MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
74, 2 (2022), 110–118
June 2022

research paper
оригинални научни рад

$\mathcal{I}^*$-α CONVERGENCE AND $\mathcal{I}^*$-EXHAUSTIVENESS OF SEQUENCES OF METRIC FUNCTIONS

Argha Ghosh

Abstract. By a metric function, we mean a function from a metric space (X, d) into a metric space (Y, ρ). We introduce and study the notions of $\mathcal{I}^*$-α convergence and $\mathcal{I}^*$-exhaustiveness of sequences of metric functions, and we establish an inter-relationship between these two concepts. Moreover, we establish some relationship between our concepts with some well-established concepts such as $\mathcal{I}$-α convergence and $\mathcal{I}$-exhaustiveness of sequences of metric functions.

1. Introduction

The idea of statistical convergence was introduced in [9] and [17] independently. This notion of convergence is more than a generalization of the usual notion of convergence of sequences and has many applications in modern mathematics [3, 10, 15].

Further, the notion of statistical convergence was extended to the concepts of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of sequences on metric spaces by Kostyrko, Šalát and Wilczyński [13]. Current development of these notions can be seen from the book edited by Dutta and Rhoades [8] and from [6, 7, 14], etc.

On the other hand, from the last century, some articles [4, 12, 18] can be seen in the notion of α-convergence (or continuous convergence) of sequences of real-valued functions. Further, Gregoriades and Papanastassiou [11] introduced the notion of exhaustiveness, and established a relationship between α-convergence and exhaustiveness of sequences of functions on metric spaces.

In 2010, Papachristodoulos, Papanastassiou and Wilczyński [16] introduced and studied the notions of $\mathcal{I}$-α convergence and $\mathcal{I}$-exhaustiveness. Later in 2012, Caserta and Kočinac [5] extended the notions of α-convergence and exhaustiveness to statistical α-convergence and statistical exhaustiveness respectively and established some relationship between these concepts. The notions of $\mathcal{I}$-α convergence and $\mathcal{I}$-exhaustive-

2020 Mathematics Subject Classification: 40A35, 26A03.
Keywords and phrases: $\mathcal{I}^*$-α convergence; $\mathcal{I}^*$-exhaustiveness; sequence of metric functions; ideal convergence.

ness are generalizations of the notions of statistical α-convergence and statistical exhaustiveness, respectively.

The notion of $\mathcal{I}^*$-convergence strongly resemble to that of $\mathcal{I}$-convergence; however, it seems that there has been no studies to find out the $\mathcal{I}^*$ version of the notions of α-convergence and exhaustiveness. Thus, in Section 3, we introduce the concept of $\mathcal{I}^*$-α convergence, and we do a comparative study of $\mathcal{I}^*$-α convergence and $\mathcal{I}$-α convergence of sequences of metric functions. Moreover, we obtain a necessary and sufficient condition of $\mathcal{I}^*$-α convergence of sequences of metric functions for P-ideals.

In Section 4, we introduce the notion of $\mathcal{I}^*$-exhaustiveness and we do a comparative study between $\mathcal{I}$-exhaustiveness and $\mathcal{I}^*$-exhaustiveness, and between $\mathcal{I}^*$-α convergence and $\mathcal{I}^*$-exhaustiveness.

2. Preliminaries

In this section, we discuss some basic definitions and ideas which we need throughout the paper. At first, we recall some definitions and concepts related to the ideals and filters of a non-empty set.

Definition 2.1 ([13]). If X is a non-empty set, then a family $\mathcal{I} \subset 2^X$ is said to be an ideal of X if
(i) $\emptyset \in \mathcal{I}$, (ii) $A; B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (iii) $A \in \mathcal{I}$; $B \subset A$ implies $B \in \mathcal{I}$.
The ideal $\mathcal{I}$ is said to be a non-trivial ideal if $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$.

Definition 2.2 ([13]). If X is a non-empty set, then a family $\mathcal{F} \subset 2^X$ is said to be a filter of X if
(i) $\emptyset \notin \mathcal{F}$, (ii) $A; B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) $A \in \mathcal{F}$; $A \subset B$ implies $B \in \mathcal{F}$.

Clearly, if $\mathcal{I} \subset 2^X$ is a non-trivial ideal of X, then $\mathcal{F}(\mathcal{I}) = \{A \subset X \ : \ X \setminus A \in \mathcal{I}\}$ is a filter of X, called the filter associated with $\mathcal{I}$ or the dual filter with respect to $\mathcal{I}$.

A non-trivial ideal of $X(\neq \emptyset)$ is said to be admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

In the rest of the paper, we assume $\mathcal{I}$ as a non-trivial admissible ideal of $\mathbb{N}$ unless otherwise stated.

Definition 2.3 ([13]). A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X, d) is said to be $\mathcal{I}$-convergent to $a \in X$ if for every $\varepsilon > 0$: $\{n \in \mathbb{N} : d(x_n, a) \geq \epsilon\} \in \mathcal{I}$. In this case, we write $\mathcal{I} - \lim x_n = a$.

Definition 2.4 ([13]). A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X, d) is said to be $\mathcal{I}^*$-convergent to $a \in X$ if there exists a set $K \in \mathcal{F}(\mathcal{I})$ (that is, $\mathbb{N} \setminus K \in \mathcal{I}$), $K = \{k_1 < k_2 < \ldots < k_n < \ldots\}$ such that $\lim_{n\to\infty} d(x_{k_n}, a) = 0$. In this case, we write $\mathcal{I}^* - \lim x_n = a$.

Definition 2.5 ([16]). An admissible ideal $\mathcal{I}$ is said to be *good* if for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets such that $A_n \notin \mathcal{I}$ there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of pairwise disjoint sets such that $B_n \subset A_n$, $B_n \in \mathcal{I}$ and $\bigcup_{n=1}^{\infty} B_n \notin \mathcal{I}$.

Definition 2.6 ([3]). An ideal $\mathcal{I}$ is said to be a P-ideal if for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets from $\mathcal{I}$ there exists an $A^\infty \in \mathcal{I}$ such that $A_n \setminus A^\infty$ is finite for each $n \in \mathbb{N}$.

Definition 2.7 ([13]). An ideal $\mathcal{I}$ is said to satisfy the condition (AP) if for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets from $\mathcal{I}$ there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that the symmetric difference $A_n \Delta B_n$ is finite for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{I}$.

Theorem 2.8 ([3]). *An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is a P-ideal if and only if $\mathcal{I}$ satisfies (AP).*

Theorem 2.9 ([13]). *Let (X, d) be a metric space and $\mathcal{I}$ be a P-ideal. Then for an arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, $\mathcal{I}^* - \lim x_n = \mathcal{I} - \lim x_n$.*

Now we recall some definitions and ideas related to α-convergence and exhaustiveness. In the rest of the paper, we assume $X = (X, d)$ and $Y = (Y, \rho)$ are arbitrary metric spaces; given spaces X and Y, we write Y^X to denote the set of all functions from X into Y, unless otherwise stated.

Definition 2.10 ([11]). A sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is said to be α-convergent to $f \in Y^X$ if for every $x \in X$ and for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converging to x, the sequence $\{f_n(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$.

Theorem 2.11 ([2]). *A sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is α-convergent to $f \in Y^X$ at $x' \in X$ if and only if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x') > 0$ and $n' = n'(\varepsilon, x') \in \mathbb{N}$ such that $d(x, x') < \delta$ implies $\rho(f_n(x), f(x')) < \varepsilon$, for all $n \geq n'$.*

Definition 2.12 ([11]). A sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is said to be exhaustive at $x' \in X$ if for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, x') > 0$ and $n' = n'(\varepsilon, x') \in \mathbb{N}$ such that $d(x, x') < \delta$ implies $\rho(f_n(x), f_n(x')) < \varepsilon$, for all $n \geq n'$. The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is exhaustive on X if $\{f_n\}_{n\in\mathbb{N}}$ is exhaustive at each $x \in X$.

Definition 2.13 ([16]). A sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is said to be $\mathcal{I}$-α convergent to $f \in Y^X$ if for every $x \in X$ and for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X $\mathcal{I}$-converging to x, the sequence $\{f_n(x_n)\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to $f(x)$.

Definition 2.14 ([16]). A sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is said to be $\mathcal{I}$-exhaustive at $x' \in X$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x') > 0$ and $A \in \mathcal{I}$ such that $d(x, x') < \delta$ implies $\rho(f_n(x), f_n(x')) < \varepsilon$, for all $n \in \mathbb{N} \setminus A$.

We say, the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive on X if and only if $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive at each $x \in X$.

Theorem 2.15 ([16]). *Let $\mathcal{I}$ be a good ideal of $\mathbb{N}$. Then a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is $\mathcal{I}$-α convergent to $f \in Y^X$ at $x_0 \in X$ if and only if $\{f_n(x_0)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to $f(x_0)$ and $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive at x_0.*

3. $\mathcal{I}^*$-α convergence

In this section, we introduce and study the notion of $\mathcal{I}^*$-α convergence. The result of [2, Theorem 2.3] suggests the introduction of the following definition.

Definition 3.1. A sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is said to be $\mathcal{I}^*$-α convergent to $f \in Y^X$ if for every $x' \in X$ there exists $A = A(x') \in \mathcal{I}$ such that for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, x') > 0$ and $n' = n'(\varepsilon, x') \in \mathbb{N}$ such that $d(x, x') < \delta$ implies $\rho(f_n(x), f(x')) < \varepsilon$, for all $n \in \mathbb{N} \setminus A$ and $n \geq n'$.

Remark 3.2. From Theorem 2.11, we can say that the notion of α-convergence implies the notion of $\mathcal{I}^*$-α convergence for every admissible ideal $\mathcal{I}$, and the notion of $\mathcal{I}^*$-α convergence and α-convergence coincide for the ideal consisting of all finite subsets of $\mathbb{N}$.

Theorem 3.3. *$\mathcal{I}^*$-α convergence implies $\mathcal{I}$-α convergence of a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$.*

Proof. Let the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-α convergent to $f \in Y^X$. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X such that $\mathcal{I} - \lim x_n = x_0$. Now it is sufficient to show that $\mathcal{I} - \lim f_n(x_n) = f(x_0)$. Let $\varepsilon > 0$ be given. Since $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-α convergent to f, so there exists $A = A(x_0) \in \mathcal{I}$ such that for the given $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, x_0) > 0$ and $n_0 = n_0(\varepsilon, x_0) > 0$ such that $d(x, x_0) < \delta$ implies $\rho(f_n(x), f(x_0)) < \varepsilon$, for all $n \in \mathbb{N} \setminus A$ and $n \geq n_0$.

Again $\mathcal{I} - \lim x_n = x_0$ implies $\{n \in \mathbb{N} : d(x_n, x_0) \geq \delta\} \in \mathcal{I}$. Let $B = \{n \in \mathbb{N} : d(x_n, x_0) \geq \delta\}$. Then $\mathbb{N} \setminus B \in \mathcal{F}(\mathcal{I})$. Thus $(\mathbb{N} \setminus A) \cap (\mathbb{N} \setminus B) \in \mathcal{F}(\mathcal{I})$. Choose $n \in (\mathbb{N} \setminus A) \cap (\mathbb{N} \setminus B)$ so that $n \geq n_0$. Now $n \in (\mathbb{N} \setminus B)$ implies $d(x_n, x_0) < \delta$, and $n \in (\mathbb{N} \setminus A)$, $n \geq n_0$ together with $d(x_n, x_0) < \delta$ imply $\rho(f_n(x_n), f(x_0)) < \varepsilon$. Therefore, $\{n \in \mathbb{N} : \rho(f_n(x_n), f(x_0)) \geq \varepsilon\} \subset A \cup B \cup \{1, 2, \ldots, n_0\}$.

Since $A \cup B \cup \{1, 2, \ldots, n_0\} \in \mathcal{I}$, so $\{n \in \mathbb{N} : \rho(f_n(x_n), f(x_0)) \geq \varepsilon\} \in \mathcal{I}$. Hence $\mathcal{I} - \lim f_n(x_n) = f(x_0)$. □

Remark 3.4. The converse of Theorem 3.3 is not always true. To justify our claim, we cite the following example.

Example 3.5. Let $\mathbb{N} = \bigcup_{j=1}^{\infty} A_j$ be a decomposition of $\mathbb{N}$, where A_j's are pairwise disjoint infinite subsets of $\mathbb{N}$. Consider a class $\mathcal{E}$ consisting of all subsets A of $\mathbb{N}$ which intersect only finite numbers of A_j's. Then $\mathcal{E}$ is a non-trivial admissible ideal of $\mathbb{N}$. Let (X, d) be a metric space with an accumulation point $x_0 \in X$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\lim x_n = x_0$. Since $\mathcal{E}$ is an admissible ideal, so $\mathcal{E} - \lim x_n = x_0$. Put $d(x_n, x_0) = \varepsilon_n$. Then $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converges to 0.

Now, consider a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset X^X$ defined by $f_n(x) = x_j$, $\forall x \in X$ whenever $n \in A_j$. Also, we define a function $f \in X^X$ by $f(x) = x_0$, $\forall x \in X$. Let $\varepsilon > 0$ be given. Since $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converges to 0, so there exists $k \in \mathbb{N}$ such that $\varepsilon_n < \varepsilon$ for all $n \geq k$. Again $A(\varepsilon) = \{n \in \mathbb{N} : d(f_n(x_n), f(x_0)) \geq \varepsilon\} \subset A_1 \cup A_2 \cup \ldots \cup A_k$. Since $A_1 \cup A_2 \cup \ldots \cup A_k \in \mathcal{E}$, so $A(\varepsilon) \in \mathcal{E}$. Thus $\{f_n(x_n)\}_{n\in\mathbb{N}}$ is $\mathcal{E}$-convergent to $f(x_0)$. Hence $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{E}$-α convergent to f at $x_0 \in X$.

If possible, let $\{f_n\}_{n\in\mathbb{N}}$ be $\mathcal{E}^*$-α convergent to f at x_0. Then there exists $A = A(x_0) \in \mathcal{E}$ such that for all $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ and $n_0 \in \mathbb{N}$ such that $d(f_n(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$, $n \geq n_0$ and $n \in \mathbb{N} \setminus A$.

Now, since $A \in \mathcal{E}$, so there exists $m \in \mathbb{N}$ such that $A \subset A_1 \cup A_2 \cup \ldots \cup A_m$. Then $A_{m+1} \subset \mathbb{N} \setminus A$. Choose $\varepsilon' < \varepsilon_{m+1}$. Then there exists $\delta' > 0$ and $n_0' \in \mathbb{N}$ such that $d(f_n(x), f(x_0)) < \varepsilon'$ whenever $d(x, x_0) < \delta'$, $n \geq n_0'$ and $n \in \mathbb{N} \setminus A$.

Now, for $n \in A_{m+1}$, we have $d(f_n(x), f(x_0)) = d(x_{m+1}, x_0) = \varepsilon_{m+1} > \varepsilon'$ for all $x \in X$. Thus, there are infinitely many $n \in \mathbb{N} \setminus A$ for which $d(f_n(x), f(x_0)) > \varepsilon'$ whenever $d(x, x_0) < \delta'$, which is a contradiction. Hence $\{f_n\}_{n\in\mathbb{N}}$ is not $\mathcal{E}^*$-α convergent to f at x_0.

Now, we state a necessary and sufficient condition of $\mathcal{I}^*$-α convergence for the P-ideals.

THEOREM 3.6. *Let $\mathcal{I}$ be a P-ideal. Then a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is $\mathcal{I}^*$-α-convergent to $f \in Y^X$ at $x_0 \in X$ if and only if the following condition holds:*

$\forall \varepsilon > 0\ \exists\, \delta = \delta(x_0, \varepsilon) > 0\ \exists\, A = A(x_0, \varepsilon) \in \mathcal{I}$ *such that*

$$x \in B(x_0, \delta) = \{x \in X : d(x, x_0) < \delta\} \Rightarrow \rho(f_n(x), f(x_0)) < \varepsilon \text{ for all } n \in \mathbb{N} \setminus A.$$

Proof. Let the condition hold. Then for every $\varepsilon > 0$ there exist $\delta = \delta(x_0, \varepsilon) > 0$ and $A = A(x_0, \varepsilon) \in \mathcal{I}$ such that $\rho(f_n(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$ and $n \in \mathbb{N} \setminus A$.

Now consider $\varepsilon = \frac{1}{i} > 0$ for each $i \in \mathbb{N}$. Then there exist $\delta_i^{x_0} = \delta_i^{x_0}(\frac{1}{i}, x_0) > 0$ and $A_i(x_0) \in \mathcal{I}$ such that $\rho(f_n(x), f(x_0)) < \frac{1}{i}$ whenever $d(x, x_0) < \delta_i^{x_0}$ and $n \in \mathbb{N} \setminus A_i(x_0)$.

Since $\mathcal{I}$ is a P-ideal and $A_i(x_0) \in \mathcal{I}$ for all $i \in \mathbb{N}$, so there exists $A^\infty \in \mathcal{I}$ such that $A_i(x_0) \setminus A^\infty$ is finite for all $i \in \mathbb{N}$. Clearly, A^∞ depends only on $x_0 \in X$. Let $\varepsilon_0 > 0$ be given. Then there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon_0$. Let $A_k(x_0) \setminus A^\infty \subset \{1, 2, \ldots, n_0\}$. Then for all $n \geq n_0 + 1$ and $n \in A^\infty$, we have $n \in A_k(x_0)$. Thus $\rho(f_n(x), f(x_0)) < \frac{1}{k} < \varepsilon_0$ whenever $d(x, x_0) < \delta_k^{x_0}$, $n \geq n_0 + 1$ and $n \in A^\infty$. Hence $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-α convergent to f at $x_0 \in X$.

Conversely, let $\{f_n\}_{n\in\mathbb{N}}$ be $\mathcal{I}^*$-α convergent to f at $x_0 \in X$. Let $\varepsilon > 0$ be given. Since $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-α convergent to f at x_0, so there exists $A = A(x_0) \in \mathcal{I}$ such that for the given $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, x_0) > 0$ and $n_0 = n_0(\varepsilon, x_0) > 0$ such that $d(x, x_0) < \delta$ implies $\rho(f_n(x), f(x_0)) < \varepsilon$, for all $n \in \mathbb{N} \setminus A$ and $n \geq n_0$.

Let us consider $B = (\mathbb{N} \setminus A) \cap \{n \in \mathbb{N} : n \geq n_0\}$. Since $\mathcal{I}$ is an admissible ideal, so $B \in \mathcal{F}(\mathcal{I})$. Clearly, $\mathbb{N} \setminus B \in \mathcal{I}$ (it depends on both ε and x_0). Now, for all $n \in \mathbb{N} \setminus (\mathbb{N} \setminus B)$ and $d(x, x_0) < \delta$, we have $\rho(f_n(x), f(x_0)) < \varepsilon$. This completes the proof. □

We state a necessary and sufficient condition of $\mathcal{I}$-α convergence for *good* ideals. We note that the main frame of the proof (converse part) is similar to that of [16, Theorem 2.7].

THEOREM 3.7. *Let $\mathcal{I}$ be a good ideal. Then a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is $\mathcal{I}$-α convergent at $x_0 \in X$ to $f \in Y^X$ if and only if the following condition holds:*

$$\forall \varepsilon > 0\ \exists\, \delta = \delta(x_0, \varepsilon) > 0\ \exists\, A = A(x_0, \varepsilon) \in \mathcal{I} \text{ such that}$$
$$x \in B(x_0, \delta) \Rightarrow \rho(f_n(x), f(x_0)) < \varepsilon \text{ for all } n \in \mathbb{N} \setminus A.$$

Proof. Let the condition be satisfied. Let $\varepsilon > 0$ be given. Then there exist $\delta = \delta(x_0, \varepsilon) > 0$ and $A = A(x_0, \varepsilon) \in \mathcal{I}$ such that $\rho(f_n(x), f(x_0)) < \varepsilon$ whenever $x \in B(x_0, \delta)$ and $n \in \mathbb{N} \setminus A$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $\mathcal{I} - \lim x_n = x_0$. Then it is sufficient to show that $\mathcal{I} - \lim f_n(x_n) = f(x_0)$. Since $\mathcal{I} - \lim x_n = x_0$, so $\{n \in \mathbb{N} : d(x_n, x) \geq \delta\} \in \mathcal{I}$.

Let $B = \{n \in \mathbb{N} : d(x_n, x) < \delta\}$. Then $B \in \mathcal{F}(\mathcal{I})$. Thus $B \cap (\mathbb{N} \setminus A) \in \mathcal{F}(\mathcal{I})$. Let $k \in B \cap (\mathbb{N} \setminus A)$. Then $k \in B$ implies $d(x_k, x) < \delta$, and $k \in \mathbb{N} \setminus A$ together with $d(x_k, x) < \delta$ imply $\rho(f_k(x_k), f(x_0)) < \varepsilon$. Therefore, $\{n \in \mathbb{N} : \rho(f_n(x_n), f(x_0)) \geq \varepsilon\} \subset \mathbb{N} \setminus (B \cap (\mathbb{N} \setminus A))$. Since $\mathbb{N} \setminus (B \cap (\mathbb{N} \setminus A)) \in \mathcal{I}$, so $\{n \in \mathbb{N} : \rho(f_n(x_n), f(x_0)) \geq \varepsilon\} \in \mathcal{I}$. Hence $\mathcal{I} - \lim f_n(x_n) = f(x_0)$.

Conversely, let $\{f_n\}_{n\in\mathbb{N}}$ be $\mathcal{I}$-α convergent at x_0 to f. If possible, let the condition do not hold. Then there exists $\varepsilon' > 0$ such that $\forall\ \delta > 0\ \forall\ A \in \mathcal{I}\ \exists\ x \in B(x_0, \delta)\ \exists\ n \in \mathbb{N} \setminus A$ such that $\rho(f_n(x), f(x_0)) \geq \varepsilon'$. Now consider $\delta = \frac{1}{k}$ for $k \in \mathbb{N}$, and let A_k be the collection of all natural numbers n^k so that $\rho(f_{n^k}(x^k), f(x_0)) \geq \varepsilon'$ for some $x^k \in B(x_0, \frac{1}{k})$. Now, if $A_k \in \mathcal{I}$ then $\exists n_*^k \in \mathbb{N} \setminus A_k$ and $\exists x_*^k \in B(x_0, \frac{1}{k})$ such that $\rho(f_{n_*^k}(x_*^k), f(x_0)) \geq \varepsilon'$, which contradicts the definition of A_k. Thus $A_k \notin \mathcal{I}$ for each $k \in \mathbb{N}$. Since $\mathcal{I}$ is a *good* ideal then there exists a countable sequence $\{B_k\}_{k\in\mathbb{N}}$ of pairwise disjoint sets such that $B_k \subset A_k$, $B_k \in \mathcal{I}$ for each $k \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} B_k \notin \mathcal{I}$.

Now, let $B_k = \{n_1^k < n_2^k < \ldots\}$. Consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ as follows: $y_n = x_0$ if $n \notin \bigcup_{k=1}^{\infty} B_k$ and $y_n = x_i^k$ if $n \in B_k$ and $n = n_i^k$ (we consider only one such x_i^k corresponding to each such n_i^k).

Let $\delta > 0$ be given. Then there exists the least $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \delta$. Now, $\{n \in \mathbb{N} : d(y_n, x_0) \geq \delta\} \subset \bigcup_{k=1}^{k_0-1} B_k$. Since $\bigcup_{k=1}^{k_0-1} B_k \in \mathcal{I}$, so $\{n \in \mathbb{N} : d(y_n, x_0) \geq \delta\} \in \mathcal{I}$. Thus $\mathcal{I} - \lim y_n = x_0$.

Again, since $\{n \in \mathbb{N} : \rho(f_n(y_n), f(x_0)) \geq \varepsilon'\} = \bigcup_{k=1}^{\infty} B_k \notin \mathcal{I}$, so $\{f_n(y_n)\}$ does not $\mathcal{I}$-converge to $f(x_0)$, which is a contradiction. Hence the condition holds. □

COROLLARY 3.8. *If $\mathcal{I}$ is a good and P-ideal of $\mathbb{N}$, then the notions of $\mathcal{I}$-α-convergence and $\mathcal{I}^*$-α convergence coincide.*

REMARK 3.9. The ideal we have considered in Example 3.5 is a *good* ideal but not a P-ideal, and the notions of $\mathcal{I}$-α convergence and $\mathcal{I}^*$-α convergence do not coincide over there.

Now we give an example of *good* P-ideal.

EXAMPLE 3.10. The ideal $\mathcal{I}_f$ consisting of all finite subsets of $\mathbb{N}$ is a *good* as well as a P-ideal.

4. $\mathcal{I}^*$-exhaustiveness

In this section, we introduce and study the notion of $\mathcal{I}^*$-exhaustiveness.

DEFINITION 4.1. A sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is said to be $\mathcal{I}^*$-exhaustive at $x' \in X$ if there exists $A = A(x') \in \mathcal{I}$ such that for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, x') > 0$ and $n' = n'(\varepsilon, x') \in \mathbb{N}$ such that $d(x, x') < \delta$ implies $\rho(f_n(x), f_n(x')) < \varepsilon$, for all $n \in \mathbb{N} \setminus A$ and $n \geq n'$.

THEOREM 4.2. *If a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is $\mathcal{I}^*$-α convergent to $f \in Y^X$, then $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-exhaustive at each $x \in X$.*

Proof. Let $x_0 \in X$. Let $\varepsilon > 0$ be given. Since $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-α convergent to f, so there exists $A = A(x_0) \in \mathcal{I}$ such that for the given $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, x_0) > 0$ and $n_0 = n_0(\varepsilon, x_0) \in \mathbb{N}$ such that $d(x, x_0) < \delta$ implies $\rho(f_n(x), f(x_0)) < \frac{\varepsilon}{2}$, for all $n \in \mathbb{N} \setminus A$ and $n \geq n_0$.

Also, $\rho(f_n(x_0), f(x_0)) < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N} \setminus A$ and $n \geq n_0$. Then $d(x, x_0) < \delta$ implies $\rho(f_n(x), f_n(x_0)) \leq \rho(f_n(x), f(x_0)) + \rho(f(x_0), f_n(x_0)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, for all $n \in \mathbb{N} \setminus A$ and $n \geq n_0$. Hence $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-exhaustive at each $x \in X$. □

THEOREM 4.3. *$\mathcal{I}^*$-exhaustiveness implies $\mathcal{I}$-exhaustiveness.*

Proof. Since $\mathcal{I}$ is an admissible ideal, the proof is obvious. □

THEOREM 4.4. *Let $\mathcal{I}$ is an P-ideal. Then $\mathcal{I}$-exhaustiveness implies $\mathcal{I}^*$-exhaustiveness.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive at $x_0 \in X$. Then it is sufficient to show that $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-exhaustive at x_0. Since $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive at x_0, so for every $\varepsilon > 0$ there exist $\delta = \delta(x_0, \varepsilon) > 0$ and $A = A(x_0, \varepsilon) \in \mathcal{I}$ such that $\rho(f_n(x), f_n(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$ and $n \in \mathbb{N} \setminus A$.

Now consider $\varepsilon = \frac{1}{i} > 0$ for each $i \in \mathbb{N}$. Then there exist $\delta_i^{x_0} = \delta_i^{x_0}(\frac{1}{i}, x_0) > 0$ and $A_i(x_0) \in \mathcal{I}$ such that $\rho(f_n(x), f_n(x_0)) < \frac{1}{i}$ whenever $d(x, x_0) < \delta_i^{x_0}$ and $n \in \mathbb{N} \setminus A_i$.

Since $\mathcal{I}$ is a P-ideal and $A_i(x_0) \in \mathcal{I}$ for all $i \in \mathbb{N}$, so there exists $A^\infty \in \mathcal{I}$ such that $A_i(x_0) \setminus A^\infty$ is finite for all $i \in \mathbb{N}$. Clearly A^∞ depends only on $x_0 \in X$. Let $\varepsilon_0 > 0$ be given. Then there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon_0$. Let $A_k(x_0) \setminus A^\infty \subset \{1, 2, \ldots, n_0\}$. Then for all $n \geq n_0 + 1$ and $n \in A^\infty$, we have $n \in A_k(x_0)$. Thus $\rho(f_n(x), f_n(x_0)) < \frac{1}{k} < \varepsilon_0$ whenever $d(x, x_0) < \delta_k^{x_0}$, $n \geq n_0 + 1$ and $n \in A^\infty$. Hence $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-exhaustive at $x_0 \in X$. □

THEOREM 4.5. *If a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset Y^X$ is pointwise $\mathcal{I}^*$-convergent to $f \in Y^X$ and $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-exhaustive at each $x \in X$, then $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-α convergent to f.*

Proof. Let $x_0 \in X$. Let $\varepsilon > 0$ be given. Since $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-exhaustive at x_0, so there exists $A = A(x_0) \in \mathcal{I}$ such that for given $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, x_0) > 0$ and $n_0 = n_0(\varepsilon, x_0) \in \mathbb{N}$ such that $d(x, x_0) < \delta$ implies $\rho(f_n(x), f_n(x_0)) < \frac{\varepsilon}{2}$, for all $n \in \mathbb{N} \setminus A$ and $n \geq n_0$.

Again, $\{f_n(x_0)\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-convergent to $f(x_0)$. Thus there exists a set $K \in \mathcal{F}(\mathcal{I})$, such that for the given $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that $\rho(f_n(x_0), f(x_0)) < \frac{\varepsilon}{2}$, for all $n \in K$ and $n \geq n_1$. Let $B = A \cup (\mathbb{N} \setminus K)$. Clearly, B depends only on x_0 and $B \in \mathcal{I}$. Let $n_2 = \max\{n_0, n_1\}$ Now, for $n \in \mathbb{N} \setminus B$ and $n \geq n_2$, we have

$$\rho(f_n(x), f(x_0)) \leq \rho(f_n(x), f_n(x_0)) + \rho(f_n(x_0), f(x_0)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $d(x, x_0) < \delta$. Therefore, $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-α convergent to f. □

THEOREM 4.6. *Let a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-α convergent to $f \in Y^X$ at $x_0 \in X$ and $\{f_n(x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-convergent to $f(x)$ for each $x \in X \setminus \{x_0\}$. Then f is continuous at $x_0 \in X$.*

Proof. Let $\varepsilon > 0$ be given. Since $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-α convergent to f at x_0, so there exists $A = A(x_0) \in \mathcal{I}$ such that for the given $\varepsilon > 0$ there exist $\delta = \delta(x_0, \varepsilon) > 0$ and $n_0 = n_0(x_0, \varepsilon) \in \mathbb{N}$ such that $\rho(f_n(x), f(x_0)) < \frac{\varepsilon}{2}$ whenever $d(x, x_0) < \delta$, $n \in \mathbb{N} \setminus A$ and $n \geq n_0$.

Again, since $\{f_n(x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*$-convergent to $f(x)$ for each $x \in X \setminus \{x_0\}$, so for every $x \in X \setminus \{x_0\}$ there exists $A_x \in \mathcal{I}$ such that for the given $\varepsilon > 0$ there exists $n_x = n_x(x, \varepsilon) \in \mathbb{N}$ such that $\rho(f_n(x), f(x)) < \frac{\varepsilon}{2}$ whenever $n \in \mathbb{N} \setminus A_x$ and $n \geq n_x$.

Let $x \in B(x_0, \delta) \setminus \{x_0\}$. Let $k_x = \max\{n_0, n_x\}$. Now $A \cap A_x \in \mathcal{I}$. Since $\mathcal{I}$ is admissible, so $\mathbb{N} \setminus (A \cap A_x)$ is an infinite set. Choose $n' \in \mathbb{N} \setminus (A \cap A_x)$ and $n' > k_x$. Then

$$\rho(f(x), f(x_0)) \leq \rho(f(x), f_{n'}(x)) + \rho(f_n(x_0), f_{n'}(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Also, $\rho(f(x_0), f(x_0)) = 0 < \varepsilon$. Thus f is continuous at x_0. □

ACKNOWLEDGEMENT. The author is thankful to the referees for their comments, which helped in improving the paper.

REFERENCES

[1] E. Athanassiadou, A. Boccuto, X. Dimitriou, N. Papanastassiou, *Ascoli-type theorems and ideal (α)-convergence*, Filomat, **26(2)** (2012), 397–405.

[2] E. Athanassiadou, C. Papachristodoulos, N. Papanastassiou, *α and hyper α-convergence in function spaces*, Q. and A. in General Topology, **33** (2015), 1–16.

[3] M. Balcerzak, K. Dems, A. Komisarski, *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl., **328** (2007), 715-–729.

[4] C. Caratheodory, *Stetige Konvergenz und normale Familien von Funktionen*, Math. Ann., **101** (1929), 515–533.

[5] A. Caserta, L. D. R. Kočinac, *On statistical exhaustiveness*, Appl. Math. Lett., **25(10)** (2012), 1447–1451.

[6] P. Das, S. Kr. Ghosal, *On $\mathcal{I}$-Cauchy nets and completeness*, Topology Appl., **157** (2010), 1152–1156.

[7] P. Das, S. K. Ghosal, *Some further results on $\mathcal{I}$-Cauchy sequences and condition (AP)*, Comp. Math. Appl., **59** (2010), 2597–2600.

[8] H. Dutta, B. E. Rhoades, eds. *Current topics in summability theory and applications*, Springer, Singapore, (2016).

[9] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244.

[10] J. A. Fridy, *Statistical limit points*, Proc. Am. Math. Soc., **118(4)** (1993), 1187–1192.

[11] V. Gregoriades, N. Papanastassiou, *The notion of exhaustiveness and Ascoli-type theorems*, Topology Appl., **155** (2008), 1111—1128.

[12] H. Hahn, *Theorie der reellen Funktionen*, J. Springer, Berlin, (1921).

[13] P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exchange, **26(2)** (2000-2001), 669–685.

[14] B. K. Lahiri, P. Das, *$\mathcal{I}$ and $\mathcal{I}^*$-convergence of nets*, Real Anal. Exchange, **33(2)** (2007-2008), 431–442.

[15] G. Di. Maio, L. D. R. Kočinac, *Statistical convergence in topology*, Topology Appl., **156(1)** (2008), 28–45.

[16] Ch. Papachristodoulos, N. Papanastassiou, W. Wilczyński, *$\mathcal{I}$-exhaustive sequences of functions*, Selected papers of the ICTA, (2010).

[17] I.J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375.

[18] S. Stoilov, *Continuous convergence*, Rev. Math. Pures Appl., **4** (1959), 341–344.

(received 26.08.2020; in revised form 31.05.2021; available online 31.01.2022)

Department of Mathematics, Brainware University, Barasat, Kolkata-700125, India

E-mail: buagbu@yahoo.co.in