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YOUNG TYPE INEQUALITIES AND REVERSES FOR MATRICES

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Abstract. In this paper, we give some Young type inequalities for scalars. By using these inequalities we establish corresponding Young type inequalities for matrices. In addition, we present some reverses of the Young type inequalities and give several refinements for matrices.

1. Introduction

In what follows, $M_n(\mathbb{C})$ denotes the space of $n \times n$ complex matrices and $M_n^+(\mathbb{C})$ denotes the class of positive semi-definite matrices in $M_n(\mathbb{C})$. A norm $|||\cdot|||$ on $M_n(\mathbb{C})$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. For $A = [a_{ij}] \in M_n(\mathbb{C})$, the Hilbert-Schmidt (or Frobenius) norm and the trace norm of A are defined by

$$\|A\|_2 = \left(\sum_{j=1}^{n} s_j^2(A)\right)^{\frac{1}{2}}, \quad \|A\|_1 = \operatorname{tr}(|A|) = \sum_{j=1}^{n} s_j(A)$$

respectively, where $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive matrix $|A| = \sqrt{A^*A}$, arranged in decreasing order and repeated according to multiplicity and $\operatorname{tr}(\cdot)$ is the usual trace. Moreover, it is well known that $\|\cdot\|_2$ and $\|\cdot\|_1$ are unitarily invariant.

The most common form of Young's inequality, often used to prove the well-known inequality for L_p functions is as follows:

$$a^{\nu} b^{1-\nu} \leq \nu a + (1-\nu) b, \tag{1}$$

where $a, b > 0$ and $0 \leq \nu \leq 1$, or equivalently $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, where $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$.

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A matrix Young's inequality due to Ando [1] asserts that $s_j(A^\nu B^{1-\nu}) \le s_j(\nu A + (1-\nu)B)$. The above singular value inequality entails the following unitarily invariant norm inequality $\left|\left|\left| A^\nu B^{1-\nu} \right|\right|\right| \le |||\nu A + (1-\nu)B|||$.

A determinant version of Young's inequalities is also known [12, p. 467]: For positive semidefinite matrices A, B and $0 \le \nu \le 1$, $\det(A^\nu B^{1-\nu}) \le \det(\nu A + (1-\nu)B)$. Although inequality (1) seems simple, it is very useful in operator theory. We refer the interested readers to [2–5, 9, 17–21].

Refinements and generalizations of this inequality has attracted the attention of many researchers in the field. The first refinement of Young's inequality is the squared version proved in [11]:

$$(a^\nu b^{1-\nu})^2 + \min\{\nu, 1-\nu\}^2(a-b)^2 \le (\nu a + (1-\nu)b)^2. \tag{2}$$

Later, Kittaneh and Manasrah [14], obtained another interesting refinement of Young's inequality:

$$a^\nu b^{1-\nu} + \min\{\nu, 1-\nu\}(\sqrt{a} - \sqrt{b})^2 \le \nu a + (1-\nu)b. \tag{3}$$

The inequalities (2) and (3) are special cases of a more general refinement stating that for $m = 1, 2, 3, \ldots$ one has:

$$\left(a^\nu b^{1-\nu}\right)^m + r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2 \le \left(\nu a^r + (1-\nu)b^r\right)^{\frac{m}{r}}, \quad r \ge 1,$$

where $r_0 = \min\{\nu, 1-\nu\}$ (see [16]).

A related refinement has been recently proved in [2], with some interesting applications in $M_n(\mathbb{C})$.

These inequalities are generalized to the space of operators on finite dimensional Hilbert spaces. In the setting of matrices, the above refinement reads as follows: $\left|\left|\left| A^\nu B^{1-\nu} \right|\right|\right|^m + r_0^m \left(|||AX|||^{\frac{m}{2}} - |||XB|||^{\frac{m}{2}} \right)^2 \le (\nu|||AX||| + (1-\nu)|||XB|||)^m$, when A, $B \in M_n^+(\mathbb{C})$ and $m \in \mathbb{N}$ (see [16]).

Kai in [13] gave the following Young type inequalities:

$$\nu^2 a^2 + (1-\nu)^2 b^2 \ge \nu^2(a-b)^2 + \nu^{2\nu} a^{2\nu} b^{2-2\nu}, \quad \text{if } 0 \le \nu \le \frac{1}{2}, \tag{4}$$

$$\nu^2 a^2 + (1-\nu)^2 b^2 \ge (1-\nu)^2(a-b)^2 + (1-\nu)^{2(1-\nu)} a^{2\nu} b^{2-2\nu}, \quad \text{if } \frac{1}{2} \le \nu \le 1. \tag{5}$$

Based on the refined Young's inequalities (4) and (5), it has been shown in [13] that if $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite, then

$$\begin{aligned} \|\nu AX + (1-\nu)XB\|_2^2 \ge{}& \nu^2 \|AX - XB\|_2^2 + \nu^{2\nu} \left\| A^\nu X B^{1-\nu} \right\|_2^2 \\ &+ 2\nu(1-\nu) \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2, \quad \text{if } 0 \le \nu \le \frac{1}{2} \\ \|\nu AX + (1-\nu)XB\|_2^2 \ge{}& (1-\nu)^2 \|AX - XB\|_2^2 + (1-\nu)^{2-2\nu} \left\| A^\nu X B^{1-\nu} \right\|_2^2 \\ &+ 2\nu(1-\nu) \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2, \quad \text{if } \frac{1}{2} \le \nu \le 1. \end{aligned}$$

Recently, Burqan and Khandaqji [8] gave the following reverse of the scalar Young type inequality:

$$\nu^2 a+(1-\nu)^2 b \leq (1-\nu)^2(\sqrt{a}-\sqrt{b})^2+a^\nu[(1-\nu)^2 b]^{1-\nu}, \quad \text{for } 0 \leq \nu \leq \frac{1}{2}, \tag{6}$$

$$\nu^2 a+(1-\nu)^2 b \leq \nu^2(\sqrt{a}-\sqrt{b})^2+\nu^{2\nu} a^\nu b^{1-\nu}, \quad \text{for } \frac{1}{2} \leq \nu \leq 1. \tag{7}$$

In this paper, we present refinements of inequalities (4) and (5) and reverses of inequalities (6) and (7). Based on the spectral theorem for positive matrices, we use these inequalities to establish corresponding inequalities for matrices. We give the trace norm, the Hilbert-Schmidt norm, and determinant versions of Young type inequalities based on the Young type inequalities (8) and (9).

2. Refinements of Young type inequalities

In this section, we give some Young type inequalities for scalars. Then by using these inequalities we establish corresponding Young type inequalities for matrices.

Lemma 2.1. *Suppose that $a, b \geq 0$ and r is any positive real number.*
(a) If $0 \leq \nu \leq \frac{1}{2}$, then

$$\left[(\nu a^r)^\nu\, b^{r(1-\nu)}\right]^2+\nu^2(a^r-b^r)^2 \leq \nu^2 a^{2r}+(1-\nu)^2 b^{2r}. \tag{8}$$

(b) If $\frac{1}{2} \leq \nu \leq 1$, then

$$\left[a^{r\nu}\left((1-\nu) b^r\right)^{1-\nu}\right]^2+(1-\nu)^2(a^r-b^r)^2 \leq \nu^2 a^{2r}+(1-\nu)^2 b^{2r}. \tag{9}$$

Proof. (a) Suppose that $0 \leq \nu \leq \frac{1}{2}$ and by the classical Young inequality, it holds that

$$\begin{aligned}
&\nu^2 a^{2r}+(1-\nu)^2 b^{2r}-\nu^2(a^r-b^r)^2 \\
=&\nu^2 a^{2r}+b^{2r}-2\nu b^{2r}+\nu^2 b^{2r}-\nu^2 a^{2r}+2\nu^2 a^r b^r-\nu^2 b^{2r} \\
=&b^r\left[(1-2\nu) b^r+2\nu\left(\nu a^r\right)\right] \geq b^r\left[b^{r(1-2\nu)}\left(\nu a^r\right)^{2\nu}\right]=\left[b^{r(1-\nu)}\left(\nu a^r\right)^\nu\right]^2 .
\end{aligned}$$

(b) If $\frac{1}{2} \leq \nu \leq 1$, then

$$\begin{aligned}
&\nu^2 a^{2r}+(1-\nu)^2 b^{2r}-(1-\nu)^2(a^r-b^r)^2 \\
=&\nu^2 a^{2r}+(1-\nu)^2 b^{2r}-(1-\nu)^2 a^{2r}+2(1-\nu)^2 a^r b^r-(1-\nu)^2 b^{2r} \\
=&\nu^2 a^{2r}-a^{2r}+2\nu a^{2r}-\nu^2 a^{2r}+2(1-\nu)^2 a^r b^r=a^r\left[(2\nu-1) a^r+2(1-\nu)^2 b^r\right] \\
\geq& a^r\left[a^{r(2\nu-1)}\left((1-\nu) b^r\right)^{2(1-\nu)}\right]=\left[a^{r\nu}\left((1-\nu) b^r\right)^{1-\nu}\right]^2 .
\end{aligned}$$

□

In this section, we give the trace norm, the Hilbert-Schmidt norm, and determinant versions of Young type inequalities based on the Young type inequalities (8) and (9). To do this, we need the following lemma.

Lemma 2.2 ([6]). *If $A, B \in M_n(\mathbb{C})$, then*

$$\sum_{i=1}^{n} s_i(AB) \leq \sum_{i=1}^{n} s_i(A)s_i(B).$$

Theorem 2.3. *Let $A, B \in M_n(\mathbb{C})$ be positive semi-definite and r be any positive real number.*
(a) If $0 \leq \nu \leq \frac{1}{2}$, then

$$\nu^{\nu} \left\| A^{r\nu} B^{r(1-\nu)} \right\|_1 \leq \sqrt{\nu^2 \left\| A^r \right\|_2^2 + (1-\nu)^2 \left\| B^r \right\|_2^2 - \nu^2 \left(\left\| A^r \right\|_2 - \left\| B^r \right\|_2 \right)^2}. \tag{10}$$

(b) If $\frac{1}{2} \leq \nu \leq 1$, then

$$(1-\nu)^{1-\nu} \left\| A^{r\nu} B^{r(1-\nu)} \right\|_1 \leq \sqrt{\nu^2 \left\| A^r \right\|_2^2 + (1-\nu)^2 \left\| B^r \right\|_2^2 - (1-\nu)^2 \left(\left\| A^r \right\|_2 - \left\| B^r \right\|_2 \right)^2}. \tag{11}$$

Proof. (a) If $0 \leq \nu \leq \frac{1}{2}$ and r is any positive real number, then by inequality (8), it holds that $\left[\left(\nu s_j^r(A) \right)^{\nu} \left(s_j^r(B^r) \right)^{1-\nu} \right]^2 + \nu^2 \left(s_j^r - s_j^r \right)^2 \leq \nu^2 s_j^{2r}(A) + (1-\nu)^2 s_j^{2r}(B)$, for all $j = 1, \ldots, n$. Thus, by the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&(\nu^2 A^{2r} + (1-\nu)^2 B^{2r}) = \nu^2 \operatorname{tr}(A^{2r}) + (1-\nu)^2 \operatorname{tr}(B^{2r}) = \sum_{j=1}^{n} \left[\nu^2 s_j^2(A^r) + (1-\nu)^2 s_j^2(B^r) \right] \\
&\geq \sum_{j=1}^{n} \left[(\nu s_j(A^r))^{\nu} (s_j(B^r))^{1-\nu} \right]^2 + \nu^2 \left(\sum_{j=1}^{n} s_j^2(A^r) + \sum_{j=1}^{n} s_j^2(B^r) - 2 \sum_{j=1}^{n} s_j(A^r) s_j(B^r) \right) \\
&\geq \sum_{j=1}^{n} \left[(\nu s_j(A^r))^{\nu} (s_j(B^r))^{1-\nu} \right]^2 \\
&\quad + \nu^2 \left[\left\| A^r \right\|_2^2 + \left\| B^r \right\|_2^2 - 2 \left(\sum_{j=1}^{n} s_j^2(A^r) \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} s_j^2(B^r) \right)^{\frac{1}{2}} \right] \\
&= \nu^{\nu} \sum_{j=1}^{n} \left[s_j(A^{r\nu}) s_j^r(B^{r(1-\nu)}) \right]^2 + \nu^2 \left(\left\| A^r \right\|_2 - \left\| B^r \right\|_2 \right)^2.
\end{aligned} \tag{12}
$$

On the other hand,

$$\nu^2 \operatorname{tr}(A^{2r}) + (1-\nu)^2 \operatorname{tr}(B^{2r}) = \nu^2 \left\| A^r \right\|_2^2 + (1-\nu)^2 \left\| B^r \right\|_2^2. \tag{13}$$

Therefore, it follows from (12) and (13) that

$$\nu^2 \left\| A^r \right\|_2^2 + (1-\nu)^2 \left\| B^r \right\|_2^2 - \nu^2 \left(\left\| A^r \right\|_2 - \left\| B^r \right\|_2 \right)^2 \geq \nu^{\nu} \sum_{j=1}^{n} \left[s_j(A^{r\nu}) s_j(B^{r(1-\nu)}) \right]^2. \tag{14}$$

By Lemma 2.2 and (14), it holds that

$$\nu^{\nu} \left\| A^{r\nu} B^{r(1-\nu)} \right\|_1 \leq \sqrt{\nu^2 \left\| A^r \right\|_2^2 + (1-\nu)^2 \left\| B^r \right\|_2^2 - \nu^2 \left(\left\| A^r \right\|_2 - \left\| B^r \right\|_2 \right)^2}.$$

(a) If $\frac{1}{2} \leq \nu \leq 1$, then by employing inequality (9) and the same reasoning as above, we obtain inequality (11). $\square$

THEOREM 2.4. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semi-definite and r be any positive real number.*
(a) If $0 \leq \nu \leq \frac{1}{2}$, then

$$\|\nu A^r X+(1-\nu)XB^r\|_2^2 \geq \nu^2 \|A^r X-XB^r\|_2^2 +\nu^{2\nu} \left\|A^{r\nu} XB^{r(1-\nu)}\right\|_2^2 +2\nu(1-\nu) \left\|A^{\frac{r}{2}} XB^{\frac{r}{2}}\right\|_2^2.$$

(b) If $\frac{1}{2} \leq \nu \leq 1$, then

$$\|\nu A^r X+(1-\nu)XB^r\|_2^2 \geq$$
$$(1-\nu)^2 \|A^r X-XB^r\|_2^2 +(1-\nu)^{2(1-\nu)} \left\|A^{r\nu} XB^{r(1-\nu)}\right\|_2^2 +2\nu(1-\nu) \left\|A^{\frac{r}{2}} XB^{\frac{r}{2}}\right\|_2^2. \quad (15)$$

Proof. Since every positive semi-definite matrix is unitarily diagonalizable, it follows that there are unitary matrices $U, V \in M_n(\mathbb{C})$ such that $A = UD_1U^*$ and $B = VD_2V^*$, where $D_1 = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $D_2 = \text{diag}(\mu_1, \ldots, \mu_n)$ and $\lambda_i, \mu_i \geq 0$ $(i = 1, \ldots, n)$. Let $Y = U^*XV = [y_{ij}]$. Then

$$\begin{aligned}
\nu A^r X+(1-\nu)XB^r &= U\left(\nu D_1^r+(1-\nu)YD_2^r\right)V^* = U\left(\left(\nu\lambda_i^r+(1-\nu)\mu_j^r\right)y_{ij}\right)V^* \\
A^r X-XB^r &= U\left(\left(\lambda_i^r-\mu_j^r\right)y_{ij}\right)V^* \\
A^{\frac{r}{2}} XB^{\frac{r}{2}} &= U\left(\lambda_i^{\frac{r}{2}}\mu_j^{\frac{r}{2}}y_{ij}\right)V^* \\
\text{and} \qquad A^{r\nu} XB^{r(1-\nu)} &= U\left(\lambda_i^{r\nu}\mu_j^{r(1-\nu)}y_{ij}\right)V^*
\end{aligned}$$

(a) If $0 \leq \nu \leq \frac{1}{2}$, then by inequality (8)) and the unitary invariance of the Hilbert-Schmidt norm, we have

$$\|\nu A^r X + (1-\nu)XB^r\|_2^2 = \sum_{i,j=1}^{n} \left(\nu\lambda_i^r + (1-\nu)\mu_j^r\right)^2 |y_{ij}|^2$$
$$= \sum_{i,j=1}^{n} \left(\nu^2\lambda_i^{2r} + (1-\nu)^2\mu_j^{2r} + 2\nu(1-\nu)\lambda_i^r\mu_j^r\right)|y_{ij}|^2$$
$$\geq \nu^2 \sum_{i,j=1}^{n} \left(\lambda_i^r - \mu_j^r\right)^2 |y_{ij}|^2 + \nu^{2\nu} \sum_{i,j=1}^{n} \left(\lambda_i^{r\nu}\mu_j^{r(1-\nu)}\right)^2 |y_{ij}|^2 + 2\nu(1-\nu) \sum_{i,j=1}^{n} \left(\lambda_i^r\mu_j^r\right)|y_{ij}|^2$$
$$\geq \nu^2 \|A^r X - XB^r\|_2^2 + \nu^{2\nu} \left\|A^{r\nu} XB^{r(1-\nu)}\right\|_2^2 + 2\nu(1-\nu) \left\|A^{\frac{r}{2}} XB^{\frac{r}{2}}\right\|_2^2.$$

(b) If $\frac{1}{2} \leq \nu \leq 1$, then by employing inequality (9) and the the same reasoning as above, we have inequality (15). $\square$

THEOREM 2.5. *Let $A, B \in M_n(\mathbb{C})$ be positive definite and r be any positive real number.*
(a) If $0 \leq \nu \leq \frac{1}{2}$, then

$$\det\left[\nu A^r + (1-\nu)B^r\right]^2 \geq \nu^{2n\nu} \det\left(A^{r\nu}B^{r(1-\nu)}\right)^2$$

$$+\,\nu^{2n}\det\left(A^r-B^r\right)^2+\left(2\nu\left(1-\nu\right)\right)^n\det\left(B^{\frac{r}{2}}A^rB^{\frac{r}{2}}\right).$$

(b) If $\frac{1}{2}\le\nu\le1$*, then*

$$\det\left[\nu A^r+(1-\nu)B^r\right]^2\ge(1-\nu)^{2n(1-\nu)}\det\left(A^{r\nu}B^{r(1-\nu)}\right)^2$$
$$+(1-\nu)^{2n}\det\left(A^r-B^r\right)^2+\left(2\nu\left(1-\nu\right)\right)^n\det\left(B^{\frac{r}{2}}A^rB^{\frac{r}{2}}\right).\quad(16)$$

Proof. (a) By inequality (8), we have
$\nu^{2\nu}\left[s_j^{\nu}\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)\right]^2+\nu^2\left[s_j\left(\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)\right)-1\right]^2\le\nu^2s_j^2\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)+(1-\nu)$
for all $j=1,\ldots,n$. Therefore

$$\det\left[\nu\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)+(1-\nu)I\right]^2=\prod_{j=1}^{n}\left[\nu s_j\left(\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)\right)+(1-\nu)\right]^2$$
$$\ge\prod_{j=1}^{n}\left[\nu^{2\nu}s_j^2\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)+\nu^2\left(s_j\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)-1\right)^2+2\nu(1-\nu)s_j\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)\right]$$
$$\ge\nu^{2n\nu}\prod_{j=1}^{n}s_j^{2\nu}\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)+\nu^{2n}\prod_{j=1}^{n}\left[s_j\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)-1\right]^2$$
$$+\,(2\nu(1-\nu))^n\prod_{j=1}^{n}s_j\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)$$
$$=\nu^{2n\nu}\det\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)^{2\nu}+\nu^{2n}\det\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}-I\right)^2+\left(2\nu\left(1-\nu\right)\right)^n\det\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right).$$

Thus, it holds that

$$\det\left(\nu A^r+(1-\nu)B^r\right)^2\ge\nu^{2n\nu}\det\left(A^{r\nu}B^{r(1-\nu)}\right)^2+\nu^{2n}\det\left(A^r-B^r\right)^2$$
$$+\,(2\nu(1-\nu))^n\det\left(B^{\frac{r}{2}}A^rB^{\frac{r}{2}}\right).$$

(b) If $\frac{1}{2}\le\nu\le1$ and r is any positive number, then by utilizing inequality (9) and following the same method as in the proof of (a), we arrive at (16). □

3. Reverses of Young type inequalities

In this section, we present some reverses of the Young type inequalities and give several refinements for matrices.

Lemma 3.1. *Let* $a,b\ge0$ *and* r *be any positive real number.*
(a) If $0\le\nu\le\frac{1}{2}$*, then*

$$\nu^{2(1-\nu)}\left[(2\nu-1)a^r+2(1-\nu)b^r\right]+a^{2r(1-\nu)}\nu^{2\nu}b^{(2\nu-1)r}\ge2\nu\sqrt{a^rb^r}.\quad(17)$$

(b) If $\frac{1}{2}\le\nu\le1$*, then*

$$\nu^{2\nu}\left[2(1-\nu)a^r+(2\nu-1)b^r\right]+a^{(2\nu-1)r}\nu^{2(1-\nu)}b^{2r(1-\nu)}\ge2\nu\sqrt{a^rb^r}.\quad(18)$$

Proof. (a) If $0 \le \nu \le \frac{1}{2}$ and r is any positive real number, then by inequality (1), we get

$$\begin{aligned} &\nu^{2(1-\nu)}\left[(2\nu-1)a^r+2(1-\nu)b^r\right]+a^{2r(1-\nu)}\nu^{2\nu}b^{(2\nu-1)r}-2\nu\sqrt{a^rb^r} \\ \ge&\nu^{2(1-\nu)}a^{(2\nu-1)r}b^{2r(1-\nu)}+a^{2r(1-\nu)}\nu^{2\nu}b^{(2\nu-1)r}-2\nu\sqrt{a^rb^r} \\ =&\left[\nu^{1-\nu}a^{\left(\frac{2\nu-1}{2}\right)r}b^{(1-\nu)r}-a^{(1-\nu)r}\nu^{\nu}b^{\left(\frac{2\nu-1}{2}\right)r}\right]^2\ge 0. \end{aligned}$$

(b) If $0 \le \nu \le \frac{1}{2}$ and r is any positive real number, then by inequality (1)), we have

$$\begin{aligned} &\nu^{2\nu}\left[2(1-\nu)a^r+(2\nu-1)b^r\right]+a^{(2\nu-1)r}\nu^{2(1-\nu)}b^{2(1-\nu)r}-2\nu\sqrt{a^rb^r} \\ \ge&\nu^{2\nu}a^{2r(1-\nu)}b^{(2\nu-1)r}+a^{(2\nu-1)r}\nu^{2(1-\nu)}b^{2r(1-\nu)}-2\nu\sqrt{a^rb^r} \\ =&\left[\nu^{\nu}a^{(1-\nu)r}b^{\left(\frac{2\nu-1}{2}\right)r}-a^{\left(\frac{2\nu-1}{2}\right)r}\nu^{1-\nu}b^{(1-\nu)r}\right]^2\ge 0. \end{aligned}$$

□

COROLLARY 3.2. *Let $a, b \ge 0$ and r be any positive real number.*
(a) If $0 \le \nu \le \frac{1}{2}$, then

$$\nu^{2(1-\nu)}\left[\frac{a^r+b^r}{2}\right]+\nu^{2\nu}\left[\frac{a^{2r(1-\nu)}b^{(2\nu-1)r}+a^{(2\nu-1)r}b^{2r(1-\nu)}}{2}\right]\ge 2\nu\sqrt{a^rb^r}. \qquad (19)$$

(b) If $\frac{1}{2} \le \nu \le 1$, then

$$\nu^{2\nu}\left[\frac{a^r+b^r}{2}\right]+\nu^{2(1-\nu)}\left[\frac{a^{(2\nu-1)r}b^{2r(\nu-1)}+a^{2r(\nu-1)}b^{(2\nu-1)r}}{2}\right]\ge 2\nu\sqrt{a^rb^r}.$$

Now, we obtain a matrix version of inequalities (17) and (18).

THEOREM 3.3. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive definite, $0 \le \nu \le 1$ and r be any positive real number.*
(a) If $0 \le \nu \le \frac{1}{2}$, then

$$2\nu\left\|A^{\frac{r}{2}}XB^{\frac{r}{2}}\right\|_2\le\nu^{1-\nu}\left\|(2\nu-1)A^rX+2(1-\nu)XB^r\right\|_2+\nu^{\nu}\left\|A^{2(1-\nu)r}XB^{(2\nu-1)r}\right\|_2.$$

(b) If $\frac{1}{2} \le \nu \le 1$, then

$$2\nu\left\|A^{\frac{r}{2}}XB^{\frac{r}{2}}\right\|_2\le\nu^{\nu}\left\|2(1-\nu)A^rX+(2\nu-1)XB^r\right\|_2+\nu^{\nu}\left\|A^{(2\nu-1)r}XB^{2r(\nu-1)}\right\|_2. \qquad (20)$$

Proof. Since every positive semi-definite matrix is unitarily diagonalizable, it follows that there are unitary matrices $U, V \in M_n(\mathbb{C})$ such that $A = UD_1U^*$ and $B = VD_2V^*$, where $D_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $D_2 = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ and $\lambda_i, \mu_i \ge 0$ $(i = 1, \ldots, n)$. Let $Y = U^*XV = [y_{ij}]$. Then

$$\begin{aligned} (2\nu-1)A^rX+2(1-\nu)XB^r &= U\left((2\nu-1)D_1^r+2(1-\nu)YD_2^r\right)V^* \\ &= U\left(\left((2\nu-1)\lambda_i^r+2(1-\nu)\mu_j^r\right)y_{ij}\right)V^* \\ A^{\frac{r}{2}}XB^{\frac{r}{2}} &= U\left(\lambda_i^{\frac{r}{2}}\mu_j^{\frac{r}{2}}y_{ij}\right)V^* \end{aligned}$$

and

$$A^{(2\nu-1)r}XB^{2r(1-\nu)} = U\left(\lambda_i^{(2\nu-1)r}\mu_j^{2r(1-\nu)}y_{ij}\right)V^*$$

(a) If $0 \le \nu \le \frac{1}{2}$, then by inequality (17) and the Minkowski inequality and the unitary invariance of the Hilbert-Schmidt norm, we obtain

$$\left\|2\nu A^{\frac{r}{2}}XB^{\frac{r}{2}}\right\|_2 = \sqrt{\sum_{i,j=1}^{n}\left[2\nu\lambda_i^{\frac{r}{2}}\mu_j^{\frac{r}{2}}\right]^2|y_{ij}|^2}$$

$$\le\sqrt{\sum_{i,j=1}^{n}\left[\nu^{2(1-\nu)}\left((2\nu-1)\lambda_i^r+2(1-\nu)\mu_j^r\right)+\nu^{2\nu}\lambda_i^{2r(1-\nu)}\mu_j^{(2\nu-1)r}\right]^2|y_{ij}|^2}$$

$$\le\nu^{1-\nu}\sqrt{\sum_{i,j=1}^{n}\left[(2\nu-1)\lambda_i^r+2(1-\nu)\mu_j^r\right]^2|y_{ij}|^2}+\nu^{1-\nu}\sqrt{\sum_{i,j=1}^{n}\left[\lambda_i^{2r(1-\nu)}\mu_j^{(2\nu-1)r}\right]^2|y_{ij}|^2}$$

$$=\nu^{1-\nu}\left\|(2\nu-1)A^rX+2(1-\nu)XB^r\right\|_2+\nu^{\nu}\left\|A^{2r(1-\nu)}XB^{(2\nu-1)r}\right\|_2.$$

(b) If $\frac{1}{2} \le \nu \le 1$, then by inequality (18) and the Minkowski inequality and the same method of proof of case (a), we get inequality (20). □

The Heinz mean is defined by $H_\nu(a,b) = \frac{1}{2}(a^\nu b^{1-\nu} + a^{1-\nu}b^\nu)$. The function H_ν is symmetric about the point $\nu = \frac{1}{2}$, i.e., $H_\nu(a,b) = H_{1-\nu}(a,b)$. Note that $H_0(a,b) = H_1(a,b) = \frac{a+b}{2}$, $H_{\frac{1}{2}}(a,b) = \sqrt{ab}$ and $H_{\frac{1}{2}}(a,b) \le H_\nu(a,b) \le H_0(a,b)$.

If $A, B \in M_n(\mathbb{C})$ are positive definite, the geometric mean of A and B, denoted by $A\sharp B$, is defined by $A\sharp B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}$. For $0 \le \nu \le 1$, the ν-weighted geometric mean, denoted by $A\sharp_\nu B$, is defined by $A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{1-\nu}A^{\frac{1}{2}}$. The ν-weighted geometric mean was introduced by Kubo and Ando [15], and when $\nu = \frac{1}{2}$ this is just the geometric mean. One can show that $A\sharp_\nu B = B\sharp_{1-\nu}A$ for $0 \le \nu \le 1$. When A and B commute, $A\sharp_\nu B = A^\nu B^{1-\nu}$. The ν-weighted arithmetic mean is $\nu A + (1-\nu)B$ and the Heinz mean with parameter ν is $\frac{1}{2}\left(A\sharp_\nu B + A\sharp_{1-\nu}\right)B$. The ν-weighted arithmetic mean of A and B, denoted by $A\nabla_\nu B$, is defined by $A\nabla_\nu B = (1-\nu)A + \nu B$.

THEOREM 3.4. *Let $A, B \in M_n(\mathbb{C})$ such that A and B are positive definite and r be any positive real number.*
(a) If $0 \le \nu \le \frac{1}{2}$, then $\nu^{2(1-\nu)}(A^r\nabla B^r) + \nu^{2\nu}H_{2\nu-1}(A^r,B^r) \ge 2\nu A^r\sharp B^r$, where $H_{2\nu-1}(A^r,B^r) = \frac{A^{(2\nu-1)r}B^{2r(1-\nu)}+A^{2r(1-\nu)}B^{(2\nu-1)r}}{2}$.
(b) If $\frac{1}{2} \le \nu \le 1$, then $\nu^{2\nu}(A^r\nabla B^r) + \nu^{2(1-\nu)}H_{2(1-\nu)}(A^r,B^r) \ge 2\nu A^r\sharp B^r$, where $H_{2(1-\nu)}(A^r,B^r) = \frac{A^{2r(1-\nu)}B^{(2\nu-1)r}+A^{(2\nu-1)r}B^{2r(1-\nu)}}{2}$.

Proof. (a) If $\nu \in [0,\frac{1}{2}]$, then by inequality (17) for $a > 0$ and $b = 1$ becomes $\nu^{2(1-\nu)}\left[(2\nu-1)a^r + 2(1-\nu)\right] + \nu^{2\nu}a^{2r(1-\nu)} \ge 2\nu\sqrt{a^r}$. Hence

$$\nu^{2(1-\nu)}\left[(2\nu-1)B^{-\frac{r}{2}}AB^{-\frac{r}{2}}+2(1-\nu)I\right]+\nu^{2\nu}\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)^{2(1-\nu)} \ge 2\nu\left(B^{-\frac{r}{2}}A^rB^{-\frac{r}{2}}\right)^{\frac{1}{2}}. \quad (21)$$

If we multiply inequality (21) by $B^{\frac{r}{2}}$ on the left and right, we get

$$\nu^{2(1-\nu)}\left[(2\nu-1)A^r+2(1-\nu)B^r\right]+\nu^{2\nu}\left(A^r\sharp_{1-2\nu}B^r\right)\geq 2\nu A^r\sharp B^r. \tag{22}$$

Now, replacing A^r by B^r and B^r by A^r, it holds that

$$\nu^{2(1-\nu)}\left[(2\nu-1)B^r+2(1-\nu)A^r\right]+\nu^{2\nu}\left(A^r\sharp_{2(1-\nu)}B^r\right)\geq 2\nu A^r\sharp B^r. \tag{23}$$

Adding the inequalities (22) and (23), we obtain

$$\nu^{2(1-\nu)}\left[A^r\nabla B^r\right]+\nu^{2\nu}H_{2\nu-1}(A^r,B^r)\geq 2\nu A^r\sharp B^r.$$

The proof of (b) is similar to (a), so we omit it. □

The matrix version of inequality $a\sharp b\leq H_\nu(a,b)\leq a\nabla b$ was proved by Bhatia and Davis [7], saying that if $0\leq\nu\leq 1$, then

$$\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|\leq\left\|\frac{A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu}{2}\right\|\leq\left\|\frac{AX+XB}{2}\right\|. \tag{24}$$

Bakherad and Moslehian [5] improved the Young inequality and obtained the following: $\|AX+XB\|_2^2+2(\nu-1)\|AX-XB\|_2^2\leq\left\|A^\nu XB^{1-\nu}+A^{1-\nu}XB^\nu\right\|_2^2$, where A and B are positive definite matrices, $X\in M_n(\mathbb{C})$ and $\nu>1$.

THEOREM 3.5. *Let $A,B,X\in M_n(\mathbb{C})$ such that A and B are positive definite and r be any positive real number.*

(a) If $\nu\in[0,\frac{1}{2}]$, then

$$\left\|\nu^{2(1-\nu)}\left(\frac{A^rX+XB^r}{2}\right)+\nu^{2\nu}\left(\frac{A^{(2\nu-1)r}XB^{2r(1-\nu)}+A^{2r(1-\nu)}XB^{(2\nu-1)r}}{2}\right)\right\|_2 \geq 2\nu\left\|A^{\frac{r}{2}}XB^{\frac{r}{2}}\right\|_2.$$

(b) If $\nu\in[\frac{1}{2},1]$, then

$$\left\|\nu^{2\nu}\left(\frac{A^rX+XB^r}{2}\right)+\nu^{2(1-\nu)}\left(\frac{A^{(2\nu-1)r}XB^{2r(1-\nu)}+A^{2r(1-\nu)}XB^{(2\nu-1)r}}{2}\right)\right\|_2 \geq 2\nu\left\|A^{\frac{r}{2}}XB^{\frac{r}{2}}\right\|_2.$$

Proof. Since every positive semi-define is unitarily diagonalizable, it follows that there are unitary matrices $U,V\in M_n(\mathbb{C})$ such that $A=UD_1U^*$ and $B=VD_2V^*$, where $D_1=\operatorname{diag}(\lambda_1,\ldots,\lambda_n)$, $D_2=\operatorname{diag}(\mu_1,\ldots,\mu_n)$ and $\lambda_i,\mu_i\geq 0$ $(i=1,\ldots,n)$. Let $Y=U^*XV=[y_{ij}]$. Then

$$\begin{aligned}
&\frac{A^{(2\nu-1)r}XB^{2r(1-\nu)}+A^{2r(1-\nu)}XB^{(2\nu-1)r}}{2}\\
=&\frac{(UD_1U^*)^{(2\nu-1)r}X(VD_2V^*)^{2r(1-\nu)}+(UD_1U^*)^{2r(1-\nu)}X(VD_2V^*)^{(2\nu-1)r}}{2}\\
=&\frac{UD_1^{(2\nu-1)r}U^*XVD_2^{2r(1-\nu)}V^*+UD_1^{2r(1-\nu)}U^*XVD_2^{(2\nu-1)r}V^*}{2}\\
=&U\left(\frac{D_1^{(2\nu-1)r}XD_2^{2r(1-\nu)}+D_1^{2r(1-\nu)}XD_2^{(2\nu-1)r}}{2}\right)V^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\| \frac{A^{(2\nu-1)r}XB^{2r(1-\nu)} + A^{2r(1-\nu)}XB^{(2\nu-1)r}}{2} \right\|_2^2 \\
&= \left\| \frac{D_1^{(2\nu-1)r}YD_2^{2r(1-\nu)} + D_1^{2r(1-\nu)}YD_2^{(2\nu-1)r}}{2} \right\|_2^2 \\
&= \sum_{i,j=1}^{n} \left[\frac{\lambda_i^{(2\nu-1)r}\mu_j^{2r(1-\nu)} + \mu_i^{(2\nu-1)r}\lambda_j^{2r(1-\nu)}}{2} \right]^2 |y_{ij}|^2.
\end{aligned}$$

Similarly, it holds that

$$\left\|A^{\frac{r}{2}}XB^{\frac{r}{2}}\right\|_2^2 = \sum_{i,j=1}^{n} \left[\lambda_i^{\frac{r}{2}}\mu_j^{\frac{r}{2}}\right]^2 |y_{ij}|^2, \qquad \left\|\frac{A^rX + XB^r}{2}\right\|_2^2 = \sum_{i,j=1}^{n} \left[\frac{\lambda_i^r + \mu_j^r}{2}\right]^2 |y_{ij}|^2.$$

It follows from inequality (19) that

$$\begin{aligned}
&\left\| \nu^{2(1-\nu)} \left(\frac{A^rX + XB^r}{2} \right) + \nu^{2\nu} \left(\frac{A^{(2\nu-1)r}XB^{2r(1-\nu)} + A^{2r(1-\nu)}XB^{(2\nu-1)r}}{2} \right) \right\|_2^2 \\
&= \sum_{i,j=1}^{n} \left[\nu^{2(1-\nu)} \left(\frac{\lambda_i^r + \mu_j^r}{2} \right) + \nu^{2\nu} \left(\frac{\lambda_i^{(2\nu-1)r}\mu_j^{2r(1-\nu)} + \mu_i^{(2\nu-1)r}\lambda_j^{2r(1-\nu)}}{2} \right) \right]^2 |y_{ij}|^2 \\
&\geq \sum_{i,j=1}^{n} \left[2\nu \left(\lambda_i^{\frac{r}{2}}\mu_j^{\frac{r}{2}} \right) \right]^2 |y_{ij}|^2 = (2\nu)^2 \left\|A^{\frac{r}{2}}XB^{\frac{r}{2}}\right\|_2^2.
\end{aligned}$$

The proof of part (b) is similar, so we omit it. □

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