MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
73, 4 (2021), 268–281
December 2021

research paper
оригинални научни рад

AN INTRODUCTION TO $\mathfrak{U}$-METRIC SPACE AND NON-LINEAR CONTRACTION WITH APPLICATION TO THE STABILITY OF FIXED POINT EQUATION

Kushal Roy, Mantu Saha and Debashis Dey

Abstract. In this paper, we introduce the notion of $\mathfrak{U}$-metric space of n-tuples which generalizes several known metric type spaces. Also we study the topological properties of such newly constructed spaces and prove Cantor's intersection like theorem therein. Banach contraction principle theorem has been proved in this space and finally we apply the theorem to obtain the stability of a fixed point equation.

1. Introduction and preliminaries

Generalizations of metric structure are an interesting topic in analysis and are covered by a vast literature. Recently, Abbas et al. [1] have introduced the notion of A-metric space by extending S-metric space of three variables to n-tuples ($n \geq 2$).

Definition 1.1. Let X be a nonempty set and $A : X^n \to [0, +\infty)$ be a mapping. Then the function A is said to be an A-metric, if it satisfies the following conditions:
(A1) $A(x_1, x_2, \dots, x_n) = 0$ if and only if $x_1 = x_2 = \dots = x_n$,

(A2) $A(x_1, x_2, \dots, x_n) \leq \sum_{i=1}^{n} A(x_i, x_i, \dots, (x_i)_{n-1}, a)$ for all $x_1, x_2, \dots, x_n, a \in X$.

Example 1.2. Let $X = \mathbb{R}$. Define $A : X^n \to [0, +\infty)$ by

$$A(x_1, x_2, \dots, x_n) = \Big|\sum_{i=2}^{n} x_i - (n-1)x_1\Big| + \Big|\sum_{i=3}^{n} x_i - (n-2)x_2\Big| + \dots + \Big|\sum_{i=n-1}^{n} x_i - 2x_{n-2}\Big| + |x_n - x_{n-1}|$$

for all $x_1, x_2, \dots, x_n \in X$. Then (X, A) is an A-metric space.

2020 Mathematics Subject Classification: 47H10, 54H25.
Keywords and phrases: $\mathfrak{U}$-metric space; Cantor's intersection like theorem; fixed point; stability of fixed point equation.

In 2017, Ughade et al. [9] investigated the concept of A_b-metric space involving the concepts of A-metric space and b-metric space as follows.

Definition 1.3. Let X be a nonempty set and $A_b : X^n \to [0, +\infty)$ be a mapping. The mapping A_b is said to be an A_b-metric with coefficient $s \geq 1$, if it satisfies the following conditions:
(A_b1) $A_b(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_1 = x_2 = \ldots = x_n$,

(A_b2) $A_b(x_1, x_2, \ldots, x_n) \leq s \sum_{i=1}^{n} A_b(x_i, x_i, \ldots, (x_i)_{n-1}, a)$
for all $x_1, x_2, \ldots, x_n, a \in X$.

Example 1.4. Let $X = [1, +\infty)$. Let us define $A_b : X^n \to [0, +\infty)$ by

$$A_b(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n-1} \sum_{i<j} |x_i - x_j|^2$$

for all $x_1, x_2, \ldots, x_n \in X$. Then (X, A_b) is an A_b-metric space with coefficient $s = 2$.

In the same year, Kamran et al. [7] generalized the concept of b-metric spaces by using a function instead of a constant coefficient in the definition.

Definition 1.5. Let X be a nonempty set and $\theta : X \times X \to [1, +\infty)$. A function $d_\theta : X^2 \to [0, +\infty)$ is called an extended b-metric if for all $x, y, z \in X$ it satisfies:
(d_θ1) $d_\theta(x, y) = 0$ if and only if $x = y$,

(d_θ2) $d_\theta(x, y) = d_\theta(y, x)$,

(d_θ3) $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.
The pair (X, d_θ) is called an extended b-metric space.

Example 1.6. [7] Let $X = \{1, 2, 3\}$. Define $\theta : X^2 \to \mathbb{R}^+$ and $d_\theta : X \times X \to \mathbb{R}^+$ as:

$\theta(x, y) = 1 + x + y$

$d_\theta(1, 1) = d_\theta(2, 2) = d_\theta(3, 3) = 0$

$d_\theta(1, 2) = d_\theta(2, 1) = 80, d_\theta(1, 3) = d_\theta(3, 1) = 1000, d_\theta(2, 3) = d_\theta(3, 2) = 600$

Then (X, d_θ) is an extended b-metric space.

The above example is also a b-metric space but here we give an example which is an extended b-metric space without being a b-metric space.

Example 1.7. Let $X = \mathbb{N}$ and $\overline{d} : X^2 \to [0, \infty)$ be given by

$$\overline{d}(y, x) = \overline{d}(x, y) = \begin{cases} 0, & \text{if } x = y; \\ \frac{1}{n}, & \text{if } x = 1, y = n > 1; \\ l(> 0), & \text{if } x = n > 1, y = m > 1. \end{cases}$$

Then $\overline{d}$ is an extended b-metric with

$$\theta(y, x) = \theta(x, y) = \begin{cases} 1, & \text{if } x = y; \\ 1 + \frac{1}{nl}, & \text{if } x = 1, y = n > 1; \\ 1 + \frac{nml}{n+m}, & \text{if } x = n > 1, y = m > 1. \end{cases}$$

But $\overline{d}$ is not a b-metric for any $s > 1$ since $l = \overline{d}(n, m) \leq s[\overline{d}(n, 1) + \overline{d}(1, m)] \to 0$ as $n(> 1), m(> 1) \to \infty$.

On the other hand, Hussain et al. [4] defined and studied the properties of a parametric metric space as follows.

Definition 1.8. Let X be a nonempty set and $d_{\mathcal{P}} : X^2 \times (0, +\infty) \to [0, +\infty)$ be a mapping. Then $d_{\mathcal{P}}$ is said to be a parametric metric if for all $x, y, z \in X$
($d_{\mathcal{P}}1$) $d_{\mathcal{P}}(x, y, r) = 0$ for all $r > 0$ if and only if $x = y$,

($d_{\mathcal{P}}2$) $d_{\mathcal{P}}(x, y, r) = d_{\mathcal{P}}(y, x, r)$ for all $r > 0$,

($d_{\mathcal{P}}3$) $d_{\mathcal{P}}(x, z, r) \leq d_{\mathcal{P}}(x, y, r) + d_{\mathcal{P}}(y, z, r)$ for all $r > 0$.

Example 1.9. Let X be the collection of all functions $g : (0, +\infty) \to \mathbb{R}$. If one defines $d_{\mathcal{P}} : X^2 \times (0, +\infty) \to [0, +\infty)$ by $d_{\mathcal{P}}(g, h, r) = |g(r) - h(r)|$ for all $r > 0$ and for all $g, h \in X$ then it can be easily seen that $d_{\mathcal{P}}$ is a parametric metric on X.

In view of the above considerations, the aim of this paper is to define a proper generalization of previous structures by a unified approach. For this purpose, we first introduce a generalized metric type space called $\mathfrak{U}$-metric space and then study some of its topological properties.

2. Introduction of $\mathfrak{U}$-metric spaces

Definition 2.1. Let X be a nonempty set and let $\mathfrak{U} : X^n \times (0, +\infty) \to [0, +\infty)$ be a mapping. Then the function $\mathfrak{U}$ is called a $\mathfrak{U}$-metric if it satisfies the following conditions:
($\mathfrak{U}1$) $\mathfrak{U}(x_1, x_2, \ldots, x_n; \theta) = 0$ if and only if $x_1 = x_2 = \ldots = x_n$, for any $\theta \in (0, +\infty)$,

($\mathfrak{U}2$) $\mathfrak{U}(x_1, x_2, \ldots, x_n; \theta) \leq \alpha(x_1, x_2, \ldots, x_n; \theta) \sum_{i=1}^{n} \mathfrak{U}(x_i, x_i, \ldots, (x_i)_{n-1}, a; \theta)$ for all $x_1, x_2, \ldots, x_n, a \in X$ and $\theta > 0$, where $\alpha : X^n \times (0, +\infty) \to [1, +\infty)$ is a given function.

The $\mathfrak{U}$-metric is called symmetric if, for any $x_1, x_2 \in X$, $\mathfrak{U}(x_1, x_1, \ldots, x_1, x_2; \theta) = \mathfrak{U}(x_2, x_2, \ldots, x_2, x_1; \theta)$ for all $\theta \in (0, +\infty)$.

Example 2.2. Let X be the set of all real valued continuous functions with domain $(0, +\infty)$ and $\mathfrak{U} : X^n \times (0, +\infty) \to [0, +\infty)$ be defined by $\mathfrak{U}(f_1, f_2, \ldots, f_n; \theta) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |f_i(\theta) - f_j(\theta)|^2$ for all $f_1, f_2, \ldots, f_n \in X$ and $\theta > 0$. Then $\mathfrak{U}$ is an $\mathfrak{U}$-metric on X with $\alpha(f_1, f_2, \ldots, f_n; \theta) = 2$ for all $f_1, f_2, \ldots, f_n \in X$ and $\theta \in (0, +\infty)$.

Example 2.3. Let $X = C^0(0, +\infty)$, the space of all real valued continuous mappings with domain $(0, +\infty)$ and let us define

$$\mathfrak{U}(f_1, f_2, \ldots, f_n; \theta) = \sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - f_j(\theta)|^2 + (\sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - f_j(\theta)|^2)^2$$

for all $f_1, f_2, \dots, f_n \in X$ and for all $\theta \in (0, \infty)$. Let us choose $f_1, f_2, \dots, f_n, g \in X$ arbitrarily. Then

$$\mathfrak{U} * (f_i, f_i, \dots, (f_i)_{n-1}, g; \theta) = (1 + (n-1)|f_i(\theta) - g(\theta)|^2)(n-1)|f_i(\theta) - g(\theta)|^2 \quad (1)$$

for all $i = 1(1)n$. Then, from (1) we get

$$\begin{aligned}\sum_{i=1}^{n} \mathfrak{U}(f_i, f_i, \dots, (f_i)_{n-1}, g; \theta) =& \sum_{i=1}^{n} (1 + (n-1)|f_i(\theta) - g(\theta)|^2)(n-1)|f_i(\theta) - g(\theta)|^2 \\ \geq& (n-1) \sum_{i=1}^{n} |f_i(\theta) - g(\theta)|^2. \quad (2)\end{aligned}$$

Thus using (2) we have

$$\begin{aligned}\mathfrak{U}(f_1, f_2, \dots, f_n; \theta) &= \sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - f_j(\theta)|^2 + (\sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - f_j(\theta)|^2)^2 \\ &\leq (1 + \sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - f_j(\theta)|^2) \sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - g(\theta) + g(\theta) - f_j(\theta)|^2 \\ &\leq 2(1 + \sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - f_j(\theta)|^2) \sum_{i=1}^{n-1} \sum_{i<j} [|f_i(\theta) - g(\theta)|^2 + |g(\theta) - f_j(\theta)|^2] \\ &\leq 2\frac{n(n-1)}{2} (1 + \sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - f_j(\theta)|^2) \sum_{i=1}^{n} |f_i(\theta) - g(\theta)|^2 \\ &\leq n(1 + \sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - f_j(\theta)|^2) \sum_{i=1}^{n} \mathfrak{U}(f_i, f_i, \dots, (f_i)_{n-1}, g; \theta).\end{aligned}$$

Hence $\mathfrak{U}$ is a $\mathfrak{U}$-metric space with $\alpha(f_1, f_2, \dots, f_n; \theta) = n(1 + \sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - f_j(\theta)|^2)$ for all $f_1, f_2, \dots, f_n \in X$ and for all $\theta \in (0, +\infty)$.

Example 2.4. Let us consider the space $X = C^0(0, +\infty)$ and define the mapping

$$\mathfrak{U}(f_1, \dots, f_n; \theta) = \begin{cases} (1 + \frac{1}{\sum_{i=1}^{n} |f_i(\theta)|}) \sum_{i=1}^{n-1} \sum_{i<j} |f_i(\theta) - f_j(\theta)|^2, & \text{if any one of } f_i(\theta) \text{ is non zero} \\ 0, & \text{if } f_1(\theta) = \dots = f_n(\theta) = 0 \end{cases}$$

for all $f_1, f_2, \dots, f_n \in X$ and for all $\theta \in (0, +\infty)$. Then one can verify that $\mathfrak{U}$ is a $\mathfrak{U}$-metric space with

$$\alpha(f_1, \dots, f_n; \theta) = \begin{cases} n(1 + \frac{1}{\sum_{i=1}^{n} |f_i(\theta)|}), & \text{if any one of } f_i(\theta) \text{ is non zero} \\ n, & \text{if } f_1(\theta) = \dots = f_n(\theta) = 0 \end{cases}$$

for all $f_1, f_2, \dots, f_n \in X$ and for all $\theta \in (0, \infty)$.

The notion of $\mathfrak{U}$-metric space generalizes several of known metric type spaces, for example:

(i) For $n = 2$ and for each $x_1, x_2 \in X$ if $\mathfrak{U}(x_1, x_2; \theta)$ is a constant function with $\alpha(x_1, x_2; \theta) = 1$ for all $x_1, x_2 \in X$, $\theta \in (0, \infty)$, then a symmetric $\mathfrak{U}$-metric is the usual metric.

(ii) For $n = 2$ and for each $x_1, x_2 \in X$ if $\mathfrak{U}(x_1, x_2; \theta)$ is a constant function with $\alpha(x_1, x_2; \theta) = s > 1$ for all $x_1, x_2 \in X$, $\theta \in (0, \infty)$, then a symmetric $\mathfrak{U}$-metric is a b-metric [2].

(iii) For $n = 2$ if $\alpha(x_1, x_2; \theta) = 1$ for all $x_1, x_2 \in X$, $\theta \in (0, \infty)$, then a symmetric $\mathfrak{U}$-metric is a parametric metric space [5].

(iv) For $n = 2$ if $\alpha(x_1, x_2; \theta) = s > 1$ for all $x_1, x_2 \in X$, $\theta \in (0, \infty)$, then a symmetric $\mathfrak{U}$-metric is a parametric b-metric.

(v) For $n = 2$ and for each $x_1, x_2 \in X$ if $\mathfrak{U}(x_1, x_2; \theta)$ is a constant function with $\alpha(x_1, x_2; \theta)$ is independent of θ for all $x_1, x_2 \in X$, then a symmetric $\mathfrak{U}$-metric is an extended b-metric [7].

(vi) For $n = 3$ and for each $x_1, x_2, x_3 \in X$ if $\mathfrak{U}(x_1, x_2, x_3; \theta)$ is a constant function with $\alpha(x_1, x_2, x_3; \theta) = 1$ for all $x_1, x_2, x_3 \in X$, $\theta \in (0, \infty)$, then an $\mathfrak{U}$-metric is an S-metric [3, 13].

(vii) For $n = 3$ and for each $x_1, x_2, x_3 \in X$ if $\mathfrak{U}(x_1, x_2, x_3; \theta)$ is a constant function with $\alpha(x_1, x_2, x_3; \theta) = s > 1$ for all $x_1, x_2, x_3 \in X$, $\theta \in (0, \infty)$, then an $\mathfrak{U}$-metric is an S_b-metric [14].

(viii) For $n = 3$ if $\alpha(x_1, x_2, x_3; \theta) = 1$ for all $x_1, x_2, x_3 \in X$, $\theta \in (0, \infty)$, then an $\mathfrak{U}$-metric is a parametric S-metric [15].

(ix) For $n = 3$ if $\alpha(x_1, x_2, x_3; \theta) = s > 1$ for all $x_1, x_2, x_3 \in X$, $\theta \in (0, \infty)$, then an $\mathfrak{U}$-metric is a parametric S_b-metric.

(x) For $n = 3$ and for each $x_1, x_2, x_3 \in X$ if $\mathfrak{U}(x_1, x_2, x_3; \theta)$ is a constant function with $\alpha(x_1, x_2, x_3; \theta)$ independent of θ, for all $x_1, x_2, x_3 \in X$, then an $\mathfrak{U}$-metric is an extended S_b-metric [8].

(xi) If for each $x_1, x_2, \ldots, x_n \in X$, $\mathfrak{U}(x_1, x_2, \ldots, x_n; \theta)$ is a constant function with $\alpha(x_1, x_2, \ldots, x_n; \theta) = 1$ for all $x_1, x_2, \ldots, x_n \in X$, $\theta \in (0, \infty)$, then an $\mathfrak{U}$-metric is an A-metric [1].

(xii) If for each $x_1, x_2, \ldots, x_n \in X$, $\mathfrak{U}(x_1, x_2, \ldots, x_n; \theta)$ is a constant function with $\alpha(x_1, x_2, \ldots, x_n; \theta) = s > 1$ for all $x_1, x_2, \ldots, x_n \in X$ and $\theta \in (0, \infty)$, then an $\mathfrak{U}$-metric is an A_b-metric [9].

(xiii) If $\alpha(x_1, x_2, \ldots, x_n; \theta) = 1$ for all $x_1, x_2, \ldots, x_n \in X$, $\theta \in (0, \infty)$, then an $\mathfrak{U}$-metric is a parametric A-metric [10].

Definition 2.5. Let $(X, \mathfrak{U})$ be a $\mathfrak{U}$-metric space. A sequence $\{x_k\} \subset X$ is said to be (i) convergent to an element $x \in X$ if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $k \geq N$ we have $\mathfrak{U}(x_k, x_k, \ldots, (x_k)_{n-1}, x; \theta) < \epsilon$ for all $\theta > 0$ that is $\mathfrak{U}(x_k, x_k, \ldots, (x_k)_{n-1}, x; \theta) \to 0$ as $k \to \infty$ for all $\theta \in (0, \infty)$.

(ii) Cauchy sequence if for any $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that for any $k, m \geq M$ we have $\mathfrak{U}(x_k, x_k, \dots, (x_k)_{n-1}, x_m; \theta) < \epsilon$ for all $\theta > 0$ that is $\mathfrak{U}(x_k, x_k, \dots, (x_k)_{n-1}, x_m; \theta) \to 0$ as $k, m \to \infty$ for all $\theta \in (0, \infty)$.

(iii) X is called complete if every Cauchy sequence in X is convergent.

Definition 2.6. Let $(X, \mathfrak{U}_1)$ and $(Y, \mathfrak{U}_2)$ be two $\mathfrak{U}$-metric spaces and $T : X \to Y$ be a mapping. Then T is said to be continuous at $x_0 \in X$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\mathfrak{U}_2(Tx, Tx, \dots, (Tx)_{n-1}, Tx_0) < \epsilon$ whenever $\mathfrak{U}_1(x, x, \dots, (x)_{n-1}, x_0) < \delta$ for all $\theta > 0$.

Proposition 2.7. *Let $(X, \mathfrak{U})$ be a $\mathfrak{U}$-metric space. Then*

$$\mathfrak{U}(x, x, \dots, (x)_{n-1}, y; \theta) \leq \alpha(x, x, \dots, (x)_{n-1}, y; \theta)\mathfrak{U}(y, y, \dots, (y)_{n-1}, x; \theta)$$

for all $x, y \in X$ and for all $\theta > 0$.

Proof. Let us choose $x, y \in X$. Then

$$\begin{aligned}
&\mathfrak{U}(x, x, \dots, (x)_{n-1}, y; \theta) \\
&\overset{[\text{by } (\mathfrak{U}2)]}{\leq} \alpha(x, x, \dots, (x)_{n-1}, y; \theta)[\sum_{\text{(n-1) times}} \mathfrak{U}(x, x, \dots, x; \theta) + \mathfrak{U}(y, y, \dots, (y)_{n-1}, x; \theta)] \\
&= \alpha(x, x, \dots, (x)_{n-1}, y; \theta)\mathfrak{U}(y, y, \dots, (y)_{n-1}, x; \theta).
\end{aligned}$$

□

Lemma 2.8. *Let $(X, \mathfrak{U})$ be a $\mathfrak{U}$-metric space which is symmetric or there exists some $s > 1$ such that $\alpha(x_1, x_2, \dots, x_n; \theta) \leq s$ for all $x_1, \dots, x_n \in X$ and $\theta \in (0, \infty)$. Then any convergent sequence in X has a unique limit.*

Proof. Let a sequence $\{x_k\} \subset X$ converges to two elements $x, y \in X$. Then, for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathfrak{U}(x, x, ..., (x)_{n-1}, y; \theta) \\
&\overset{[\text{by } (\mathfrak{U}2)]}{\leq} \alpha(x, x, ..., (x)_{n-1}, y; \theta)[\sum_{\text{(n-1) times}} \mathfrak{U}(x, x, ..., (x)_{n-1}, x_k; \theta) + \mathfrak{U}(y, y, ..., (y)_{n-1}, x_k; \theta)] \\
&= \alpha(x, x, ..., (x)_{n-1}, y; \theta)[(n-1)\mathfrak{U}(x, x, ..., (x)_{n-1}, x_k; \theta) + \mathfrak{U}(y, y, ..., (y)_{n-1}, x_k; \theta)]. \quad (3)
\end{aligned}$$

Case I: If X is symmetric then from (3) we get

$$\begin{aligned}
&\mathfrak{U}(x, x, ..., (x)_{n-1}, y; \theta) \\
&\leq \alpha(x, x, ..., (x)_{n-1}, y; \theta)[(n-1)\mathfrak{U}(x_k, x_k, ..., (x_k)_{n-1}, x; \theta) + \mathfrak{U}(x_k, x_k, ..., (x_k)_{n-1}, y; \theta)].
\end{aligned}$$

for all $k \in \mathbb{N}$. By taking $k \to \infty$, we get $\mathfrak{U}(x, x, ..., (x)_{n-1}, y; \theta) = 0$ for all $\theta > 0$. Thus $x = y$.

Case II: If there exists some $s>1$ such that $\alpha(x_1, x_2, ..., x_n; \theta) \leq s$ for all $x_1, ..., x_n \in X$ and $\theta \in (0, \infty)$ then (3) gives

$$\begin{aligned}
&\mathfrak{U}(x, x, ..., (x)_{n-1}, y; \theta) \\
&\leq \alpha(x, x, ..., (x)_{n-1}, y; \theta)[(n-1)\mathfrak{U}(x, x, ..., (x)_{n-1}, x_k; \theta) + \mathfrak{U}(y, y, ..., (y)_{n-1}, x_k; \theta)] \\
&\leq \alpha(x, x, ..., (x)_{n-1}, y; \theta)[(n-1)\alpha(x, x, ..., (x)_{n-1}, x_k; \theta)\mathfrak{U}(x_k, x_k, ..., (x_k)_{n-1}, x; \theta)
\end{aligned}$$

$+\alpha(y, y, ..., (y)_{n-1}, x_k; \theta)\mathfrak{U}(x_k, x_k, ..., (x_k)_{n-1}, y; \theta)]$ [From Proposition 2.7]
$\leq \alpha(x, x, ..., (x)_{n-1}, y; \theta)[(n-1)s\mathfrak{U}(x_k, x_k, ..., (x_k)_{n-1}, x; \theta) + s\mathfrak{U}(x_k, x_k, ..., (x_k)_{n-1}, y; \theta)]$.
for all $k \in \mathbb{N}$. If we let $k \to \infty$, we see that $\mathfrak{U}(x, x, ..., (x)_{n-1}, y; \theta) = 0$ for all $\theta > 0$ and hence $x = y$. □.

PROPOSITION 2.9. *Let $(X, \mathfrak{U})$ be a $\mathfrak{U}$-metric space. Then*

$$\begin{aligned}&\mathfrak{U}(x, x, \dots, (x)_{n-1}, z; \theta)\\ &\leq (n-1)\alpha(x, x, \dots, (x)_{n-1}, z; \theta)\mathfrak{U}(x, x, \dots, (x)_{n-1}, y; \theta)\\ &\quad + \alpha(x, x, \dots, (x)_{n-1}, z; \theta)\alpha(z, z, \dots, (z)_{n-1}, y; \theta)\mathfrak{U}(y, y, \dots, (y)_{n-1}, z; \theta)\end{aligned}$$

for all $x, y, z \in X$ and for all $\theta > 0$.

Proof. For any $x, y, z \in X$ and for all $\theta > 0$ we get,

$$\begin{aligned}&\mathfrak{U}(x, x, \dots, (x)_{n-1}, z; \theta)\\ &\overset{[\text{by } (\mathfrak{U}2)]}{\leq} \alpha(x, x, \dots, (x)_{n-1}, z; \theta)[\sum_{\text{(n-1) times}} \mathfrak{U}(x, x, \dots, (x)_{n-1}, y; \theta) + \mathfrak{U}(z, z, \dots, (z)_{n-1}, y; \theta)]\\ &= (n-1)\alpha(x, x, \dots, (x)_{n-1}, z; \theta)\mathfrak{U}(x, x, \dots, (x)_{n-1}, y; \theta)\\ &\quad + \alpha(x, x, \dots, (x)_{n-1}, z; \theta)\alpha(z, z, \dots, (z)_{n-1}, y; \theta)\mathfrak{U}(y, y, \dots, (y)_{n-1}, z; \theta).\end{aligned}$$ □

PROPOSITION 2.10. *In an $\mathfrak{U}$-metric space $(X, \mathfrak{U})$, for any $x_1, x_2, \dots, x_n, a \in X$ and for any $\theta > 0$ we have*

(i) $\mathfrak{U}(x_1, x_2, \dots, x_n; \theta) \leq$
$\alpha(x_1, x_2, \dots, x_n; \theta) \sum_{i=2}^{n} \{\alpha(x_i, x_i, \dots, (x_i)_{n-1}, x_1; \theta)\mathfrak{U}(x_1, x_1, \dots, (x_1)_{n-1}, x_i; \theta)\}$;

(ii) $\mathfrak{U}(x_1, x_2, \dots, x_n; \theta) \leq \alpha(x_1, x_2, \dots, x_2; \theta)\mathfrak{U}(x_1, \dots, (x_1)_{n-1}, x_2; \theta)$;

(iii) $\mathfrak{U}(x_1, x_2, \dots, x_n; \theta) \leq (n-1)\alpha(x_1, x_2, \dots, x_2; \theta)\mathfrak{U}(x_2, \dots, (x_2)_{n-1}, x_1; \theta)$;

(iv) $\mathfrak{U}(x_1, x_2, \dots, x_n; \theta) \leq$
$\alpha(x_1, x_2, \dots, x_n; \theta) \sum_{i=1}^{n} \{\alpha(x_i, x_i, \dots, (x_i)_{n-1}, a; \theta)\mathfrak{U}(a, a, \dots, (a)_{n-1}, x_i; \theta)\}$.

Proof. Let $x_1, x_2, ..., x_n, a \in X$; then we get

(i) $\mathfrak{U}(x_1, x_2, ..., x_n; \theta)$

$$\begin{aligned}&\overset{[\text{by } (\mathfrak{U}2)]}{\leq} \alpha(x_1, x_2, ..., x_n; \theta) \sum_{i=1}^{n} \mathfrak{U}(x_i, x_i, ..., (x_i)_{n-1}, x_1; \theta)\\ &= \alpha(x_1, x_2, ..., x_n; \theta) \sum_{i=2}^{n} \mathfrak{U}(x_i, x_i, ..., (x_i)_{n-1}, x_1; \theta)\\ &\overset{[\text{by Prop. 2.7}]}{\leq} \alpha(x_1, x_2, ..., x_n; \theta) \sum_{i=2}^{n} \{\alpha(x_i, x_i, ..., (x_i)_{n-1}, x_1; \theta)\mathfrak{U}(x_1, x_1, ..., (x_1)_{n-1}, x_i; \theta)\}\end{aligned}$$

(ii) $\mathfrak{U}(x_1, x_2, ..., x_2; \theta)$

$$\overset{[\text{by } (\mathfrak{U}2)]}{\leq} \alpha(x_1, x_2, ..., x_2; \theta)[\mathfrak{U}(x_1, ..., (x_1)_{n-1}, x_2; \theta) + \sum_{\text{(n-1) times}} \mathfrak{U}(x_2, x_2, ..., x_2; \theta)]$$

$$= \alpha(x_1, x_2, ..., x_2; \theta)\mathfrak{U}(x_1, ..., (x_1)_{n-1}, x_2; \theta).$$

(iii) $\mathfrak{U}(x_1, x_2, ..., x_2; \theta)$

$$\overset{[\text{by } (\mathfrak{U}2)]}{\leq} \alpha(x_1, x_2, ..., x_2; \theta)[\mathfrak{U}(x_1, ..., x_1, x_1; \theta) + \sum_{\text{(n-1) times}} \mathfrak{U}(x_2, x_2, ..., (x_2)_{n-1}, x_1; \theta)]$$

$$= (n-1)\alpha(x_1, x_2, ..., x_2; \theta)\mathfrak{U}(x_2, ..., (x_2)_{n-1}, x_1; \theta).$$

(iv) $\mathfrak{U}(x_1, x_2, ..., x_n; \theta)$

$$\overset{[\text{by } (\mathfrak{U}2)]}{\leq} \alpha(x_1, x_2, ..., x_n; \theta) \sum_{i=1}^{n} \mathfrak{U}(x_i, x_i, ..., (x_i)_{n-1}, a; \theta)$$

$$\overset{[\text{by Prop. 2.7}]}{=} \alpha(x_1, x_2, ..., x_n; \theta) \sum_{i=1}^{n} \{\alpha(x_i, x_i, ..., (x_i)_{n-1}, a; \theta)\mathfrak{U}(a, a, ..., (a)_{n-1}, x_i; \theta)\}$$

for all $\theta \in (0, \infty)$. □

Proposition 2.11. *In an $\mathfrak{U}$-metric space $(X, \mathfrak{U})$, for any $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, a, b \in X$ and for any $\theta > 0$ one has*

(i) $|\mathfrak{U}(x_1, x_2, \dots, x_{n-1}, a; \theta) - \mathfrak{U}(x_1, x_2, \dots, x_{n-1}, b; \theta)| \leq [\alpha(x_1, x_2, \dots, x_{n-1}, a; \theta) + \alpha(x_1, x_2, \dots, x_{n-1}, b; \theta)] \sum_{i=1}^{n-1} \{\alpha(x_i, x_i, \dots, (x_i)_{n-1}, a; \theta)\alpha(x_i, x_i, \dots, (x_i)_{n-1}, b; \theta) \times [\mathfrak{U}(a, a, \dots, (a)_{n-1}, x_i; \theta) + \mathfrak{U}(b, b, \dots, (b)_{n-1}, x_i; \theta)]\}$;

(ii) $|\mathfrak{U}(x_1, x_2, \dots, x_{n-1}, a; \theta) - \mathfrak{U}(y_1, y_2, \dots, y_{n-1}, a; \theta)| \leq [\alpha(x_1, x_2, \dots, x_{n-1}, a; \theta) + \alpha(y_1, y_2, \dots, y_{n-1}, a; \theta)] \sum_{i=1}^{n-1} \{\alpha(x_i, x_i, \dots, (x_i)_{n-1}, a; \theta)\alpha(y_i, y_i, \dots, (y_i)_{n-1}, a; \theta) [\mathfrak{U}(a, a, \dots, (a)_{n-1}, x_i; \theta) + \mathfrak{U}(a, a, \dots, (a)_{n-1}, y_i; \theta)]\}$;

(iii) $|\mathfrak{U}(x_1, x_2, \dots, x_{n-1}, x_n; \theta) - \mathfrak{U}(y_1, y_2, \dots, y_{n-1}, y_n; \theta)| \leq \alpha(x_1, x_2, \dots, x_{n-1}, x_n; \theta) \times \alpha(y_1, y_2, \dots, y_{n-1}, y_n; \theta) \sum_{i=1}^{n} [\mathfrak{U}(x_i, x_i, \dots, (x_i)_{n-1}, a; \theta) + \mathfrak{U}(y_i, y_i, \dots, (y_i)_{n-1}, a; \theta)]$.

Proof. (i) Let $x_1, x_2, \dots, x_{n-1}, a, b \in X$ and $\theta \in (0, \infty)$. Then by Proposition 2.10 (iv) we have

$$|\mathfrak{U}(x_1, x_2, \dots, x_{n-1}, a; \theta) - \mathfrak{U}(x_1, x_2, \dots, x_{n-1}, b; \theta)|$$
$$\leq \mathfrak{U}(x_1, x_2, \dots, x_{n-1}, a; \theta) + \mathfrak{U}(x_1, x_2, \dots, x_{n-1}, b; \theta)$$
$$\leq \alpha(x_1, x_2, \dots, x_{n-1}, a; \theta) \sum_{i=1}^{n-1} \{\alpha(x_i, x_i, \dots, (x_i)_{n-1}, a; \theta)\mathfrak{U}(a, a, \dots, (a)_{n-1}, x_i; \theta)\}$$
$$+ \alpha(x_1, x_2, \dots, x_{n-1}, b; \theta) \sum_{i=1}^{n-1} \{\alpha(x_i, x_i, \dots, (x_i)_{n-1}, b; \theta)\mathfrak{U}(b, b, \dots, (b)_{n-1}, x_i; \theta)\}$$
$$\leq [\alpha(x_1, x_2, \dots, x_{n-1}, a; \theta) + \alpha(x_1, x_2, \dots, x_{n-1}, b; \theta)] \sum_{i=1}^{n-1} \{\alpha(x_i, x_i, \dots, (x_i)_{n-1}, a; \theta)$$
$$\times \alpha(x_i, x_i, \dots, (x_i)_{n-1}, b; \theta)[\mathfrak{U}(a, a, \dots, (a)_{n-1}, x_i; \theta) + \mathfrak{U}(b, b, \dots, (b)_{n-1}, x_i; \theta)]\}.$$

(ii) Let $x_1, x_2, \dots, x_{n-1}, y_1, y_2, \dots, y_{n-1}, a \in X$ and $\theta \in (0, \infty)$. Then by Proposition 2.10 (iv) we have

$$\begin{aligned}
&|\mathfrak{U}(x_1,x_2,\dots,x_{n-1},a;\theta)-\mathfrak{U}(y_1,y_2,\dots,y_{n-1},a;\theta)|\\
\leq&\mathfrak{U}(x_1,x_2,\dots,x_{n-1},a;\theta)+\mathfrak{U}(y_1,y_2,\dots,y_{n-1},a;\theta)\\
\leq&\alpha(x_1,x_2,\dots,x_{n-1},a;\theta)\sum_{i=1}^{n-1}\{\alpha(x_i,x_i,\dots,(x_i)_{n-1},a;\theta)\mathfrak{U}(a,a,\dots,(a)_{n-1},x_i;\theta)\}\\
&+\alpha(y_1,y_2,\dots,y_{n-1},a;\theta)\sum_{i=1}^{n-1}\{\alpha(y_i,y_i,\dots,(y_i)_{n-1},a;\theta)\mathfrak{U}(a,a,\dots,(a)_{n-1},y_i;\theta)\}\\
\leq&[\alpha(x_1,x_2,\dots,x_{n-1},a;\theta)+\alpha(y_1,y_2,\dots,y_{n-1},a;\theta)]\sum_{i=1}^{n-1}\{\alpha(x_i,x_i,\dots,(x_i)_{n-1},a;\theta)\\
&\times\alpha(y_i,y_i,\dots,(y_i)_{n-1},a;\theta)[\mathfrak{U}(a,a,\dots,(a)_{n-1},x_i;\theta)+\mathfrak{U}(a,a,\dots,(a)_{n-1},y_i;\theta)]\}.
\end{aligned}$$

(iii) Let $x_1,x_2,\dots,x_{n-1},x_n,y_1,y_2,\dots,y_{n-1},y_n,a,b\in X$ and $\theta\in(0,\infty)$.

$$\begin{aligned}
&|\mathfrak{U}(x_1,x_2,\dots,x_{n-1},x_n;\theta)-\mathfrak{U}(y_1,y_2,\dots,y_{n-1},y_n;\theta)|\\
\leq&\mathfrak{U}(x_1,x_2,\dots,x_{n-1},x_n;\theta)+\mathfrak{U}(y_1,y_2,\dots,y_{n-1},y_n;\theta)\\
\leq&\alpha(x_1,x_2,\dots,x_{n-1},x_n;\theta)\sum_{i=1}^{n}\mathfrak{U}(x_i,x_i,\dots,(x_i)_{n-1},a;\theta)\\
&+\alpha(y_1,y_2,\dots,y_{n-1},y_n;\theta)\sum_{i=1}^{n}\mathfrak{U}(y_i,y_i,\dots,(y_i)_{n-1},a;\theta)\\
\leq&\alpha(x_1,x_2,\dots,x_{n-1},x_n;\theta)\alpha(y_1,y_2,\dots,y_{n-1},y_n;\theta)\\
&\times\sum_{i=1}^{n}[\mathfrak{U}(x_i,x_i,\dots,(x_i)_{n-1},a;\theta)+\mathfrak{U}(y_i,y_i,\dots,(y_i)_{n-1},a;\theta)]
\end{aligned}$$

□

3. Topological constructions

Now we are in a state to study some topological properties of $\mathfrak{U}$-metric spaces.

Definition 3.1. In an $\mathfrak{U}$-metric space $(X,\mathfrak{U})$, the open ball and the closed ball with center at $x_0\in X$ and radius $r>0$ are defined by

$$S_r^{\mathfrak{U}}(x_0)=\{y\in X:\sup_{\theta\in(0,\infty)}\mathfrak{U}(y,y,\dots,(y)_{n-1},x_0;\theta)<r\};$$

$$S_r^{\mathfrak{U}}[x_0]=\{y\in X:\sup_{\theta\in(0,\infty)}\mathfrak{U}(y,y,\dots,(y)_{n-1},x_0;\theta)\leq r\}.$$

Proposition 3.2. *Let $(X,\mathfrak{U})$ be a $\mathfrak{U}$-metric space. The collection $\tau_{\mathfrak{U}}=\{\emptyset\}\cup\{P(\neq\emptyset)\subset X:$ for any $x\in P$ there exists $r>0$ such that $S_r^{\mathfrak{U}}(x)\subset P\}$ forms a topology on X.*

Definition 3.3. Let $(X, \mathfrak{U})$ be a $\mathfrak{U}$-metric space. The sets that belong to $\tau_{\mathfrak{U}}$ are called $\mathfrak{U}$-open sets. A subset B of X is said to be $\mathfrak{U}$-closed if there exists $V \in \tau$ such that $B = V^c$.

Proposition 3.4. *Let V be a $\mathfrak{U}$-open set containing x_0 in an $\mathfrak{U}$-metric space $(X, \mathfrak{U})$ and $\{x_k\} \subset X$ be such that $x_k \to x_0$ as $k \to \infty$. Then $x_k \in V$ for all $k \geq N$.*

Proof. Since $x_0{\in}V$ and V is $\mathfrak{U}$-open, there exists $r{>}0$ such that $S_r^{\mathfrak{U}}(x_0){\subset}V$. Now let us take some $0{<}r'{<}r$. Then there exists $N{\in}\mathbb{N}$ such that $\mathfrak{U}(x_k, x_k, \dots, (x_k)_{n-1}, x_0; \theta){<}r'$ for all $k{\geq}N$ and for all $\theta{\in}(0,\infty)$, that is $\sup_{\theta\in(0,\infty)} \mathfrak{U}(x_k, x_k, \dots, (x_k)_{n-1}, x_0; \theta){\leq}r'{<}r$ for all $k{\geq}N$. Thus, for any $k{\geq}N$, $x_k{\in}S_r^{\mathfrak{U}}(x_0){\subset}V$. □

Proposition 3.5. *Let $(X, \mathfrak{U})$ be a $\mathfrak{U}$-metric space and $F(\neq \emptyset) \subset X$ be $\mathfrak{U}$-closed. If there exists a sequence $\{x_k\} \subset F$ such that $x_k \to z$ as $k \to \infty$ then $z \in F$.*

Proof. If possible, let $z \notin F$. Then $z \in F^c$. Since F^c is $\mathfrak{U}$-open and $x_k \to z$ as $k \to \infty$ then by the previous theorem it follows that there exists $N \in \mathbb{N}$ such that $x_k \in F^c$ for all $k \geq N$, a contradiction. Therefore $z \in F$. □

Definition 3.6. The diameter $diam(F)$ of a subset $F \subset X$, X is a $\mathfrak{U}$-metric space, is defined by $diam(F) = \sup_{x,y\in X,\theta\in(0,\infty)} \mathfrak{U}(x, x, \dots, (x)_{n-1}, y; \theta)$.

Next we prove a Cantor's Intersection-like theorem in the context of $\mathfrak{U}$-metric spaces.

Theorem 3.7. *Let $(X, \mathfrak{U})$ be a complete $\mathfrak{U}$-metric space and $\{F_k\}$ be a decreasing sequence of nonempty $\mathfrak{U}$-closed sets such that $diam(F_k)$ tends to zero as k tends to infinity. Then the intersection $\bigcap_{k=1}^{\infty} F_k$ contains exactly one point.*

Proof. Let $x_k \in F_k$ for all $k \in \mathbb{N}$ be arbitrary. Since $\{F_k\}$ is decreasing, we have $\{x_k, x_{k+1}, ...\} \subset F_k$ for any $k \geq 1$. Now for any $k, m \in \mathbb{N}$ with $k, m \geq p$ we get

$$\sup_{\theta>0} \mathfrak{U}(x_k, x_k, \dots, (x_k)_{n-1}, x_m; \theta) \leq diam(F_p),\ p \geq 1.$$

Let $\epsilon > 0$ be given. Since $diam(F_k) \to 0$ as $k \to \infty$ then there exists some $q \in \mathbb{N}$ such that $diam(F_q) < \epsilon$. From this it follows that

$$\mathfrak{U}(x_k, x_k, \dots, (x_k)_{n-1}, x_m; \theta) \leq diam(F_q) < \epsilon, \text{ for all } k.m \geq q \text{ and } \theta > 0.$$

So $\{x_k\}$ is a Cauchy sequence in X and therefore by completeness of X there exists $z \in X$ such that $x_k \to z$ as $n \to \infty$. Therefore by Proposition 3.5 it follows that $z \in F_n$ for all $n \in \mathbb{N}$.

Now let $y \in \bigcap_{k=1}^{\infty} F_k$. Then $\mathfrak{U}(y, y, \dots, (y)_{n-1}, z; \theta) \leq diam(F_k)$ for all $k \geq 1$ and for any $\theta \in (0,\infty)$. So by taking $k \to \infty$ we get $\mathfrak{U}(y, y, \dots, (y)_{n-1}, z; \theta) = 0$ for all $\theta \in (0,\infty)$, which implies that $y = z$. Hence $\bigcap_{k=1}^{\infty} F_k = \{z\}$ and this completes the proof. □

4. Banach Contraction Principle in $\mathfrak{U}$-metric spaces

Now we present the celebrated Banach Contraction Principle in the context of $\mathfrak{U}$-metric space.

THEOREM 4.1. *Let $(X, \mathfrak{U})$ be a complete symmetric $\mathfrak{U}$-metric space and $T : X \to X$ be a mapping satisfying*

$$\mathfrak{U}(Tx, Tx, \dots, (Tx)_{n-1}, Ty; \theta) \leq r\mathfrak{U}(x, x, \dots, (x)_{n-1}, y; \theta),$$

for all $x, y \in X$ and for all $\theta > 0$, where $r \in (0, 1)$ is such that

$$\lim_{k,m\to\infty} \alpha(T^k x, \dots, (T^k x)_{n-1}, T^m x; \theta) < \frac{1}{r}$$

for all $x \in X$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Let us consider the Picard iterating sequence $\{x_m\}$ where $x_m = T^m x_0$ for all $m \in \mathbb{N}$. Then for all $m \in \mathbb{N}$ we have

$$\mathfrak{U}(x_m, x_m, \dots, (x_m)_{n-1}, x_{m+1}; \theta) = \mathfrak{U}(Tx_{m-1}, Tx_{m-1}, \dots, (Tx_{m-1})_{n-1}, Tx_m; \theta)$$
$$\leq r\mathfrak{U}(x_{m-1}, x_{m-1}, \dots, (x_{m-1})_{n-1}, x_m; \theta) \cdots \leq r^m \mathfrak{U}(x_0, x_0, \dots, (x_0)_{n-1}, x_1; \theta).$$

Now for any $1 \leq k < m$ we have,

$$\mathfrak{U}(x_k, ..., (x_k)_{n-1}, x_m; \theta)$$
$$\leq \alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)[(n-1)\mathfrak{U}(x_k, ..., (x_k)_{n-1}, x_{k+1}; \theta) + \mathfrak{U}(x_m, ..., (x_m)_{n-1}, x_{k+1}; \theta)]$$
$$= \alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)[(n-1)\mathfrak{U}(x_k, ..., (x_k)_{n-1}, x_{k+1}; \theta) + \mathfrak{U}(x_{k+1}, ..., (x_{k+1})_{n-1}, x_m; \theta)]$$
$$\leq (n-1)\alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\mathfrak{U}(x_{k+1}, ..., (x_{k+1})_{n-1}, x_k; \theta) +$$
$$\alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\alpha(x_{k+1}, ..., (x_{k+1})_{n-1}, x_m; \theta)[(n-1)\mathfrak{U}(x_{k+1}, ..., (x_{k+1})_{n-1}, x_{k+2}; \theta)$$
$$+ \mathfrak{U}(x_m, ..., (x_m)_{n-1}, x_{k+2}; \theta)]$$
$$= (n-1)\alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\mathfrak{U}(x_{k+1}, ..., (x_{k+1})_{n-1}, x_k; \theta)$$
$$+ (n-1)\alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\alpha(x_{k+1}, ..., (x_{k+1})_{n-1}, x_m; \theta)\mathfrak{U}(x_{k+2}, ..., (x_{k+2})_{n-1}, x_{k+1}; \theta)$$
$$+ \alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\alpha(x_{k+1}, ..., (x_{k+1})_{n-1}, x_m; \theta)\mathfrak{U}(x_{k+2}, ..., (x_{k+2})_{n-1}, x_m; \theta)$$
$$\dots$$
$$\leq (n-1)[\alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\mathfrak{U}(x_{k+1}, ..., (x_{k+1})_{n-1}, x_k; \theta)$$
$$+ \alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\alpha(x_{k+1}, ..., (x_{k+1})_{n-1}, x_m; \theta)\mathfrak{U}(x_{k+2}, ..., (x_{k+2})_{n-1}, x_{k+1}; \theta)$$
$$+ ... + \alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\alpha(x_{k+1}, ..., (x_{k+1})_{n-1}, x_m; \theta)...$$
$$\alpha(x_{m-2}, ..., (x_{m-2})_{n-1}, x_m; \theta)\mathfrak{U}(x_{m-1}, ..., (x_{m-1})_{n-1}, x_{m-2}; \theta)]$$
$$+ \alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\alpha(x_{k+1}, ..., (x_{k+1})_{n-1}, x_m; \theta)...$$
$$\alpha(x_{m-2}, ..., (x_{m-2})_{n-1}, x_m; \theta)\mathfrak{U}(x_{m-1}, ..., (x_{m-1})_{n-1}, x_m; \theta)$$
$$\leq (n-1)[\alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\mathfrak{U}(x_{k+1}, ..., (x_{k+1})_{n-1}, x_k; \theta)$$
$$+ \alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\alpha(x_{k+1}, ..., (x_{k+1})_{n-1}, x_m; \theta)\mathfrak{U}(x_{k+2}, ..., (x_{k+2})_{n-1}, x_{k+1}; \theta)$$
$$+ ... + \alpha(x_k, ..., (x_k)_{n-1}, x_m; \theta)\alpha(x_{k+1}, ..., (x_{k+1})_{n-1}, x_m; \theta)...$$
$$\alpha(x_{m-2}, ..., (x_{m-2})_{n-1}, x_m; \theta)\mathfrak{U}(x_{m-1}, ..., (x_{m-1})_{n-1}, x_{m-2}; \theta) +$$

$\alpha(x_k,...,(x_k)_{n-1},x_m;\theta)\alpha(x_{k+1},...,(x_{k+1})_{n-1},x_m;\theta)...$
$\alpha(x_{m-2},...,(x_{m-2})_{n-1},x_m;\theta)\alpha(x_{m-1},...,(x_{m-1})_{n-1},x_m;\theta)\mathfrak{U}(x_m,...,(x_m)_{n-1},x_{m-1};\theta)]$
$\leq(n-1)[\alpha(x_k,...,(x_k)_{n-1},x_m;\theta)r^k+\alpha(x_k,...,(x_k)_{n-1},x_m;\theta)\alpha(x_{k+1},...,(x_{k+1})_{n-1},x_m;\theta)r^{k+1}$
$+...+\alpha(x_k,...,(x_k)_{n-1},x_m;\theta)\alpha(x_{k+1},...,(x_{k+1})_{n-1},x_m;\theta)...$
$\alpha(x_{m-2},...,(x_{m-2})_{n-1},x_m;\theta)\alpha(x_{m-1},...,(x_{m-1})_{n-1},x_m;\theta)r^{m-1}]\mathfrak{U}(x_1,...,(x_1)_{n-1},x_0;\theta).$

Let us denote $P_j^{(m)}(x_0;\theta) = (n-1)r^j \prod_{i=1}^{j} \alpha(x_i,\dots,(x_i)_{n-1},x_m;\theta)$ for all $j \in \mathbb{N}$ and for all $\theta > 0$. Then from the previous we get

$$\mathfrak{U}(x_k,\dots,(x_k)_{n-1},x_m;\theta) \leq \left(\sum_{j=k}^{m-1} P_j^{(m)}(x_0;\theta)\right)\mathfrak{U}(x_1,\dots,(x_1)_{n-1},x_0;\theta), \tag{4}$$

where $1 \leq k < m$. Now since $\lim_{k,m\to\infty} \alpha(T^k x_0,\dots,(T^k x_0)_{n-1},T^m x_0;\theta) < \frac{1}{r}$, by ratio test we see that the series $\lim_{m\to\infty}\sum_{t=1}^{\infty} P_t^{(m)}(x_0;\theta)$ is convergent and thus $\lim_{k\to\infty}\sum_{j=k}^{m-1} P_j^{(m)}(x_0;\theta) = 0$. So (4) implies that x_m is a Cauchy sequence in X and by the completeness of X we get some $z \in X$ such that $x_m \to z$ as $m \to \infty$. Hence, for any $\theta > 0$, $\mathfrak{U}(x_{m+1},x_{m+1},\dots,(x_{m+1})_{n-1},Tz;\theta) \leq r\mathfrak{U}(x_m,x_m,\dots,(x_m)_{n-1},z;\theta) \to 0$ as $m \to \infty$. Thus by Lemma 2.8 we have $Tz = z$ and T possesses a fixed point in X.

Now if u and v are two fixed points of T then we have

$$\mathfrak{U}(u,u,\dots,(u)_{n-1},v;\theta) = \mathfrak{U}(Tu,Tu,\dots,(Tu)_{n-1},Tv;\theta) \leq r\,\mathfrak{U}(u,u,\dots,(u)_{n-1},v;\theta).$$

Since $0 < r < 1$ we get $\mathfrak{U}(u,u,\dots,(u)_{n-1},v;\theta) = 0$ and hence $u = v$, which shows that the fixed point of T is unique in X. □

Example 4.2. Let $X = C^0(0,\infty)$, the space of all real valued continuous mappings with domain $(0,\infty)$ and let us define $\mathfrak{U} : X^3 \times (0,\infty) \to [0,\infty)$ by

$$\mathfrak{U}(f_1,f_2,f_3;\theta) = \sum_{i=1}^{2}\sum_{i<j}|f_i(\theta)-f_j(\theta)|^2 + (\sum_{i=1}^{2}\sum_{i<j}|f_i(\theta)-f_j(\theta)|^2)^2$$

for all $f_1,f_2,f_3 \in X$ and for all $\theta \in (0,\infty)$. Then $\mathfrak{U}$ is a $\mathfrak{U}$-metric with $\alpha(f_1,f_2,f_3;\theta) = 3(1+\sum_{i=1}^{2}\sum_{i<j}|f_i(\theta)-f_j(\theta)|^2)$ for all $f_1,f_2,f_3 \in X$ and for all $\theta \in (0,\infty)$.

Let us take a fixed element $f_0 \in X$ and define $T : X \to X$ by $Tf = \frac{f}{2} + f_0$ for all $f \in X$. Then $\mathfrak{U}(Tf,Tf,Tg;\theta) \leq \frac{1}{4}\mathfrak{U}(f,f,g;\theta)$ for any $f,g \in X$ and for all $\theta > 0$. Also for any $\theta \in (0,\infty)$, $\alpha(T^k f,T^k f,T^m f;\theta) = 3[1+2(f(\theta)-2f_0(\theta))^2\{\frac{1}{2^k}-\frac{1}{2^m}\}^2] \to 3$ as $k,m \to \infty$. Therefore all the conditions of Theorem 4.1 are satisfied and therefore T has a unique fixed point in X which is $2f_0$.

5. Application of fixed point theorem to the stability of fixed point equation

In 1940, Ulam has raised an open problem concerning the stability of homomorphisms in algebra. Hyers was the first who gave an answer to the stability of functional equations in the context of Banach spaces (see [6]). Very recently, Roy et al [11,12] have studied Ulam-Hyers stability of fixed point problems under different generalized metric structures. Here we discuss Ulam-Hyers stability of fixed point problems in an $\mathfrak{U}$-metric space.

Let $(X, \mathfrak{U})$ be a $\mathfrak{U}$-metric space and $T : X \to X$ be a mapping. Let us consider the fixed point equation

$$Tx = x \tag{5}$$

and the inequality

$$\mathfrak{U}(y, y, \dots, (y)_{n-1}, Ty; \theta) < \epsilon \text{ for all } \theta > 0 \text{ and for any } \epsilon > 0. \tag{6}$$

Definition 5.1. The fixed point problem (5) is said to be Ulam-Hyers stable if there exists a mapping $\zeta : X^2 \times (0, \infty) \to [0, \infty)$ such that for each $\epsilon > 0$ and an ϵ-solution (a solution of (6)) $v \in X$ there exists a solution $u \in X$ of the fixed point equation (5) such that for all $\theta \in (0, \infty)$, $\mathfrak{U}(v, v, \dots, (v)_{n-1}, u; \theta) < \zeta(v, u; \theta)\epsilon$.

Theorem 5.2. *Let $(X, \mathfrak{U})$ be a complete symmetric $\mathfrak{U}$-metric space and $T : X \to X$ be a mapping satisfying all the conditions of Theorem 4.1 with the Lipschitz constant r such that $\alpha(x, x, \dots, (x)_{n-1}, y; \theta) < \frac{1}{r}$ for all $x, y \in X$ and $\theta \in (0, \infty)$. Then the fixed point equation of T is Ulam-Hyers stable.*

Proof. From Theorem 4.1, we see that T has a unique fixed point u in X, that is the fixed point equation (6) of T has a unique solution. Let $\epsilon > 0$ be arbitrary and v be an ϵ-solution of T. Then

$$\begin{aligned}
&\mathfrak{U}(v, v, ..., (v)_{n-1}, u; \theta)\\
\leq &\alpha(v, v, ..., (v)_{n-1}, u; \theta)[(n-1)\mathfrak{U}(v, v, ..., (v)_{n-1}, Tv; \theta) + \mathfrak{U}(u, u, ..., (u)_{n-1}, Tv; \theta)]\\
= &\alpha(v, v, ..., (v)_{n-1}, u; \theta)[(n-1)\mathfrak{U}(v, v, ..., (v)_{n-1}, Tv; \theta) + \mathfrak{U}(Tv, Tv, ..., (Tv)_{n-1}, u; \theta)]\\
= &\alpha(v, v, ..., (v)_{n-1}, u; \theta)[(n-1)\mathfrak{U}(v, v, ..., (v)_{n-1}, Tv; \theta) + \mathfrak{U}(Tv, Tv, ..., (Tv)_{n-1}, Tu; \theta)]\\
\leq &\alpha(v, v, ..., (v)_{n-1}, u; \theta)[(n-1)\mathfrak{U}(v, v, ..., (v)_{n-1}, Tv; \theta) + r\,\mathfrak{U}(v, v, ..., (v)_{n-1}, u; \theta)].
\end{aligned}$$

This implies $\mathfrak{U}(v, v, \dots, (v)_{n-1}, u; \theta) \leq \frac{(n-1)\alpha(v,v,\dots,(v)_{n-1},u;\theta)}{1-r\alpha(v,v,\dots,(v)_{n-1},u;\theta)}\mathfrak{U}(v, v, \dots, (v)_{n-1}, Tv; \theta)$ $< \zeta(v, u; \theta)\epsilon$, where $\zeta(v, u; \theta) = \frac{(n-1)\alpha(v,v,\dots,(v)_{n-1},u;\theta)}{1-r\alpha(v,v,\dots,(v)_{n-1},u;\theta)}$ for all $\theta > 0$. Therefore the fixed point equation of T is Ulam-Hyers stable. □

Acknowledgement. Kushal Roy acknowledges financial support awarded by the Council of Scientific and Industrial Research, New Delhi, India, through research fellowship for carrying out research work leading to the preparation of this manuscript.

References

[1] M. Abbas, B. Ali, Y. I Suleiman, *Generalized coupled common fixed point results in partially ordered A-metric spaces*, Fixed Point Theory Appl., (2015), 24 pages.

[2] I.A. Bakhtin, *The contraction principle in quasimetric spaces*, Func. An., Ulianowsk, Gos. Ped. Ins., **30** (1989), 26–37.

[3] D. Dey, K. Roy, M. Saha, *On generalized contraction principles over S-metric spaces with application to homotopy*, J. New Theory, **31** (2020), 95–103.

[4] N. Hussain, S. Khaleghizadeh, P. Salimi, A.A.N. Abdou, *A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces*, Abstr. Appl. Anal., (2014), Article ID 690139, 16 pages.

[5] R. Jain, R.D. Daheriya, M. Ughade, *Fixed point, coincidence point and common fixed point theorems under various expansive conditions in parametric metric spaces and parametric b-metric spaces*, Gazi University Journal of Science, **29(1)** (2016), 95–107.

[6] S.M. Jung, *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis*, Springer Optim. Appl., **48** (2011).

[7] T. Kamran, M. Samreen and Q.U. Ain, *A generalization of b-metric space and some fixed point theorems*, Mathematics, **5(19)** doi:10.3390/math5020019, (2017).

[8] N. Mlaiki, *Extended S_b-metric spaces*, J. Math. Anal., **9(1)** (2018), 124–135.

[9] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289–297.

[10] N. Priyobarta, Y. Rohen, and S. Radenović, *Fixed point theorems on parametric A-metric space*, Amer. J. Appl. Math. Stat., **6(1)** (2018), 1–5.

[11] K. Roy, M. Saha, *Fixed points of mappings over a locally convex topological vector space and Ulam-Hyers stability of fixed point problems*, Novi Sad J. Math., **50(1)** (2020), 99–112.

[12] K. Roy, M. Saha, *On fixed points of Ćirić-type contractive mappings over a C^*-algebra valued metric space and Hyers-Ulam stability of fixed point problems*, J. Adv. Math. Stud., **12(3)** (2019), 350–363.

[13] S. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesn., **64(3)** (2012), 258–266.

[14] N. Souayah, N. Mlaiki, *A fixed point theorem in S_b-metric spaces*, J. Math. Computer Sci., **16** (2016), 131–139.

[15] N. Tas, N.Y. Özgür, *On parametric S-metric spaces and fixed-point type theorems for expansive mappings*, J. Math., (2016), Article ID 4746732, 6 pages.

(received 25.03.2020; in revised form 19.12.2020; available online 29.07.2021)

Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India

E-mail: kushal.roy93@gmail.com

Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India

E-mail: mantusaha.bu@gmail.com

Sudpur High School(H.S), Purba Bardhaman-713150, West Bengal, India

E-mail: debashismathdey@gmail.com