) &= \gamma\operatorname{Int}(A) \cup (A \cap \operatorname{Cl}(\operatorname{Int}(B))) \\
&= (A \cap (\operatorname{Cl}(\operatorname{Int}(A)) \cup \operatorname{Int}(\operatorname{Cl}(A)))) \cup (A \cap \operatorname{Cl}(\operatorname{Int}(B))) \\
&= A \cap (\operatorname{Cl}(\operatorname{Int}(A)) \cup \operatorname{Int}(\operatorname{Cl}(A)) \cup \operatorname{Cl}(\operatorname{Int}(B))) \\
&= A \cap (\operatorname{Int}(\operatorname{Cl}(A)) \cup \operatorname{Cl}(\operatorname{Int}(B))) = (A \cap \operatorname{Int}(\operatorname{Cl}(A))) \cup (A \cap \operatorname{Cl}(\operatorname{Int}(B))) \\
&= p\operatorname{Int}\circ\pi_u(A,B) \cup s\operatorname{Int}_l(A,B) = p\operatorname{Int}\circ\pi_u \oplus s\operatorname{Int}_l(A,B),
\end{aligned}$$

for all $(A,B) \in \mathcal{P}(X)\vec{\times}\mathcal{P}(X)$, So, in this case the operator $\Psi_1 \oplus \Psi_2$ determines the relation noted in item (7).

Analogously, one can prove that $\gamma\operatorname{Int}\circ\pi_u \oplus p\operatorname{Int}_l = s\operatorname{Int}\circ\pi_u \oplus p\operatorname{Int}_l$. So, in the case when $\Psi_2 = p\operatorname{Int}_l$, the operator $\Psi_1 \oplus \Psi_2$ determines the relation noted in item (4). Finally, if $\Psi_2 = \beta\operatorname{Int}_l$, analogously as in the previous parts one can prove that $\Psi_1 \oplus \Psi_2 = \Psi_2$.

Finally, if we assume that $\Psi_1 = \beta\operatorname{Int}\circ\pi_u$, then for Ψ_2 equal to Int_l (resp. $\alpha\operatorname{Int}_l$, $q\operatorname{Int}_l$, $p\operatorname{Int}_l$, $\gamma\operatorname{Int}_l$), the operators $\Psi_1 \oplus \Psi_2$ determine the relations noted in item (10) (resp. (11), (12), (13), (14)), while $\Psi_1 \oplus \beta\operatorname{Int}_l = \beta\operatorname{Int}_l$, which finishes the proof of the theorem. $\square$

We can directly express in the following way the new types of continuity listed as items (1)–(14) in the above theorem, in terms of the inverse images of open subsets.

Lemma 2.2. *For any multifunction $F : (X,\tau) \to (Y,\tau^*)$, the following pairs of statements are equivalent:*

(1) F is u.$[\alpha, c]$.c. and $F^+(W) \subset \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(F^+(W)))) \cup \operatorname{Int}(F^-(W))$ for each $W \in \tau^*$,

(2) F is u.$[q, c]$.c. and $F^+(W) \subset \operatorname{Cl}(\operatorname{Int}(F^+(W))) \cup \operatorname{Int}(F^-(W))$ for each $W \in \tau^*$,

(3) F is u.$[q,\alpha]$.c. $F^+(W)\subset \mathrm{Cl}(\mathrm{Int}(F^+(W)))\cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ for each $W\in\tau^*$,

(4) F is u.$[q,p]$.c. and $F^+(W)\subset \mathrm{Cl}(\mathrm{Int}(F^+(W)))\cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ for each $W\in\tau^*$,

(5) F is u.$[p,c]$.c. and $F^+(W)\subset \mathrm{Int}(\mathrm{Cl}(F^+(W)))\cup \mathrm{Int}(F^-(W))$ for each $W\in\tau^*$,

(6) F is u.$[p,\alpha]$.c and $F^+(W)\subset \mathrm{Int}(\mathrm{Cl}(F^+(W)))\cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ for each $W\in\tau^*$,

(7) F is u.$[p,q]$.c. and $F^+(W)\subset \mathrm{Int}(\mathrm{Cl}(F^+(W)))\cup \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ for each $W\in\tau^*$,

(8) F is u.$[\gamma,c]$.c. and $F^+(W)\subset \mathrm{Cl}(\mathrm{Int}(F^+(W)))\cup \mathrm{Int}(\mathrm{Cl}(F^+(W)))\cup \mathrm{Int}(F^-(W))$ for each $W\in\tau^*$,

(9) F is u.$[\gamma,\alpha]$.c. and $F^+(W)\subset \mathrm{Cl}(\mathrm{Int}(F^+(W)))\cup \mathrm{Int}(\mathrm{Cl}(F^+(W)))$ $\cup\, \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ for each $W\in\tau^*$,

(10) F is u.$[\beta,c]$.c. and $F^+(W)\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))\cup \mathrm{Int}(F^-(W))$ for each $W\in\tau^*$,

(11) F is u.$[\beta,\alpha]$.c. and $F^+(W)\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))\cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ for each $W\in\tau^*$,

(12) F is u.$[\beta,q]$.c. and $F^+(W)\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))\cup \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ for each $W\in\tau^*$,

(13) F is u.$[\beta,p]$.c. and $F^+(W)\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))\cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ for each $W\in\tau^*$,

(14) F is u.$[\beta,\gamma]$.c and $F^+(W)\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))\cup \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ $\cup\, \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ for each $W\in\tau^*$.

The use of the lower inverse images on the left side of the sign of inclusion in the above lemma defines new types of continuity as follows.

Definition 2.3. Let $F:(X,\tau)\to(Y,\tau^*)$ be a multifunction. Then
(1) F is said to be *l.*$[\alpha,c]$*.c.* if $F^-(W)\subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(W))))\cup \mathrm{Int}(F^-(W))$ for each $W\in\tau^*$,

(2) F is said to be *l.*$[q,c]$*.c.* if $F^-(W)\subset \mathrm{Cl}(\mathrm{Int}(F^+(W)))\cup \mathrm{Int}(F^-(W))$ for each $W\in\tau^*$,

(3) F is said to be *l.*$[q,\alpha]$*.c.* if $F^-(W)\subset \mathrm{Cl}(\mathrm{Int}(F^+(W)))\cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ for each $W\in\tau^*$,

(4) F is said to be *l.*$[q,p]$*.c.* if $F^-(W)\subset \mathrm{Cl}(\mathrm{Int}(F^+(W)))\cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ for each $W\in\tau^*$,

(5) F is said to be *l.*$[p,c]$*.c.* if $F^-(W)\subset \mathrm{Int}(\mathrm{Cl}(F^+(W)))\cup \mathrm{Int}(F^-(W))$ for each $W\in\tau^*$,

(6) F is said to be *l.*$[p,\alpha]$*.c* if $F^-(W)\subset \mathrm{Int}(\mathrm{Cl}(F^+(W)))\cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ for each $W\in\tau^*$,

(7) F is said to be *l.*$[p,q]$*.c.* if $F^-(W)\subset \mathrm{Int}(\mathrm{Cl}(F^+(W)))\cup \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ for each $W\in\tau^*$,

(8) F is said to be *l.*$[\gamma,c]$*.c.* if $F^-(W)\subset \mathrm{Cl}(\mathrm{Int}(F^+(W)))\cup \mathrm{Int}(\mathrm{Cl}(F^+(W)))\cup \mathrm{Int}(F^-(W))$ for each $W\in\tau^*$,

(9) F is said to be *l.*$[\gamma,\alpha]$*.c.* if $F^-(W)\subset \mathrm{Cl}(\mathrm{Int}(F^+(W)))\cup \mathrm{Int}(\mathrm{Cl}(F^+(W)))$ $\cup\, \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ for each $W\in\tau^*$,

(10) F is said to be $l.[\beta, c].c.$ if $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) \cup \mathrm{Int}(F^-(W))$ for each $W \in \tau^*$,

(11) F is said to be $l.[\beta, \alpha].c.$ if $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ for each $W \in \tau^*$,

(12) F is said to be $l.[\beta, q].c.$ if $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) \cup \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ for each $W \in \tau^*$,

(13) F is said to be $l.[\beta, p].c.$ if $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ for each $W \in \tau^*$,

(14) F is said to be $l.[\beta, \gamma].c$ if $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) \cup \mathrm{Cl}(\mathrm{Int}(F^-(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ for each $W \in \tau^*$.

The above lemma shows that, analogously as in Theorem 2.1, these types of continuity can be presented in terms of relations i.e., as types of $\mathcal{C}(\pi_l, \Psi_1 \oplus \Psi_2)$-continuity, in the case if $\Psi_1, \Psi_2 \in \mathcal{I}_{\pi_l} \cup \mathcal{I}_u$, where

$$\mathcal{I}_{\pi_l} = \{\mathrm{Int} \circ \pi_l, \alpha \,\mathrm{Int} \circ \pi_l, s \,\mathrm{Int} \circ \pi_l, p \,\mathrm{Int} \circ \pi_l, \gamma \,\mathrm{Int} \circ \pi_l, \beta \,\mathrm{Int} \circ \pi_l\} \quad \text{and}$$
$$\mathcal{I}_u = \{\mathrm{Int}_u, \alpha \,\mathrm{Int}_u, s \,\mathrm{Int}_u, p \,\mathrm{Int}_u, \gamma \,\mathrm{Int}_u, \beta \,\mathrm{Int}_u\}.$$

LEMMA 2.4. *Let $F : (X, \tau) \to (Y, \tau^*)$ be a multifunction and suppose that Ψ_1 and Ψ_2 are operators belonging to $\mathcal{I}_{\pi_l} \cup \mathcal{I}_u$. Then $\mathcal{C}(\pi_l, \Psi_1 \oplus \Psi_2)$-continuity of F is equivalent to one of the following types of continuity:*

(1) $\mathcal{C}(\pi_l, \alpha \,\mathrm{Int}_u \oplus \mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[\alpha, c].c.$,

(2) $\mathcal{C}(\pi_l, s \,\mathrm{Int}_u \oplus \mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[q, c].c.$,

(3) $\mathcal{C}(\pi_l, s \,\mathrm{Int}_u \oplus \alpha \,\mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[q, \alpha].c.$,

(4) $\mathcal{C}(\pi_l, s \,\mathrm{Int}_u \oplus p \,\mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[q, p].c.$,

(5) $\mathcal{C}(\pi_l, p \,\mathrm{Int}_u \oplus \mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[p, c].c.$,

(6) $\mathcal{C}(\pi_l, p \,\mathrm{Int}_u \oplus \alpha \,\mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[p, \alpha].c,$,

(7) $\mathcal{C}(\pi_l, p \,\mathrm{Int}_u \oplus q \,\mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[p, q].c.$

(8) $\mathcal{C}(\pi_l, \gamma \,\mathrm{Int}_u \oplus \mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[\gamma, c].c.$,

(9) $\mathcal{C}(\pi_l, \gamma \,\mathrm{Int}_u \oplus \alpha \,\mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[\gamma, \alpha].c.$,

(10) $\mathcal{C}(\pi_l, \beta \,\mathrm{Int}_u \oplus \mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[\beta, c].c.$,

(11) $\mathcal{C}(\pi_l, \beta \,\mathrm{Int}_u \oplus \alpha \,\mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[\beta, \alpha].c.$,

(12) $\mathcal{C}(\pi_l, \beta \,\mathrm{Int}_u \oplus q \,\mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[\beta, q].c.$,

(13) $\mathcal{C}(\pi_l, \beta \,\mathrm{Int}_u \oplus p \,\mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[\beta, p].c.$,

(14) $\mathcal{C}(\pi_l, \beta \,\mathrm{Int}_u \oplus \gamma \,\mathrm{Int} \circ \pi_l)$-continuity or, equivalently, $l.[\beta, \gamma].c$,

(15) l.s.c., l.α.c., l.q.c., l.p.c., l.γ.c. or l.β.c.,

(16) l.u.s.c., l.u.α.c., l.u.q.c., l.u.p.c., l.u.γ.c. or l.u.β.c.

The relationships between the operators that are used in the characterizations given in Remark 1.3, Lemma 1.9, Lemma 2.2 and Definition 2.3 designate the appropriate relationships between types of continuity.

Below we present the relationship between these operators, where we have used the short notation k and i [31] to represent Cl and Int, respectively. So, the operators Int, $\mathrm{Int}\circ\mathrm{Cl}\circ\mathrm{Int}$, $\mathrm{Cl}\circ\mathrm{Int}$, $\mathrm{Int}\circ\mathrm{Cl}$, $\mathrm{Cl}\circ\mathrm{Int}\vee\mathrm{Int}\circ\mathrm{Cl}$ and $\mathrm{Cl}\circ\mathrm{Int}\circ\mathrm{Cl}$ take the forms i, iki, ki, ik, $ki\vee ik$ and kik, respectively.

The presented below diagram, where $(A,B)\in\mathcal{P}(X)\vec{\times}\mathcal{P}(X)$, shows the inclusion relationships between the values of topological operators used to define the types of generalized continuity of multifunctions $F:(X,\tau)\to(Y,\tau^*)$ by taking all pairs (A,B) of the form $A=F^+(W)$ and $B=F^-(W)$, where $W\in\tau^*$.

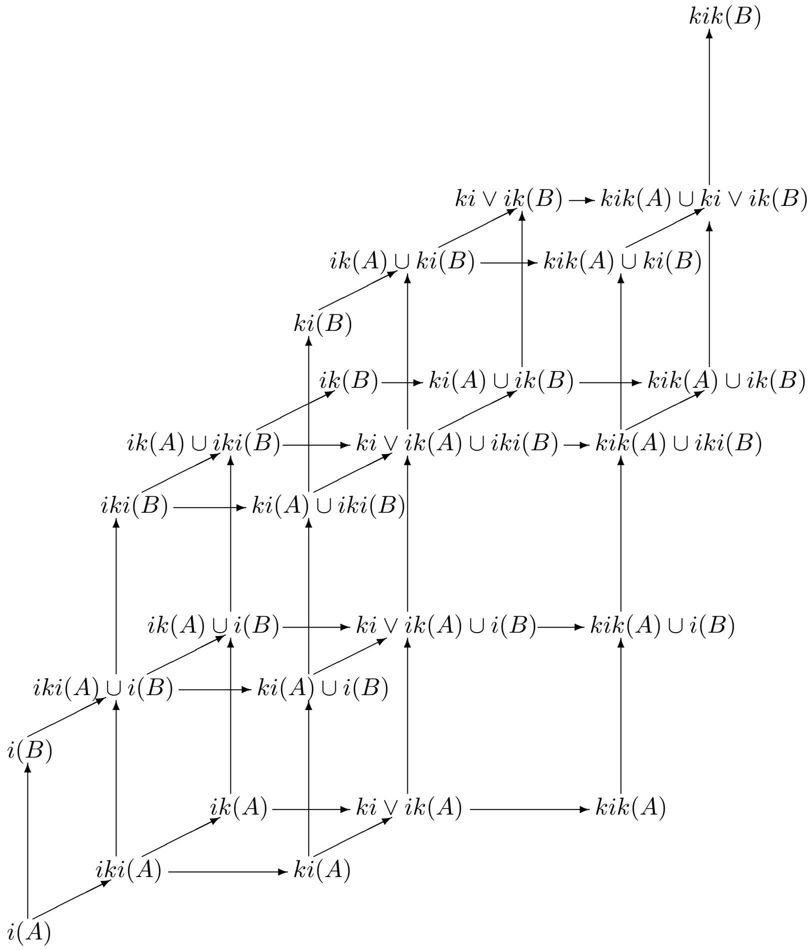

Diagram 1

The diagram below presents all types of $\mathcal{C}(\pi_l,\Psi_1\oplus\Psi_2)$-continuity listed in Lemma 2.4.

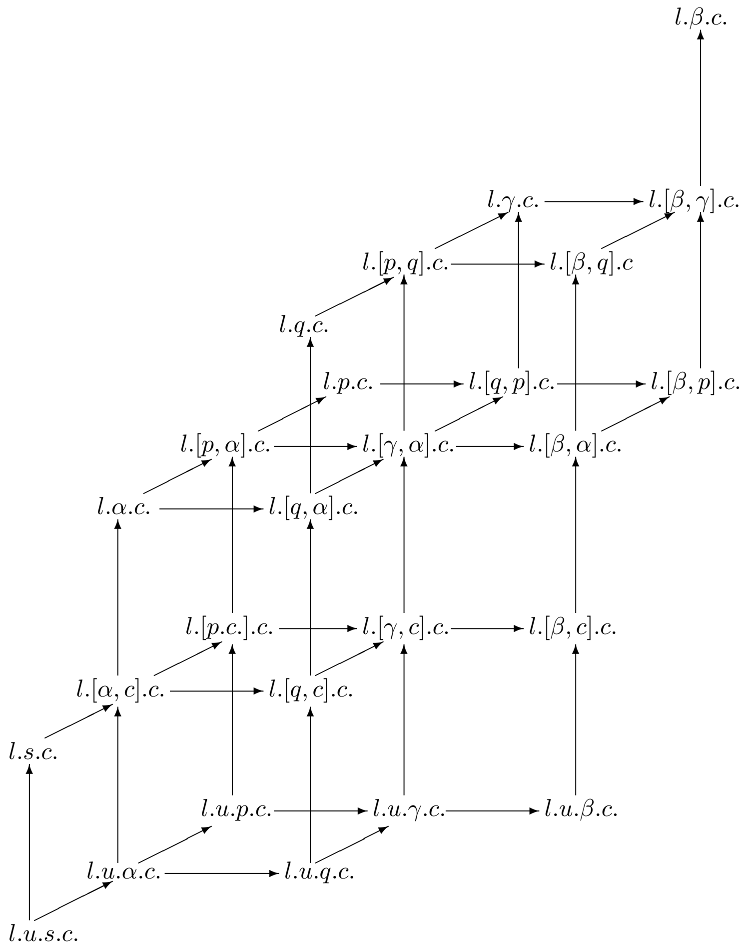

Diagram 2

The classes of multifunctions presented in the above diagram are strictly different as the following example shows.

EXAMPLE 2.5. We consider a multifunctions $F : (X,\tau) \to (X,\tau^*)$, where X is the set of all real numbers, τ is the natural topology on X and τ^* is generated by $\{(-\infty,r) : r \in R\}$. We will denote the set of all rational numbers by Q.

(E1) Let us define F by

$$F(0) = X, \qquad F(x) = \begin{cases} (-\infty, -\ln(-x)) & \text{if } x < 0, \\ (-\infty, -\ln(x)) & \text{if } x > 0. \end{cases}$$

If $W = (-\infty, r)$ for some $r \in X$, then

$$F^+(W) = (-\infty, -\exp(-r)] \cup [\exp(-r), +\infty) \quad \text{and} \quad F^-(W) = X.$$

So, the set $F^-(W)$ is open and $F^-(W) \not\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))$ which proves that F is $l.s.c.$, but not $l.u.\beta.c.$

(E2) We define F by

$$F(0) = X, \qquad F(x) = \begin{cases} \{-\ln(x)\} & \text{if } x \in \left\{\frac{1}{n} : n = 1, 2, \ldots\right\}, \\ (-\infty, -\ln(x)] & \text{if } x \in (0, +\infty) \setminus \left(\left\{\frac{1}{n} : n = 1, 2, \ldots\right\}\right), \\ \{-\ln(-x)\} & \text{if } x \in \left\{-\frac{1}{n} : n = 1, 2, \ldots\right\}, \\ (-\infty, -\ln(-x)] & \text{if } x \in (-\infty, 0) \setminus \left(\left\{-\frac{1}{n} : n = 1, 2, \ldots\right\}\right). \end{cases}$$

If $W = (-\infty, r)$ for some $r \in X$, then

$$F^+(W) = (-\infty, -\exp(-r)) \cup (\exp(-r), +\infty) \text{ and}$$

$$F^-(W) = X \setminus (\left\{-\frac{1}{n} : -\exp(-r) < -\frac{1}{n}\right\} \cup \left\{\frac{1}{n} : \frac{1}{n} < \exp(-r)\right\}).$$

So, $\mathrm{Int}(F^-(W)) = F^-(W)\setminus\{0\}$, $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}\, F^-(W))) = X$, and $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) = (-\infty, -\exp(-r)] \cup [\exp(-r), +\infty)$. Hence $F^-(W) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ and $F^-(W) \not\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))\cup \mathrm{Int}(F^-(W))$ which proves that F is $l.\alpha.c$, but not $l.[\beta, c].c$.

(E3) If we define F by

$$F(0) = X, \qquad F(x) = \begin{cases} \{-\ln(-x)\} & \text{if } x < 0, \\ (-\infty, -\ln(x)] & \text{if } x > 0, \end{cases}$$

then for any $W = (-\infty, r)$, where $r \in X$, we have

$$F^+(W) = (-\infty, -\exp(-r)) \cup (\exp(-r), +\infty) \quad \text{and} \quad F^-(W) = (-\infty, -\exp(-r)) \cup [0, +\infty).$$

So, $\mathrm{Cl}(\mathrm{Int}(F^-(W))) = (-\infty, -\exp(-r)] \cup [0, +\infty)$, $\mathrm{Int}(\mathrm{Cl}(F^-(W))) = (-\infty, -\exp(-r)) \cup (0, +\infty)$ and $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) = (-\infty, -\exp(-r)] \cup [\exp(-r), +\infty)$. Hence, $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ and $F^-(W) \not\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$. This proves that F is $l.q.c$, but not $l.[\beta, p].c$.

(E4) We define F by

$$F(0) = X, \qquad F(x) = \begin{cases} \{-\ln(x)\} & \text{if } x > 0, \\ (-\infty, -\ln(-x)] & \text{if } x \in Q \cap (-\infty, 0), \\ [-\ln(-x), \infty) & \text{if } x \in (-\infty, 0) \setminus Q. \end{cases}$$

If $W = (-\infty, r)$ for some $r \in X$, then

$$F^+(W) = ((-\infty, -\exp(-r)) \cap Q) \cup (\exp(-r), +\infty) \text{ and}$$

$$F^-(W) = (-\infty, -\exp(-r)) \cup ([-\exp(-r), 0] \cap Q) \cup (\exp(-r), \infty).$$

Then $\mathrm{Cl}(\mathrm{Int}(F^-(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W))) = (-\infty, 0) \cup [\exp(-r), \infty)$, $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) = (-\infty, -\exp(-r)) \cup [\exp(-r), \infty)$ and $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(W)))) = (-\infty, 0] \cup [\exp(-r), \infty)$. So, $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(W))))$ and $F^-(W) \not\subset \mathrm{Cl}(\mathrm{Int}(F^-(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ which proves that F is $l.\beta.c$, but not $l.[\beta, \gamma].c$.

(E5) Let us define F by

$$F(0)=X, \qquad F(x)=\begin{cases}\{-\ln(-x)\} & \text{if } x\in\left\{-\frac{1}{n}:n=1,2,\ldots\right\},\\ \{-\ln(x)\} & \text{if } x\in\left\{\frac{1}{n}:n=1,2,\ldots\right\},\\ \{\ln(-x)\} & \text{if } x\in(-\infty,0)\setminus\left\{-\frac{1}{n}:n=1,2,\ldots\right\},\\ \{\ln(x)\} & \text{if } x\in(0,+\infty)\setminus\left\{\frac{1}{n}:n=1,2,\ldots\right\}.\end{cases}$$

If $W=(-\infty,r)$ for some $r\in X$, then

$F^+(W)=(-\exp(r),\exp(r))\setminus(\left\{-\dfrac{1}{n}:-\exp(-r)<-\dfrac{1}{n}\right\}\cup\{0\}\cup\left\{\dfrac{1}{n}:\dfrac{1}{n}<\exp(-r)\right\}),$

$F^-(W)=F^+(W)\cup\{0\}$.

So $\operatorname{Int}F^-(W)=F^+(W)$ and $\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(F^+(W))))=(-\exp(r),\exp(r))$. Consequently, $F^-(W)\not\subset\operatorname{Int}(F^-(W)$ and $F^-(W)\subset\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(F^+(W))))$ which proves that F is *l.u.α.c* but not *l.s.c.*

(E6) We define F by

$$F(0)=X, \qquad F(x)=\begin{cases}\{\ln(-k)\} & \text{if } x\in[k-1,k)\quad \text{where } k=-1,-2,\ldots\\ \left\{\ln(-\frac{1}{n+1})\right\} & \text{if } x\in\left[-\frac{1}{n},-\frac{1}{n+1}\right)\quad \text{where } n=1,2,\ldots\\ \{\ln(k)\} & \text{if } x\in(k,k+1]\quad \text{where } k=1,2,\ldots\\ \left\{\ln(\frac{1}{n+1})\right\} & \text{if } x\in\left(\frac{1}{n+1},\frac{1}{n}\right]\quad \text{where } n=1,2,\ldots\end{cases}$$

Let $W=(-\infty,r)$ for some $r\in X$, then $F^-(W)=[-\exp(r)-\epsilon),\exp(r)+\epsilon)]$ and $F^+(W)=F^-(W)\setminus\{0\}$ for some $\epsilon\in[0,1)$. So, $F^-(W)=\operatorname{Cl}(\operatorname{Int}(F^+(W)))$ and $F^-(W)\not\subset\operatorname{Int}(\operatorname{Cl}(F^-(W))$ which proves that F is l.u.q.c. but not l.p.c.

(E7) Let F be defined by

$$F(0)=X, \qquad F(x)=\begin{cases}\{\ln(-x)\} & \text{if } x\in Q\cap(-\infty,0),\\ \{\ln(-x)+1\} & \text{if } x\in(-\infty,0)\setminus Q,\\ \{\ln(x)\} & \text{if } x\in Q\cap(0,+\infty),\\ \{\ln(x)+1\} & \text{if } x\in(0,+\infty)\setminus Q.\end{cases}$$

If $W=(-\infty,r)$ for some $r\in X$, then

$$\begin{aligned}F^+(W)=&((-\exp(r),-\exp(r-1)]\cap Q)\cup(-\exp(r-1),0)\\ &\cup(0,\exp(r-1))\cup([\exp(r-1),\exp(r))\cap Q)\\ F^-(W)=&F^+(W)\cup\{0\}.\end{aligned}$$

Hence, $\operatorname{Int}(\operatorname{Cl}(F^+(W)))=(-\exp(r),\exp(r))$ and $\operatorname{Cl}(\operatorname{Int}(F^-(W)))$ $=[-\exp(r-1),\exp(r-1)]$. So, $F^-(W)\subset\operatorname{Int}(\operatorname{Cl}(F^+(W)))$ and $F^-(W)\not\subset\operatorname{Cl}(\operatorname{Int}(F^-(W)))$ which proves that F is *l.u.p.c.* but not *l.q.c.*

(E8) We define F by

$$F(0)=X, \qquad F(x)=\begin{cases}\{-\ln(-x)\} & \text{if } x\in(-\infty,0),\\ \{-\ln(x)\} & \text{if } x\in(0,+\infty)\cap Q,\\ \{\ln(x)\} & \text{if } x\in(0,+\infty)\setminus Q.\end{cases}$$

If $W = (-\infty, r)$ for some $r \in X$, then

$$\begin{aligned} F^+(W) &= (-\infty, -\exp(-r)) \cup ((0, \exp(-r)] \setminus Q) \cup (\exp(-r), \exp(r)) \\ &\qquad \cup ([\exp(r), +\infty) \cap Q), \\ F^-(W) &= F^+(W) \cup \{0\}, \\ \mathrm{Cl}(F^+(W)) &= (-\infty, -\exp(-r)] \cup [0, +\infty) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))), \\ \mathrm{Cl}(\mathrm{Int}(F^-(W))) &= (-\infty, -\exp(-r)] \cup [\exp(-r), \exp(r)] \end{aligned}$$

and $\mathrm{Int}(\mathrm{Cl}(F^-(W))) = (-\infty, \exp(-r)) \cup (0, +\infty)$. So, $F^-(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))$ and $F^-(W) \not\subset \mathrm{Cl}(\mathrm{Int}(F^-(W))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ which proves that F is $l.u.\beta.c.$, but not $l.\gamma.c.$

(E9) Let us define F by

$$F(0) = X, \qquad F(x) = \begin{cases} (-\infty, \ln(-x)] & \text{if } x \in (-\infty, 0) \\ (-\infty, \ln(x)] & \text{if } x \in (0, +\infty) \cap Q \\ [\ln(x), +\infty,) & \text{if } x \in (0, +\infty) \setminus Q \end{cases}.$$

If $W = (-\infty, r)$ for some $r \in X$, then

$$\begin{aligned} F^+(W) &= (-\exp(r), 0) \cup ((0, \exp(r)) \cap Q), \\ F^-(W) &= (-\infty, \exp(r)) \cup ([\exp(r), +\infty) \cap Q) \end{aligned}$$

and consequently, $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) = [(-\exp(r), \exp(r)]$, $\mathrm{Cl}(\mathrm{Int}(F^-(W))) = (-\infty, \exp(r)]$ and $\mathrm{Int}(\mathrm{Cl}(F^-(W))) = X$. So $F^-(W) \subset \mathrm{Int}(\mathrm{Cl}(F^-(W)))$, but $F^-(W) \not\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) \cup \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ which proves that F is $l.p.c.$, but not $l.[\beta, q].c.$

Analogously as in Diagram 2, we present below all types of $\mathcal{C}(\pi_u, \Psi_1 \oplus \Psi_2)$-continuity listed in Theorem 2.1.

The following example shows that the classes of multifunctions presented in Diagram 3 are strictly different.

Example 2.6. In the first part we consider the multifunction $F : (X, \tau) \to (X, \tau^*)$, where X is the set of all real numbers, τ is the natural topology on X and τ^* is generated by $\{(-\infty, r) : r \in X\}$. In the other parts we will consider the examples of multifunctions $F : (X, \tau) \to (Y, \tau^u)$, where $Y = [0, +\infty,)$ and τ^u is generated by $\{[0, r) : r \in X\}$.

(E1) Let F be defined by

$$F(0) = X, \qquad F(x) = \begin{cases} [\ln(-x), +\infty,) & \text{if } x \in (-\infty, 0) \setminus \left\{-\frac{1}{n} : n = 1, 2, \ldots\right\}, \\ \{\ln(-x)\} & \text{if } x \in \left\{-\frac{1}{n} : n = 1, 2, \ldots\right\}, \\ [\ln(x), +\infty,) & \text{if } x \in (0, +\infty) \setminus \left\{\frac{1}{n} : n = 1, 2, \ldots\right\}, \\ \{\ln(x)\} & \text{if } x \in \left\{\frac{1}{n} : n = 1, 2, \ldots\right\}. \end{cases}$$

If $W = (-\infty, r)$ for some $r \in X$, then

$$\begin{aligned} F^+(W) &= \left\{-\frac{1}{n} : -\frac{1}{n} > -\exp(r)\right\} \cup \left\{\frac{1}{n} : \frac{1}{n} < \exp(r)\right\}, \quad \text{and} \\ F^-(W) &= (-\exp(r), \exp(r)). \end{aligned}$$

So, $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) = \emptyset$ and $F^-(W) = \mathrm{Int}(F^-(W))$. Consequently, $F^+(W) \subset \mathrm{Int}(F^-(W))$ and $F^-(W) \not\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))$ which proves that F is $u.l.s.c.$ but but not $u.\beta.c.$

(E2) Let F be defined as follows:

$$F(0) = \{0\}\,,$$
$$F(\frac{1}{n}) = \left\{\frac{1}{n}\right\}, \quad \text{for } n = 1, 2, \ldots,$$
$$F(x) = \begin{cases} [x, +\infty) & \text{if } x \in (0, +\infty) \setminus \left\{\frac{1}{n} : n = 1, 2, \ldots\right\}, \\ [-x, +\infty,) & \text{if } x \in (-\infty, 0) \setminus \left\{-\frac{1}{n} : n = 1, 2, \ldots\right\}, \end{cases}$$
$$F(-\frac{1}{n}) = \{n\}\,, \quad \text{for } n = 1, 2, \ldots$$

If $W = [0, r)$ for some $r \in X$, then

$$F^+(W) = \left\{-\frac{1}{n} : -\frac{1}{n} < -\frac{1}{r}\right\} \cup \{0\} \cup \left\{\frac{1}{n} : \frac{1}{n} < r\right\}, \quad \text{and}$$
$$F^-(W) = (-r, r) \setminus \left\{-\frac{1}{n} : -\frac{1}{n} > -\frac{1}{r}\right\}.$$

So, $\mathrm{Int}(F^-(W)) = F^-(W)\setminus\{0\}$, $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W)))) = (-r, r)$ and $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) = \emptyset$. Consequently, $F^+(W) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(W))))$ but $F^+(W) \not\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) \cup \mathrm{Int}(F^-(W))$. So, F is $u.l.\alpha.c.$ but not $u.[\beta.c].c.$

(E3) Define F by

$$F(0) = \{0\}\,,$$
$$F(\frac{1}{n}) = \left\{\frac{1}{n}\right\}, \quad \text{for } n = 1, 2, \ldots,$$
$$F(x) = [x, +\infty)\,, \quad \text{if } x \in (0, +\infty) \setminus \left\{\frac{1}{n} : n = 1, 2, \ldots\right\},$$
$$F(k) = \left\{-\frac{1}{k}\right\}, \quad \text{for } k = -1, -2, \ldots, \quad \text{and}$$
$$F(x) = \left[-\frac{1}{x}, +\infty, \right), \quad \text{if } x \in (-\infty, 0) \setminus \{k = -1, -2, \ldots\}\,.$$

If $W = [0, r)$, where $r \in X$, then $F^+(W) = \left\{-k : -k < -\frac{1}{r}\right\} \cup \{0\} \cup \left\{\frac{1}{n} : \frac{1}{n} < r\right\}$ and $F^-(W) = \left(\infty, -\frac{1}{r}\right) \cup [0, r)$. So, $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) = \emptyset$ and $\mathrm{Cl}(\mathrm{Int}(F^-(W)))) = \mathrm{Cl}(F^-(W))$.

Consequently, $F^+(W) \subset \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ but $F^+(W) \not\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) \cup \mathrm{Int}(\mathrm{Cl}(F^-(W)))$. Therefore, F is $u.l.q.c.$ but not $u.[\beta.p].c$

(E4) We define F by

$$F(0) = \{0\}$$
$$F(\frac{1}{n}) = \left\{\frac{1}{n}\right\}, \quad \text{for } n = 1, 2, \ldots,$$

$$F(x) = [x, +\infty)\,, \quad \text{if } x \in (Q \cap (0,1)) \setminus \left\{ \frac{1}{n} : n = 1, 2, \dots \right\},$$
$$F(x) = \left[\frac{1}{x}, +\infty \right), \quad \text{if } x \in ((0,1) \setminus Q) \cup (1, +\infty)\,,$$
$$F(x) = \left\{ -\frac{1}{x} \right\}, \quad \text{if } x \in (-\infty, 0)\,.$$

If $W = [0, r)$, where $r \in X$, then

$F^+(W) = \left(-\infty, -\frac{1}{r} \right) \cup \{0\} \cup \left\{ \frac{1}{n} : \frac{1}{n} < r \right\}$

$F^-(W) = F^+(W) \cup ([0, r) \cap Q) \cup \left(\frac{1}{r}, +\infty \right)$

$\mathrm{Int}(F^-(W)) = \left(-\infty, -\frac{1}{r} \right) \cup \left(\frac{1}{r}, +\infty \right)$

$\mathrm{Cl}(F^-(W)) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(W)))) = \left(\infty, -\frac{1}{r} \right] \cup [0, r] \cup \left[\frac{1}{r}, +\infty \right)$

$\mathrm{Int}(\mathrm{Cl}(F^-(W))) = \left(\infty, -\frac{1}{r} \right) \cup (0, r) \cup \left(\frac{1}{r}, +\infty \right)$, and $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))) = \left(\infty, -\frac{1}{r} \right]$.

So, $F^+(W) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^-(W))))$ and $F^+(W) \not\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))\cup \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ $\cup\, \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ which proves that F is $u.l.\beta.c.$ but not $u.[\beta.\gamma].c.$

(E5) Let us define F as follows:

$$F(0) = \{0\}$$
$$F(\frac{1}{n}) = F(-\frac{1}{n}) = [n, +\infty)\,, \quad \text{for } n = 1, 2, \dots,$$
$$F(x) = \{x\}\,, \quad \text{if } x \in (0,1) \setminus \left\{ \frac{1}{n} : n = 1, 2, \dots \right\}$$
$$F(x) = \{-x\}\,, \quad \text{if } x \in (-1, 0) \setminus \left\{ -\frac{1}{n} : n = 1, 2, \dots \right\}$$
$$F(x) = \left[\frac{1}{x}, +\infty \right), \quad \text{if } x \in (1, +\infty) \quad \text{and}$$
$$F(x) = \left[-\frac{1}{x}, +\infty \right), \quad \text{if } x \in (-\infty, -1)\,.$$

If $W = [0, r)$, where $r \in X$, then $F^+(W) = (-1,1) \setminus (\{-\frac{1}{n} : -r < -\frac{1}{n}\} \cup \{\frac{1}{n} : \frac{1}{n} < r\})$ and $F^-(W) = F^+(W) \cup \left(-\infty, -\frac{1}{r}\right) \cup \left(\frac{1}{r}, \infty\right)$.

So, $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(W)))) = (-1, 1)$ and $0 \notin \mathrm{Int}(F^-(W))$, and consequently, $F^+(W) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+)))$ but $F^+(W) \not\subset \mathrm{Int}(F^-(W))$. This proves that F is $u.\alpha.c.$ but not $u.l.s.c.$

(E6) Let F be defined by

$$F(0) = \{0\}\,,$$

$$F(x) = \{x\}\,, \quad \text{if } x \in Q \cap (0,1) \setminus \left\{\frac{1}{n} : n = 1, 2, \dots\right\},$$
$$F(x) = \left[\frac{1}{x}, +\infty\right), \quad \text{if } x \in ((0,1) \setminus Q) \cup \left\{\frac{1}{n} : n = 1, 2, \dots\right\},$$
$$F(x) = [0, +\infty)\,, \quad \text{if } x \in [1, +\infty) \quad \text{and}$$
$$F(x) = \left\{-\frac{1}{x}\right\}, \quad \text{if } x < 0.$$

If $W = [0, r)$, where $r \in X$, then

$$F^+(W) = \left(-\infty, -\frac{1}{r}\right) \cup ((Q \cap [0, r)) \setminus \left\{\frac{1}{n} : \frac{1}{n} < r, n = 1, 2, \dots\right\}), \quad \text{if } r \leq 1, \quad \text{or}$$
$$F^+(W) = \left(-\infty, -\frac{1}{r}\right) \cup ((Q \cap [0, 1)) \setminus \left\{\frac{1}{n} : n = 1, 2 \dots\right\}), \quad \text{if } r > 1, \quad \text{or}$$
$$F^-(W) = F^+(W) \cup [1, +\infty)\,, \quad \text{if } r \leq 1, \quad \text{or}$$
$$F^-(W) = F^+(W) \cup \left(\frac{1}{r}, +\infty\right), \quad \text{if } r > 1.$$

So, $\mathrm{Cl}(F^+(W)) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) = \left(-\infty, -\frac{1}{r}\right] \cup [0, r]$ if $r \leq 1$, or $\mathrm{Cl}(F^+(W)) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W)))) = \left(-\infty, -\frac{1}{r}\right] \cup [0, 1]$ if $r > 1$ and, $\mathrm{Cl}(\mathrm{Int}(F^-(W))) = \left(-\infty, -\frac{1}{r}\right] \cup [1, +\infty)$ if $r \leq 1$, or $\mathrm{Cl}(\mathrm{Int}(F^-(W))) = \left(-\infty, -\frac{1}{r}\right] \cup \left[\frac{1}{r}, +\infty\right)$ if $r > 1$. Consequently, $F^+(W) \subset \mathrm{Cl}(In(\mathrm{Cl}(F^+(W))))$ but, since $0 \notin \mathrm{Int}(\mathrm{Cl}(F^-(W))) \cup \mathrm{Cl}(\mathrm{Int}(F^-(W)))$, we have $F^+(W) \not\subset \mathrm{Int}(\mathrm{Cl}(F^-(W))) \cup \mathrm{Cl}(\mathrm{Int}(F^-(W)))$ which proves that F is *u.β.c.* but not *u.l.γ.c.*

(E7) Let F be defined as follows:

$$F(0) = \{0\}\,,$$
$$F(\frac{1}{n}) = F(-\frac{1}{n}) = \left\{\frac{1}{n}\right\}, \quad \text{for } n = 1, 2, \dots,$$
$$F(x) = [x, +\infty)\,, \quad \text{if } x \in (Q \cap ((0,1)) \setminus \left\{\frac{1}{n} : n = 1, 2, \dots\right\},$$
$$F(x) = \left[\frac{1}{x}, +\infty\right), \quad \text{if } x \in (((0,1) \setminus Q) \cup (1, +\infty)) \setminus \left\{\frac{1}{n} : n = 1, 2, \dots\right\},$$
$$F(x) = [-x, +\infty)\,, \quad \text{if } x \in (Q \cap ((-1, 0)) \setminus \left\{-\frac{1}{n} : n = 1, 2, \dots\right\} \quad \text{and}$$
$$F(x) = \left[-\frac{1}{x}, +\infty\right), \quad \text{if } x \in (((-1, 0) - Q) \cup (-\infty, -1)) \setminus \left\{-\frac{1}{n} : n = 1, 2, \dots\right\}.$$

Let $W = [0, r)$, where $r \in X$, then we have $F^+(W) = \left\{-\frac{1}{n} : -\frac{1}{n} > -r\right\} \cup \{0\} \cup \left\{\frac{1}{n} : \frac{1}{n} < r\right\}$ and, $F^-(W)\ \left(-\infty, -\frac{1}{r}\right) \cup ((-r, r) \cap Q) \cup \left(\frac{1}{r}, +\infty\right)$ if $r \leq 1$, or $F^+(W)\ \left(-\infty, -\frac{1}{r}\right) \cup \left(\left(-\frac{1}{r}, \frac{1}{r}\right) \cap Q\right) \cup \left(\frac{1}{r}, +\infty\right)$ if $r > 1$. Consequently, $In(\mathrm{Cl}(B_W)) = \left(-\infty, -\frac{1}{r}\right) \cup (-r, r) \cup \left(\frac{1}{r}, +\infty\right)$ if $r \leq 1$, or $\mathrm{Int}(\mathrm{Cl}(F^+(W))) = X$ if $r > 1$ and, $\mathrm{Cl}(\mathrm{Int}(F^+(W))) = \left(-\infty, -\frac{1}{r}\right] \cup \left[\frac{1}{r}, +\infty\right)$. So, $F^+(W) \subset \mathrm{Int}(\mathrm{Cl}(F^-(W)))$ but $F^-(W) \not\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(W))))$

$\cup\, \mathrm{Cl}(\mathrm{Int}(F^{-}(W)))$ which proves that F is $u.l.p.c.$ but not $u.[\beta, q].c.$

(E8) The multifunction from Example 2.5 (E6) is $u.q.c.$ but not $u.l.p.c.$

(E9) The multifunction from Example 2.5 (E8) is u.p.c. but not ul.q.c.

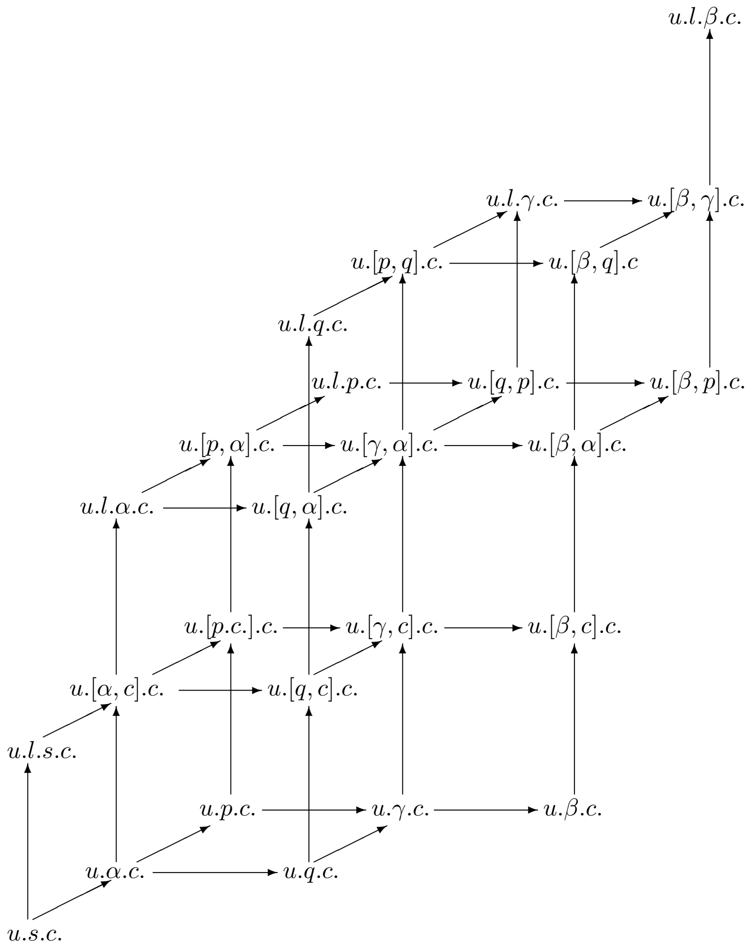

Diagram 3

References

[1] D. Andrijević, *Semi-preopen sets*, Mat. Vesn., **38(1)** (1986), 24–32.

[2] D. Andrijević, *On b-open sets*, Mat. Vesn., **48(1-2)** (1996), 59–64.

[3] A.A. El-Atik, *A study of some types of mappings on topological spaces*, Master's Thesis. Faculty of Science, Tanta University, 1997.

[4] C. Berge, *Espaces topologiques fonctions multivaques*, Dunod, Paris, 1959.

[5] C. Choquet, *Lectures on Analysis*, Benjamin, Vol. I, New York, Amsterdam 1969.

[6] J.P.R. Christensen, *Theorems of J. Namioka and R.E.Johnson type for upper semi-continuous and compact-valued set-valued maps*, Proc. Amer. Math. Soc., **86** (1982), 649–55.

[7] M.M. Coban, P.S. Kenderov, J.P. Revalski, *Densely defined selections of multivalued mappings*, Trans. Amer. Math. Soc., **344** (1994), 533–552.

[8] S.G. Crossley, S.K. Hildebrand, *Semi-closure*, Texas J. Sci., **22** (1971), 99–112.

[9] J. Dontchev, M. Przemski, *On the various decompositions of continuous and some weakly continuous functions*, Acta Math. Hungar., **71** (1996), 109–120.

[10] J. Ewert, *On quasi-continuous and cliquish maps*, Bull. Acad. Polon. Math., **32** (1984), 81–88.

[11] S. Kempisty, *Sur les functions quasicontinues*, Fund. Math., **19** (1932), 184–197.

[12] E. Klein, A.C. Thompson, *Theory of correspondences*, John Wiley Sons, 1984.

[13] K. Kuratowski, *Sur l'opération A de l'analysis situs*, Fund. Math., **3(1)** (1922), 182–199 .

[14] K. Kuratowski, *Topology, I*, New York 1966.

[15] M. Lassonde, J. Revalski, *Fragmentability of sequences of set-valued mappings with applications to variational principles*, Proc. Amer. Math. Soc., **133** (2005), 2637–2646.

[16] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.

[17] A.S. Mashhour, I.A. Hasanein, S.N. El-Deeb, *α-continuous and α-open mappings*, Acta Math. Hungar., **41** (1983), 213–218.

[18] A.S. Mashhour, M.E. El-Monsef, S.N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Phys. Soc. Egypt, **5** (1982), 47–53.

[19] M. Michael, *Topologies on spaces of sets*, Trans. Amer. Math. Soc., **71** (1951), 152-–182.

[20] M.E. El-Monsef, S.N. El-Deeb, R.A. Mahmoud, *β-Open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ., **12** (1983), 77–90.

[21] M.E. El-Monsef, M.E. Abd El-Monsef, A.A. Nasef, *On multifunctions*, Chaos Solitons Fractals, **12** (2001), 2387–2394.

[22] W.B. Moors, J.R. Gilas, *Generic continuity of minimal set-valued mappings*,. Austral. Math.Soc. Ser. A, **63** (1997), 238–262.

[23] T. Neubrunn, *Strongly quasi-continuous multivalued mappings*, Gen. Top. and its Rel. Mod. Anal. Algebra VI, (1988), 351–59.

[24] O. Njastad *On some classes of nearly open sets*, Pacific J. Math., **15** (1965), 961–970.

[25] V.I. Ponomarev, *Properties of topological spaces preserved under multivalued continuous mappings on compacta*, Amer. Math. Soc. Trans., **38(2)** (1964), 119–140.

[26] V. Popa, *Some properties of H-almost continuous multifunctions*, Problemy Mat., **10** (1990), 9–26.

[27] V. Popa, *On a decomposition of quasicontinuity for multifunctions*, Stud. Cerc. Mat., **27** (1975), 323–328.

[28] V. Popa, T. Noiri, *On upper and lower β-continuous multifunctions*, Real Anal. Exch., **22** (1996/97), 362–367.

[29] M. Przemski, *Decompositions of continuity for multifunctions*, Hacet. J. Math. Stat., **46(4)** (2017), 621–628.

[30] M. Przemski, *On the relationships between the graphs of multifunctions*, Demonstratio Math., **41** (2008), 203–224.

[31] D. Sherman, *Variations on Kuratowski's 14-set theorem*, Am. Math. Mon., **117(2)** (2010), 113–123.

(received 29.12.2019; in revised form 08.12.2020; available online 28.07.2021)

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