MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
74, 2 (2022), 119–129
June 2022

research paper
оригинални научни рад

BOUNDS FOR THE A_α-SPECTRAL RADIUS OF A DIGRAPH

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Abstract. Let D be a digraph of order n and let $A(D)$ be the adjacency matrix of D. Let $\operatorname{Deg}(D)$ be the diagonal matrix of vertex out-degrees of D. For any real $\alpha \in [0,1]$, the generalized adjacency matrix $A_\alpha(D)$ of the D is defined as $A_\alpha(D) = \alpha \operatorname{Deg}(D) + (1-\alpha)A(D)$. The largest modulus of the eigenvalues of $A_\alpha(D)$ is called the generalized adjacency spectral radius or the A_α-spectral radius of D. In this paper, we obtain some new upper and lower bounds for the spectral radius of $A_\alpha(D)$ in terms of the number of vertices n, the number of arcs, the vertex out-degrees, the average 2-out-degrees of the vertices of D and the parameter α. We characterize the extremal digraphs attaining these bounds.

1. Introduction

Let $D = (V(D), E(D))$ be a digraph, where $V(D) = \{1, 2, \ldots, n\}$ and $E(D)$ are the vertex set and arc set of D, respectively. A digraph D is simple if it has no loops and multiple arcs. A digraph D is strongly connected if for every pair of vertices $i, j \in V(D)$, there are directed paths from i to j and from j to i. In this paper, we consider finite, simple connected digraphs. We follow [2] for terminology and notations.

Two vertices u and v of a digraph D are called adjacent if they are connected by an arc $(u, v) \in E(D)$ or $(v, u) \in E(D)$ and doubly adjacent if $(u, v), (v, u) \in E(D)$. For $e = (i, j) \in E(D)$, i is the initial vertex of e, j is the terminal vertex of e and vertex i is a tail of vertex j. Let $N_D^-(i) = \{j \in V(D) \mid (j, i) \in E(D)\}$ and $N_D^+(i) = \{j \in V(D) \mid (i, j) \in E(D)\}$ denote the in-neighbors and out-neighbors of i, respectively. Let $d_i^- = |N_D^-(i)|$ denote the in-degree of the vertex i and $d_i^+ = |N_D^+(i)|$ denote the out-degree of the vertex i in D. The minimum out-degree is denoted by δ^+, the maximum out-degree is denoted by Δ^+ and the minimum in-degree by δ^-. If $d_1^+ = d_2^+ = \cdots = d_n^+$, then D is a regular digraph.

2020 Mathematics Subject Classification: 05C50, 05C12, 15A18.
Keywords and phrases: Strongly connected digraphs; generalized adjacency matrix; generalized adjacency spectral radius; digraphs.

A walk π of length l from vertex u to vertex v is a sequence of vertices $\pi : u = u_0, u_1, \ldots, u_l = v$, where (u_{k-1}, u_k) is an arc of D for any $1 \leq k \leq l$. If $u = v$ then π is called a closed walk. Denote the number of closed walks of length 2 associated to the vertex $v_i \in$ by $c_2^{(i)}$. The sequence $(c_2^{(1)}, c_2^{(2)}, \ldots, c_2^{(n)})$ is called closed walk sequence of length 2 of D. Thus $c_2 = c_2^{(1)} + c_2^{(2)} + \ldots + c_2^{(n)}$ is the total number of closed walks of length 2 of D. A digraph D is symmetric if $(u, v) \in E(D)$ implies $(v, u) \in E(D)$, where $u, v \in V(D)$. There is a one-to-one correspondence between simple graphs and symmetric digraphs given by $G \mapsto \overleftrightarrow{G}$, where $\overleftrightarrow{G}$ has the same vertex set as the graph G, and each edge uv of G is replaced by a pair of symmetric arcs (u, v) and (v, u). Under this correspondence, a graph can be identified with a symmetric digraph.

Let D be a digraph with adjacency matrix $A(D) = (a_{ij})$, where $a_{ij} = 1$ whenever $v_iv_j \in E(D)$, and $a_{ij} = 0$ otherwise. Let $\mathrm{Deg}\,(D) = (d_1^+, d_2^+, \ldots, d_n^+)$ be the diagonal matrix of vertex out-degrees of D. Recently, Liu et al. [13] following the idea of Nikiforov [14] (who proposed the generalized adjacency matrix $A_\alpha(G)$ of the graph G defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G),\ 0 \leq \alpha \leq 1$), and defined the generalized adjacency matrix $A_\alpha(D)$ of the digraph D, for any real $\alpha \in [0, 1]$ as $A_\alpha(D) = \alpha\,\mathrm{Deg}\,(D) + (1-\alpha)A(D)$. It is clear that $A_\alpha(D) = A(D)$, if $\alpha = 0$, $2A_\alpha(D) = Q(D)$, if $\alpha = \frac{1}{2}$, and $A_\alpha(D) = \mathrm{Deg}\,(D)$, if $\alpha = 1$. From this it follows that the matrix $A_\alpha(D)$ extends the spectral theory of both adjacency matrix $A(D)$ and the signless Laplacian matrix $Q(D)$ of the digraph. Therefore, it will be interesting to study the spectral properties of the matrix $A_\alpha(D)$. The eigenvalues of $A_\alpha(D)$ are called the generalized adjacency eigenvalues or the A_α-eigenvalues of the digraph D and are denoted by $\lambda_1(A_\alpha(D))$, $\lambda_2(A_\alpha(D))$, …, $\lambda_n(A_\alpha(D))$. In general, the matrix $A_\alpha(D)$ is not symmetric and so its eigenvalues can be complex numbers. The eigenvalue of $A_\alpha(D)$ with largest modulus is called generalized adjacency spectral radius or A_α-spectral radius of the digraph D and is denoted by $\lambda_1(A_\alpha(D)) = \lambda(A_\alpha(D))$. If D is a strongly connected digraph, then it follows from the Perron Frobenius Theorem [8] that $\lambda(A_\alpha(D))$ is an eigenvalue of $A_\alpha(D)$ and there is a unique positive unit eigenvector corresponding to $\lambda(A_\alpha(D))$. The positive unit eigenvector corresponding to $\lambda(A_\alpha(D))$ is called the Perron vector of $A_\alpha(D)$. For some works on the spectral properties of A_α-matrix of a digraph, we refer to [4, 5, 13, 15].

The spectral radius, the Laplacian spectral radius and the signless Laplacian spectral radius of digraphs have received a lot of attention of researchers and as such many papers can be found in this direction. For some recent papers and the related results we refer to [1, 3, 6, 7, 10, 11, 16] and the references therein.

The rest of the paper is organized as follows. In Section 2, we obtain lower bounds for the spectral radius of $A_\alpha(D)$ in terms of the number of vertices, the number of arcs, the number of closed walks at a vertex, the vertex out-degrees of the D. We also obtain upper bounds for the spectral radius of $A_\alpha(D)$ in terms of different parameters associated with the structure of the digraph D. We characterize the extremal graphs that attain these lower and upper bounds. We conclude this paper with a conclusion to highlight that our results extend some known results for the adjacency and the signless Laplacian spectral radius of a digraph D to a general setting.

2. Bounds for the generalized adjacency spectral radius

For a matrix $n \times n$, the matrix $A = (a_{ij})$, its geometric symmetrization, denoted by $S(A) = (s_{ij})$, is the $n \times n$ matrix with entries $s_{ij} = \sqrt{a_{ij}a_{ji}}$ for all $i, j = 1, 2, \ldots, n$. Let $\lambda(M)$ denotes the spectral radius of the matrix M. For the spectral radius of the matrices A and $S(A)$, it is shown in [9] that $\lambda(A) \geq \lambda(S(A)) = \sqrt{\lambda(S(A^2))}$.

Let $A_\alpha(D)$ be the generalized adjacency matrix of the digraph D of order n having a arcs and let $S(A_\alpha(D))$ be the geometric symmetrization of $A_\alpha(D)$. It is easy to see that for any vertex $v_i \in V$, we have $\sum_{j=1}^{n} s_{ij} = \alpha d_i^+ + (1-\alpha)c_2^{(i)}$.

Lemma 2.1 ([8]). *Let A and B be nonnegative matrices. If $0 \leq A \leq B$, then $\rho(A) \leq \rho(B)$. Furthermore, if B is irreducible and $0 \leq A < B$, then $\rho(A) < \rho(B)$.*

A bipartite digraph D with bipartition $V(D) = V_1 \cup V_2$ is said to be bipartite semi-regular if out-degree of every vertex in each set V_1 and V_2 is constant. If the out-degree of every vertex in V_1 is r and the out-degree of every vertex in V_2 is s, the D is said to be a (r, s)-semi-regular bipartite digraph.

The average 2-out-degree of the vertex v_i is defined as

$$m_i^+ = \frac{1}{d_i^+} \sum_{(v_i,v_j)\in E(D)} d_j^+ = \frac{t_i^+}{d_i^+}, \text{ where } t_i^+ = \sum_{(v_i,v_j)\in E(D)} d_j^+ \text{ is the 2-out-degree of the vertex } v_i.$$

Let $t_2^{(i)}$ be the sum of all closed walks of length 2 at the vertices which are both out and in-neighbors of v_i, that is, $t_2^{(i)} = \sum_{(v_i,v_j),(v_j,v_i)\in E(D)} c_2^{(i)}$.

Liu, Tain and Cui [12] showed that the adjacency spectral radius of a digraph D with at least one closed walk of length 2 is greater than or equal to $\sqrt{\frac{\sum_{i=1}^{n}\left(t_2^{(i)}\right)^2}{\sum_{i=1}^{n}\left(c_2^{(i)}\right)^2}}$, with equality if and only if $D = \overleftrightarrow{G}+\{$possibly some arcs that do not belong to cycles$\}$, where each connected component of G is a r-regular graph or an (r_1, r_2)-semiregular bipartite graph, satisfying $r^2{=}r_1r_2{=}\frac{\sum_{i=1}^{n}\left(t_2^{(i)}\right)^2}{\sum_{i=1}^{n}\left(c_2^{(i)}\right)^2}$. In the following theorem, we extend this result to A_α-spectrum of digraphs.

The following result gives lower bounds for the generalized adjacency spectral radius of a digraph, in terms of vertex our-degrees, the number of closed walks at v_i, the sum of the all closed walks of length 2 at the vertices which are both out and in-neighbors of v_i of the digraph and the parameter α.

Theorem 2.2. *Let D be a digraph of order n with a arcs and let $\alpha \in [0, 1)$. Let $(c_2^{(1)}, c_2^{(2)}, \ldots, c_2^{(n)})$ be the sequence of closed walks of length 2 and let $t_2^{(i)}$ be defined above. If D has at least one symmetric pair of arcs, then*

$$\lambda(A_\alpha(D)) \geq \sqrt{\frac{\sum_{i=1}^{n}\left(\alpha d_i^+ c_2^{(i)} + (1-\alpha)t_2^{(i)}\right)^2}{\sum_{i=1}^{n}\left(c_2^{(i)}\right)^2}}. \quad (1)$$

If $\alpha = 0$, then equality occurs in (1) *if and only if* $D = \overleftrightarrow{G} + \{$*possibly some arcs that do not belong to cycles*$\}$, *where each connected component of G is a r-regular graph or a (r_1, r_2)-semiregular bipartite graph with* $r^2 = r_1 r_2 = \frac{\sum_{i=1}^{n}\left(t_2^{(i)}\right)^2}{\sum_{i=1}^{n}\left(c_2^{(i)}\right)^2}$. *If $\alpha \neq 0$, then for a strongly connected digraph D, equality occurs in* (1) *if and only if* $D = \overleftrightarrow{G}$ *such that* $\alpha d_i^+ + (1-\alpha)m_i^+$ *is same for all i or* $D = \overleftrightarrow{G}$ *with each connected component of D has the property that both* $\lambda(A_\alpha(D))$ *and* $-\lambda(A_\alpha(D))$ *are the eigenvalues of* $A_\alpha(D)$ *with eigenvector* $c = \left(c_2^{(1)}, c_2^{(2)}, \ldots, c_2^{(n)}\right)^T$.

Proof. Let $A_\alpha(D) = (b_{ij})$ be the generalized adjacency matrix of the digraph D and let $S(A_\alpha(D)) = (s_{ij})$ be the geometric symmetrization of $A_\alpha(D)$. Then $A_\alpha(D) \geq S(A_\alpha(D)) \geq 0$ and so by Lemma 2.1, it follows that $\lambda(A_\alpha(D)) \geq \lambda(S(A_\alpha(D)))$. Since the matrix $S(A_\alpha(D))$ is symmetric, therefore by Rayleigh quotient, we have

$$\lambda(A_\alpha(D)) \geq \lambda(S(A_\alpha(D))) = \sqrt{\lambda(S(A_\alpha(D))^2)} = \sqrt{\max_{X\neq 0}\frac{X^T S(A_\alpha(D))^2 X}{X^T X}}$$

$$\geq \sqrt{\frac{c^T S(A_\alpha(D))^2 c}{c^T c}} = \sqrt{\frac{(S(A_\alpha(D))c)^T (S(A_\alpha(D))c)}{c^T c}} = \frac{\sum_{i=1}^{n}\left(\alpha d_i^+ c_2^{(i)} + (1-\alpha) t_2^{(i)}\right)^2}{\sum_{i=1}^{n}\left(c_2^{(i)}\right)^2}, \quad (2)$$

where $c = \left(c_2^{(1)}, c_2^{(2)}, \ldots, c_2^{(n)}\right)^T$ is the column vector with i-th entry the number of closed walks of length 2 at v_i. Thus the inequality (1) is proved. If $\alpha = 0$, then equality case follows from [12]. For $\alpha \neq 0$, suppose that equality holds in (1), then all the inequalities occur as equalities. From the equality in (2), we get $\lambda(A_\alpha(D)) = \lambda(S(A_\alpha(D)))$ and $\lambda(S(A_\alpha(D))^2) = \frac{c^T S(A_\alpha(D))^2 c}{c^T c}$. The second equality gives that c is an eigenvector of $S(A_\alpha(D))^2$ corresponding to the eigenvalues $\lambda(S(A_\alpha(D))^2)$, which implies that the multiplicity of the eigenvalue $\lambda(S(A_\alpha(D))^2)$ is either one or two. If D is a strongly connected digraph, then the matrix $A_\alpha(D)$ is an irreducible matrix. Since $A_\alpha(D) \geq S(A_\alpha(D))$ and $A_\alpha(D)$ is an irreducible matrix, so if $A_\alpha > S(A_\alpha)$, then by Lemma 2.1 we have $\lambda(A_\alpha(D)) > \lambda(S(A_\alpha(D)))$, a contradiction to our assumption of equality. Therefore, we must have $A_\alpha(D) = S(A_\alpha(D))$, giving that $A_\alpha(D)$ is a symmetric matrix, which implies that $D = \overleftrightarrow{G}$, where G is the underlying graph of D and $\overleftrightarrow{G}$ is the symmetric digraph corresponding to G.

If the multiplicity of $\lambda(S(A_\alpha(D))^2)$ is one, then since $S(A_\alpha(D))$ is a symmetric matrix, it follows that $\lambda(S(A_\alpha(D))^2) = \lambda^2(S(A_\alpha(D)))$. Using that c is an eigenvector corresponding to the eigenvalue $\lambda(S(A_\alpha(D))^2)$, it follows that $\lambda(S(A_\alpha(D)))$ is an eigenvalue of $S(A_\alpha(D))$ with eigenvector c, that is, $S(A_\alpha(D)c = \lambda(S(A_\alpha(D)))c$. From this it follows that $\alpha d_i^+ + (1-\alpha)\frac{t_2^{(i)}}{c_2^{(i)}} = \alpha d_i^+ + (1-\alpha)m_i^+$ is same for all i. The last equality is due to the fact that for a symmetric digraph $D = \overleftrightarrow{G}$, we have $c_2^{(i)} = d_i^+$ and $t_2^{(i)} = t_i^+$. Thus, it follows that equality occurs in (1) in this case if $D = \overleftrightarrow{G}$ and

$\alpha d_i^+ + (1-\alpha)m_i^+$ is the same for all i.

On the other hand, if the multiplicity of $\lambda(S(A_\alpha(D))^2)$ is two, then both $\lambda(S(A_\alpha(D)))$ and $-\lambda(S(A_\alpha(D)))$ are eigenvalues of $A_\alpha(D)$ giving that some of the eigenvalues of $A_\alpha(D)$ are negative in this case. This gives that equality occurs in this case if $D = \overleftrightarrow{G}$ and D has the property that both $\lambda(A_\alpha(D))$ and $-\lambda(A_\alpha(D))$ are eigenvalues of $A_\alpha(D)$ with eigenvector c.

Let D be the direct sum of its disjoint strongly connected components D_1, D_2, $\ldots$, D_s. Let $A_\alpha(D_k)$ be the generalized adjacency matrix of order $n_k \times n_k$ of the component D_k with $\sum_{k=1}^n n_k = n$. In this case, we have

$$A_\alpha(D)^2 = \begin{pmatrix} A_\alpha(D_1)^2 & & & \\ & A_\alpha(D_2)^2 & & \\ & & \ddots & \\ & & & A_\alpha(D_s)^2 \end{pmatrix},$$

where the rest of the unspecified entries are 0. Clearly the matrix $S(A_\alpha(D))$ is also a block diagonal matrix in this case. Since $S(A_\alpha(D))$ is a symmetric matrix, therefore we have $\lambda(S(A_\alpha(D))) = \max_k \lambda(S(A_\alpha(D_k)))$. Let c_{n_k} be part of the column vector c of order n_k which corresponds to block $S(A_\alpha(D_k))$ of $S(A_\alpha(D))$. Since equality holds in (1), we have

$$\lambda(A_\alpha(D))=\sqrt{\lambda(S(A_\alpha(D))^2)}=\sqrt{\frac{c^T S(A_\alpha(D))^2 c}{c^T c}}=\sqrt{\sum_{k=1}^{s}\frac{c_{n_k}^T S(A_\alpha(D_k))^2 c_{n_k}}{c_{n_k}^T c_{n_k}}\,\frac{c_{n_k}^T c_{n_k}}{c^T c}}$$

$$\leq\sqrt{\sum_{k=1}^{s}\frac{c_{n_k}^T c_{n_k}\lambda(S(A_\alpha(D_k))^2)}{c^T c}}\leq\sqrt{\max_k \lambda(S(A_\alpha(D_k))^2)}=\sqrt{\lambda(S(A_\alpha(D))^2)}=\lambda(A_\alpha(D)),$$

which implies that, for every $k = 1, 2, \ldots, s$, we have

$$\lambda(A_\alpha(D))=\sqrt{\lambda(A_\alpha(D)^2)}=\sqrt{\lambda(A_\alpha(D_k)^2)}=\sqrt{\lambda(S(A_\alpha(D_k))^2)}=\sqrt{\sum_{k=1}^{s}\frac{c_{n_k}^T S(A_\alpha(D_k))c_{n_k}}{c_{n_k}^T c_{n_k}}}.$$

Then, by the above case, the equality holds in this case for the digraphs mentioned in the statement. □

We note that the lower bound obtained by Liu, Tain and Cui [12] for the adjacency spectral radius holds for only those digraphs D which have at least one symmetric pair of arcs. For if D is a digraph with no symmetric pair of arcs then $t_2^{(i)} = c_2^{(i)} = 0$, for all i, then the quantity $\sqrt{\frac{\sum_{i=1}^n \left(t_2^{(i)}\right)^2}{\sum_{i=1}^n \left(c_2^{(i)}\right)^2}}$ does not exist. Next, we obtain another lower bound for $\lambda(A_\alpha(D))$ which holds for all digraphs.

Taking $X = (d_1^+, d_2^+, \ldots, d_n^+)^T$ in Theorem 2.2 and proceeding similarly, we obtain the following lower bound for $\lambda(A_\alpha(D))$.

Theorem 2.3. *Let D be a digraph of order n with a arcs and let $\alpha \in [0, 1)$. For each* *vertex $v_i \in V(D)$, let $T_i^+ = \sum_{(v_i,v_j),(v_j,v_i)\in E(D)} d_i^+$. Then*

$$\lambda(A_\alpha(D)) \geq \sqrt{\frac{\sum_{i=1}^n \left(\alpha(d_i^+)^2 + (1-\alpha)T_i^+\right)^2}{\sum_{i=1}^n \left(d_i^+\right)^2}}. \tag{3}$$

If $\alpha = 0$, then equality occurs in (3) if and only if $D = \overleftrightarrow{G} + \{$possibly some arcs that do not belong to cycles$\}$, where each connected component of G is an r-regular graph or a (r_1, r_2)-semiregular bipartite graph with $r^2 = r_1 r_2 = \frac{\sum_{i=1}^n (T_i^+)^2}{\sum_{i=1}^n (d_i^+)^2}$. If $\alpha \neq 0$, then for a strongly connected digraph D, equality occurs in (3) if and only if $D = \overleftrightarrow{G}$ and $\alpha d_i^+ + (1-\alpha)m_i^+$ is the same for all i or $D = \overleftrightarrow{G}$ with each connected component of D has the property that both $\lambda(A_\alpha(D))$ and $-\lambda(A_\alpha(D))$ are the eigenvalues of $A_\alpha(D)$ with eigenvector $c = (d_1^+, d_2^+, \ldots, d_n^+)^T$.

Proof. The proof follows by taking $X = (d_1^+, d_2^+, \ldots, d_n^+)^T$ in (2) and proceeding similarly as in Theorem 2.2. □

The following lemma gives the generalized adjacency spectral radius of a bipartite semi-regular digraph.

Lemma 2.4. *Let D be a strongly connected bipartite semi-regular digraph with bipartition $V(D) = V_1 \cup V_2$. If $d_i^+ = r$, for all $v_i \in V_1$ and $d_j^+ = s$, for all $v_j \in V_2$, then $\lambda(A_\alpha(D)) = \frac{1}{2}\left(\alpha(r+s) + \sqrt{\alpha^2(r+s)^2 + 4(1-2\alpha)rs}\right)$.*

Proof. Let D be a strongly connected bipartite semi-regular digraph with bipartition $V(D) = V_1 \cup V_2$ such that $d_i^+ = r$, for all $v_i \in V_1$ and $d_j^+ = s$, for all $v_j \in V_2$. Let $V(D) = \{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_l\}$, where $V_1 = \{u_1, u_2, \ldots, u_k\}$ and $V_2 = \{w_1, w_2, \ldots, w_l\}$. Under this labelling of vertices of D the generalized adjacency matrix of D can be written as

$$A_\alpha(D) = \begin{pmatrix} \alpha r I_k & (1-\alpha)B \\ (1-\alpha)C & \alpha s I_l \end{pmatrix},$$

where I_p is the identity matrix of order p, B is the part of the matrix $A_\alpha(D)$ which corresponds to the arcs having initial in V_1 and terminal in V_2 and C is the part of the matrix $A_\alpha(D)$ which corresponds to the arcs having initial in V_2 and terminal in V_1. The equitable quotient matrix of $A_\alpha(D)$ is $M = \begin{pmatrix} \alpha r & (1-\alpha)r \\ (1-\alpha)s & \alpha s \end{pmatrix}$. The spectral radius of the matrix M is $\lambda(M) = \frac{\alpha(r+s)+\sqrt{\alpha^2(r+s)^2+4(1-2\alpha)rs}}{2}$. Since the matrix $A_\alpha(D)$ is nonnegative, therefore it follows from [17, Theorem 2.5] that $\lambda(A_\alpha(D)) = \lambda(M)$. □

The following gives an upper bound for $\lambda(A_\alpha(D))$, in terms of the vertex out-degrees of the digraph and the parameter α.

Theorem 2.5. *Let D be a strongly connected digraph of order n and let $\alpha \in [0, 1)$.*

Let $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$ be the out-degree sequence of D. Then

$$\lambda(A_\alpha(D)) \leq \max_{(v_i,v_j)\in E(D)} \left\{ \frac{\alpha(d_i^+ + d_j^+) + \sqrt{\alpha^2(d_i^+ + d_j^+)^2 + 4(1-2\alpha)d_i^+ d_j^+}}{2} \right\}. \quad (4)$$

Moreover, equality holds if and only if D is a regular digraph or a bipartite semi-regular digraph

Proof. Let D be a strongly connected digraph of order n and let $E(D)$ be the arc set of D. Let $X=(x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $A_\alpha(D)$ corresponding to the eigenvalue $\lambda(A_\alpha(D))$. We assume that $x_i = \max\{x_k; v_k \in V(D)\}$ and $x_j = \max\{x_k; (v_i, v_k) \in E(D)\}$. From the i-th equation of $A_\alpha(D)X = \lambda(A_\alpha(D))X$, we have

$$\lambda(A_\alpha(D))x_i = \alpha d_i^+ x_i + (1-\alpha) \sum_{(v_i,v_j)\in E(D)} x_j,$$

i.e.
$$(\lambda(A_\alpha(D)) - \alpha d_i^+)x_i \leq (1-\alpha)d_i^+ x_j. \quad (5)$$

Also, from the j-th equation of $A_\alpha(D)X = \lambda(A_\alpha(D))X$, we have

$$\lambda(A_\alpha(D))x_j = \alpha d_j^+ x_j + (1-\alpha) \sum_{(v_j,v_k)\in E(D)} x_k,$$

i.e.
$$(\lambda(A_\alpha(D)) - \alpha d_j^+)x_j \leq (1-\alpha)d_j^+ x_i. \quad (6)$$

Multiplying the corresponding sides of (5), (6) and using the fact that $x_k > 0$ for all k, we get

$$(\lambda(A_\alpha(D)) - \alpha d_i^+)(\lambda(A_\alpha(D)) - \alpha d_j^+) \leq (1-\alpha)^2 d_i^+ d_j^+,$$

i.e.
$$\lambda(A_\alpha(D))^2 - \alpha(d_i^+ + d_j^+)\lambda(A_\alpha(D)) - (1-2\alpha)d_i^+ d_j^+ \leq 0. \quad (7)$$

From this the inequality (4) follows. Suppose that the equality holds in (4). Then, all the inequalities in the above argument must be equalities. From equality in (6), we get $x_k = x_j$ for all k such that $(v_i, v_k) \in E(D)$ and from the equality in (7), we get $x_k = x_i$ for all k such that $(v_j, v_k) \in E(D)$. Consider the sets $V_1 = \{v_k : x_k = x_i\}$ and $V_2 = \{v_k : x_k = x_j\}$. Clearly $N^+(v_i) \in V_2$ and $N^+(v_j) \in V_1$. We will show that $V(D) = V_1 \cup V_2$. Let $v_z \in N^+(v_i)$ and $v_r \in N^+(v_z)$; then $x_z = x_j$ and $x_r = x_i$. Further, if $s \in N^+(v_r)$ and $v_r \in N^+(v_z)$, then by above $x_r = x_j$. Proceeding in this way and using the fact that D is a strongly connected digraph, we conclude that $x_u = x_i$ or $x_u = x_j$, for all $u \in V(D)$. This proves that $V(D) = V_1 \cup V_2$. We first suppose that D is non-bipartite strongly connected digraph; then D contains directed odd cycles and so using above procedure we arrive at $x_i = x_j$. This gives that $X = (1, 1, \ldots, 1)$ is an eigenvector for $\lambda(A_\alpha(D))$ and so D is a r-regular digraph. On the other hand, if D is a bipartite digraph, then we have either $x_i = x_j$ or $x_i \neq x_j$. If $x_i = x_j$, then as above D is an r-regular digraph. Assume that $x_i \neq x_j$. Since, for $v_k \in V_1$, we have $x_k = x_i$ and $x_z = x_j$, for all $v_z \in N^+(v_k)$, therefore it follows that $\lambda(A_\alpha(D))x_i = \alpha d_k^+ x_i + (1-\alpha)d_k^+ x_j$. This gives that $d_k^+ = \frac{\lambda(A_\alpha(D))x_i}{\alpha x_i + (1-\alpha)x_j}$, for all $v_k \in V_1$. Similarly, for $v_k \in V_2$, we have $x_k = x_j$ and $x_z = x_i$, for all $v_z \in N^+(v_k)$, it follows that $d_k^+ = \frac{\lambda(A_\alpha(D))x_j}{\alpha x_j + (1-\alpha)x_i}$, for all $v_k \in V_2$. This shows that D is a semi-regular bipartite digraph in this case.

Conversely, if the strongly connected digraph D is a regular digraph or a bipartite semi-regular digraph then using Lemma 2.4 (in case of bipartite semi-regular), it can be seen that equality holds in (4). This completes the proof. □

The following gives a lower bound for $\lambda(A_\alpha(D))$, in terms of the vertex out-degrees of the digraph and the parameter α.

Theorem 2.6. *Let D be a strongly connected digraph of order n and let $\alpha \in [0,1)$. Let $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$ be the out-degree sequence of D. Then*

$$\lambda(A_\alpha(D)) \geq \min_{(v_i,v_j)\in E(D)} \left\{ \frac{\alpha(d_i^+ + d_j^+) + \sqrt{\alpha^2(d_i^+ + d_j^+)^2 + 4(1-2\alpha)d_i^+ d_j^+}}{2} \right\}, \quad (8)$$

provided that

$$\lambda(A_\alpha(D)) > \max_{(v_i,v_j)\in E(D)} \left\{ \frac{\alpha(d_i^+ + d_j^+) - \sqrt{\alpha^2(d_i^+ + d_j^+)^2 + 4(1-2\alpha)d_i^+ d_j^+}}{2} \right\}.$$

Moreover, equality holds if and only if D is a regular digraph or a bipartite semi-regular digraph

Proof. Let D be a strongly connected digraph of order n and let $E(D)$ be the arc set of D. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $A_\alpha(D)$ corresponding to the eigenvalue $\lambda(A_\alpha(D))$. We assume that $x_i = \min\{x_k; v_k \in V(D)\}$ and $x_j = \min\{x_k; (v_i, v_k) \in E(D)\}$. The rest of the proof is similar to Theorem 2.5 and is therefore omitted. □

Lemma 2.7 ([8]). *Let $M = (m_{ij})$ be an $n \times n$ nonnegative matrix and let $r_i(M)$ be the i-th row sum of M. Then* $\min\{r_i(M), 1 \leq i \leq n\} \leq \lambda(M) \leq \max\{r_i(M), 1 \leq i \leq n\}$. *If M is irreducible, then each equality holds if and only if $r_1 = r_2 = \cdots = r_n$.*

The following result gives an upper bound for $\lambda(A_\alpha(D))$ in terms of the maximum out-degree, the minimum out-degree, the number of arcs, the number of vertices and the parameter α.

Theorem 2.8. *Let D be a strongly connected digraph of order $n \geq 3$ with a arcs and let $\alpha \in [0,1)$. Let Δ^+ and δ^+ be respectively the maximum vertex out-degree and the minimum vertex out-degree of D. Then*

$$\lambda(A_\alpha(D)) \leq \max\left\{ \alpha\Delta^+ + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{\Delta^+} + \delta^+ - 1\right), \right.$$
$$\left. 2\alpha + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{2} + \delta^+ - 1\right), (1-\alpha)\Delta^+ + \alpha\left(\delta^+ - 1 + \frac{m-\delta^+(n-1)}{\Delta^+}\right) \right\}.$$

Moreover, if $D(\neq C_n)$ is a regular digraph, then the equality holds.

Proof. Let D be a strongly connected digraph and let $\operatorname{Deg}(D) = \operatorname{Diag}(d_1^+, d_2^+, \ldots, d_n^+)$ be the diagonal matrix of out-degrees of the vertices of the digraph D. Since D is strongly connected implies that $d_i^+ \geq 1$, it follows that the matrix $\mathrm{Deg}\,(D)^{-1} = \mathrm{Diag}\,(1/d_1^+, 1/d_2^+, \ldots, 1/d_n^+)$ exists. For a matrix M, let $r_i(M)$ denoted the sum of the entries in the i-th row. Considering the matrix $\mathrm{Deg}\,(D)^{-1}A_\alpha(D)\,\mathrm{Deg}\,(D)$, it is easy to see that

$$r_i(\mathrm{Deg}\,(D)^{-1}A_\alpha(D)\,\mathrm{Deg}\,(D)) = \alpha d_i^+ + (1-\alpha)\frac{1}{d_i^+}\sum_{(v_i,v_j)\in E} d_j^+ = \alpha d_i^+ + (1-\alpha)m_i^+.$$

Since the matrices $A_\alpha(D)$ and $\mathrm{Deg}\,(D)^{-1}A_\alpha(D)\,\mathrm{Deg}\,(D)$ are similar, it follows that $\lambda(A_\alpha(D)) = \lambda(\mathrm{Deg}\,(D)^{-1}A_\alpha(D)\,\mathrm{Deg}\,(D))$. Now, using Lemma 2.7, we obtain

$$\min\left\{\alpha d_i^+ + (1-\alpha)m_i^+, v_i \in V(D)\right\} \leq \lambda(A_\alpha(D)) \leq \max\{\alpha d_i^+ + (1-\alpha)m_i^+, v_i \in V(D)\}. \quad (9)$$

Using Lemma 2.7 and the fact D is strongly connected it is easy to see that equality holds on both sides of (9) if and only if $\alpha d_1^+ + (1-\alpha)m_1^+ = \alpha d_2^+ + (1-\alpha)m_2^+ = \cdots = \alpha d_n^+ + (1-\alpha)m_n^+$.

From the inequality (9) we know that $\lambda(D) \leq \max\{\alpha d_i^+ + (1-\alpha)m_i^+, v_i \in V(D)\}$. So we only need to prove that $\max\{\alpha d_i^+ + (1-\alpha)m_i^+, v_i \in V(D)\} \leq \max\Big\{\alpha\Delta^+ + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{\Delta^+} + \delta^+ - 1\right), 2\alpha + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{2} + \delta^+ - 1\right), (1-\alpha)\Delta^+ + \alpha\left(\delta^+ - 1 + \frac{m-\delta^+(n-1)}{\Delta^+}\right)\Big\}$. Suppose $\max\{\alpha d_i^+ + (1-\alpha)m_i^+, v_i \in V(D)\}$ occurs at vertex u. Two cases arise: $d_u^+ = 1$ or $2 \leq d_u^+ \leq \Delta^+$.

(i) $d_u^+ = 1$. Suppose that $N_u^+ = \{w\}$. Since $m_u^+ = d_w^+ \leq \Delta^+$, thus $\alpha d_u^+ + (1-\alpha)m_u^+ \leq \alpha + (1-\alpha)\Delta^+$. Since $\sum_{v_i\in V(D)} d_i^+ = m$, let $d_j^+ = \Delta^+$, then $\sum_{i\neq j} d_i^+ = m - \Delta^+ \geq (n-1)\delta^+$, so $m-(n-1)\delta^+ \geq \Delta^+$. Therefore $\delta^+ - 1 + \frac{m-\delta^+(n-1)}{\Delta^+} \geq \delta^+ - 1 + \frac{\Delta^+}{\Delta^+} = \delta^+ \geq 1$. Hence $\alpha d_u^+ + (1-\alpha)m_u^+ \leq (1-\alpha)\Delta^+ + \alpha\left(\delta^+ - 1 + \frac{m-\delta^+(n-1)}{\Delta^+}\right)$, the result follows.

(ii) $2 \leq d_u^+ \leq \Delta^+$. Note that $m-(n-1)\delta^+ \geq d_u^+ \geq 2$, and

$$m = \sum_{(u,v)\in E} d_v^+ + \sum_{(u,v)\notin E} d_v^+ \geq \sum_{(u,v)\in E} d_v^+ + d_u^+ + (n - d_u^+ - 1)\delta^+,$$

thus
$$\sum_{(u,v)\in E} d_v^+ \leq m - d_u^+ - (n - d_u^+ - 1)\delta^+ = m - (n-1)\delta^+ + (\delta^+ - 1)d_u^+,$$

hence
$$m_u^+ = \frac{\sum_{(u,v)\in E} d_v^+}{d_u^+} \leq \frac{m-(n-1)\delta^+}{d_u^+} + \delta^+ - 1.$$

It follows that $\alpha d_u^+ + (1-\alpha)m_u^+ \leq \alpha d_u^+ + (1-\alpha)\left(\frac{m-(n-1)\delta^+}{d_u^+} + \delta^+ - 1\right)$. Let $f(x) = \alpha x + (1-\alpha)\left(\frac{m-(n-1)\delta^+}{x} + \delta^+ - 1\right)$, where $x \in [2, \Delta^+]$. It is easy to see that $f'(x) = \alpha - (1-\alpha)\left(\frac{m-(n-1)\delta^+}{x^2}\right)$. Let $a = \frac{(1-\alpha)(m-(n-1)\delta^+)}{\alpha}$, then $\sqrt{a}$ is the unique positive root of $f'(x) = 0$. We consider the next three subcases:

(1) $\sqrt{a} < 2$. When $x \in [2, \Delta^+]$, since $f'(x) > 0$, then $f(x) \leq f(\Delta^+)$.

(2) $2 \leq \sqrt{a} \leq \Delta^+$. Then $f'(x) < 0$ for $x \in [2, \sqrt{a})$ and $f'(x) \geq 0$, for $x \in [\sqrt{a}, \Delta^+]$. Thus $f(x) \leq \max\{f(2), f(\Delta^+)\}$.

(3) $\Delta^+ < \sqrt{a}$. When $x \in [2, \Delta^+]$, since $f'(x) < 0$, then $f(x) \leq f(2)$. Recall that $2 \leq d_u^+ \leq \Delta^+$, thus

$$\alpha d_u^+ + (1-\alpha) m_u^+ \leq \max\{f(2), f(\Delta^+)\}$$

$$= \max\left\{\alpha\Delta^+ + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{\Delta^+} + \delta^+ - 1\right), 2\alpha + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{2} + \delta^+ - 1\right)\right\}.$$

If $D(\neq C_n)$ is a regular digraph, then $\alpha d_i^+ + (1-\alpha) m_i^+ = d_i^+ = \Delta^+$ for all $v_i \in V(D)$. We can get $\lambda(D) = \Delta^+$. Since $D(\neq C_n)$ is a strongly connected digraph, then we may assume that $\Delta^+ \geq 2$, this implies that

$$2\alpha + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{2} + \delta^+ - 1\right) = 2\alpha + (1-\alpha)\left(\frac{\Delta^+}{2} + \Delta^+ - 1\right) \leq \Delta^+$$

$$= \alpha\Delta^+ + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{\Delta^+} + \delta^+ - 1\right).$$

So $\max\left\{\alpha\Delta^+ + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{\Delta^+} + \delta^+ - 1\right), 2\alpha + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{2} + \delta^+ - 1\right)\right\} = \Delta^+$. Thus the equality holds. □

The following observation follows from Theorem 2.8.

Corollary 2.9. *Let D be a strongly connected digraph of order $n \geq 3$ with m arcs having maximum out-degree Δ^+ and the minimum out-degree δ^+. If $\Delta^+ \geq \frac{m-(n-1)}{2}$ and $\delta^+ = 1$, then $\lambda(D) \leq \max\{\alpha(\Delta^+ - 2) - 2, \Delta^+ - \alpha(\Delta^+ - 2)\}$.*

Proof. Since $\alpha\Delta^+ + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{\Delta^+} + \delta^+ - 1\right) \leq \alpha(\Delta^+ - 2) - 2$, also $2\alpha + (1-\alpha)\left(\frac{m-\delta^+(n-1)}{2} + \delta^+ - 1\right) \leq \Delta^+ - \alpha(\Delta^+ - 2)$ and $(1-\alpha)\Delta^+ + \alpha\left(\delta^+ - 1 + \frac{m-\delta^+(n-1)}{\Delta^+}\right) \leq \Delta^+ - \alpha(\Delta^+ - 2)$, hence by Theorem 2.8 the result follows. □

3. Concluding remarks

As mentioned in the introduction, for $\alpha = 0$, the generalized adjacency matrix $A_\alpha(D)$ of the digraph D is the same as the adjacency matrix $A(D)$ and for $\alpha = \frac{1}{2}$, twice the generalized adjacency matrix $A_\alpha(D)$ is the same as the signless Laplacian matrix $Q(D)$. Therefore, if in particular, we put $\alpha = 0$ and $\alpha = \frac{1}{2}$, in all the results obtained in Section 2, we obtain the corresponding bounds for the adjacency spectral radius $\lambda(A(D))$ and the signless Laplacian spectral radius $\lambda(Q(D))$, respectively. We note that most of these results we obtained in Section 2 has been already discussed for the adjacency spectral radius $\lambda(A(D))$ or/and for the signless Laplacian spectral radius $\lambda(Q(D))$. Therefore, in this setting our results are the generalization of these known results. Further if in particular $D = \overleftrightarrow{G}$, where $\overleftrightarrow{G}$ is the symmetric digraph corresponding to the underlying graph G of D, then our results obtained in Section 2 become the corresponding results for the generalized adjacency spectral radius $\lambda(A_\alpha(G))$ of the graph G. Thus our results are also the generalizations of the corresponding results for the A_α-matrix of the graph G.

Acknowledgement. We are highly thankful to the communicating editor and the anonymous referee for his valuable suggestions to improve the presentation of the paper.

References

[1] B. A. Chat, H. A. Ganie, S. Pirzada, *Bounds for the skew Laplacian spectral radius of oriented graphs*, Carpathian J. Math., **35(1)** (2019), 31–40.

[2] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graph, Theory and Application*, Academic Press, New York, 1980.

[3] H. A. Ganie, *Bounds for the skew Laplacian (skew adjacency) spectral radius of a digraph*, Trans. Combin., **8(2)** (2019), 1–12.

[4] H. A. Ganie, *On the A_α-spectrum of joined union of digraphs*, Discrete Math. Algori. Appli., doi: 10.1142/S1793830921500865

[5] H. A. Ganie, M. Baghipur, *On the generalized adjacency spectral radius of digraphs*, Linear Multilinear Algebra, doi: 10.1080/03081087.2020.1814194.

[6] H. A. Ganie, S. Pirzada, B. A. Chat, X. Li, *On skew Laplacian spectrum and energy of digraphs*, Asian-European Journal of Math., **14(04)** (2021), 2150051.

[7] W. Hong, L. You, *Spectral radius and signless Laplacian spectral radius of strongly connected digraphs*, Linear Algebra Appl., **457** (2014), 93–113.

[8] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[9] L. Yu. Kolotilina, *Lower bounds for the Perron root of a nonnegative matrix*, Linear Algebra Appl., **180** (1993), 133–151.

[10] W. Lang, L. Wang, *Sharp bounds for the signless Laplacian spectral radius of digraphs*, Appl. Math. Comput., **238** (2014), 43–49.

[11] H. Q. Lin, J. L. Shu, Y.R. Wu, G. L. Yu, *Spectral radius of strongly connected digraphs*, Discrete Math. **312** (2012), 3663–3669.

[12] M. H. Liu, G. X. Tian, S. Y. Cui, *Some sharp bounds for the spectral radius and energy of digraphs*, Ars Comb., **127** (2016), 45–55.

[13] J. P. Liu, X. Z. Wu, J. S. Chen, B. L. Liu, *The A_α spectral radius characterization of some digraphs*, Linear Algebra Appl., **563** (2019), 63–74.

[14] V. Nikiforov, *Merging the A-and Q-spectral theories*, Appl. Anal. Discrete Math., **11** (2017), 81–107.

[15] W. Xi, W. So, L. Wang, *On the A_α spectral radius of digraphs with given parameters*, Linear Multilinear Algebra, doi: 10.1080/03081087.2020.1793879.

[16] W. Xi, L. Wang, *Sharp bounds on the signless laplacian spectral radius of strongly connected digraphs*, Discuss. Math. Graph Theory, **36** (2016), 977–988.

[17] L. You, M. Yan, W. So, W. Xi, *On the spectrum of an equitable quotient matrix and its application*, Linear Algebra Appl., **577** (2019) 21–40.

(received 10.11.2020; in revised form 02.06.2021; available online 31.01.2022)

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