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AN INERTIAL BREGMAN HYBRID ALGORITHM FOR APPROXIMATING SOLUTIONS OF FIXED POINT AND VARIATIONAL INEQUALITY PROBLEM IN REAL BANACH SPACES

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Abstract. In this work, we study an inertial extragradient-like S-iteration process for approximating a common element of the set of solutions of some variational inequality problem involving a monotone Lipschitz map and a fixed point of asymptotically nonexpansive mapping in a reflexive Banach space. The result in this paper is an extension and generalization of some recently announced results.

1. Introduction

Let E be a real Banach space and E^* be its dual space. Let C be a nonempty, closed and convex subset of E, and $A : C \to E^*$ be a mapping.

The problem of finding a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall\, v \in C, \tag{1}$$

is called a *variational inequality problem*. We denote the set of solutions of the variational inequality problem (1) by $VI(C, A)$. The problem of finding common elements of the set of solutions of variational inequality problems and the set of fixed points of operators has become an interesting area of contemporary research for numerous mathematicians working in nonlinear operator theory (see, e.g. [10,17] and the references theirin).

Let $A : C \to E^*$ be a mapping. Then A is said to be

– *k-Lipschitz continuous* if there exists a constant $k > 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|,\ \forall x, y \in C.$$

– *η-strongly monotone* if there exists a constant $\eta > 0$ such that:

$$\big\langle x - y, Ax - Ay\big\rangle \geq \eta\|x - y\|^2,\ \forall x, y \in C.$$

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– *δ-inverse strongly monotone* if there exists a $\delta > 0$, such that

$$\big\langle x-y, Ax-Ay\big\rangle \geq \delta\|Ax-Ay\|^2,\ \forall x,y\in C.$$

– *Monotone* if

$$\big\langle x-y, Ax-Ay\big\rangle \geq 0,\ \forall x,y\in C.$$

It is well-known that if A is k-Lipschitz continuous and η-strongly monotone on C, then the variational inequality (1) has a unique solution. Recently, Zhou et al. [24] weakened the Lipschitz continuity condition to the hemicontinuity. However, if A is just k-Lipschitz continuous and monotone on C, but not η-strongly monotone, then the variational inequality (1) may fail to have a solution (see, [17] for counter-examples).

Let $J : E \to 2^{E^*}$ be a normalized duality mapping defined by

$$J(x) = \{u^* \in E^* : \langle x, u^*\rangle = \|x\|^2 = \|u^*\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing. It is well known that if E is a smooth, strictly convex and reflexive Banach space then J is single-valued, one-to-one and onto. Let $f : E \to (-\infty,+\infty]$ be a proper, lower semi-continuous and convex function. We denote by $\operatorname{dom} f := \{u \in E : f(u) < +\infty\}$ the domain of f. Let $u \in \operatorname{intdom} f$; the subdifferential of f at u is the convex set defined by $\partial f(u) = \{u^* \in E^* : f(u) + \langle u^*, y-x\rangle \leq f(v),\ \forall v \in E\}$, where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty,+\infty]$ defined by $f^*(u^*) = \sup\{\langle u^*, u\rangle - f(u) : u \in E\}$. It is known that the Young-Fenchel inequality holds: $\langle u^*, u\rangle \leq f(u) + f^*(u^*)$, $\forall u \in E$.

A function f on E is said to be strongly convex with constant $\sigma > 0$, if

$$f(x) - f(y) \geq \langle\nabla f(x), y-x\rangle + \frac{\sigma}{2}\|y-x\|^2, \quad \forall x,y\in E. \tag{2}$$

Given a function $f : E \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, we denote by $\operatorname{slev}(f;\alpha)$ (resp. $\operatorname{lev}(f;\alpha)$) the sublevel set $\{x \in E : f(x) \leq \alpha\}$ (resp. the level set $\{x \in E : f(x) = \alpha\}$) of f at height α.

A function f on E is coercive if the sublevel set of f is bounded; equivalently, $\lim_{\|u\|\to+\infty} f(u) = +\infty$. A function f on E is said to be strongly coercive [23] if $\lim_{\|u\|\to+\infty} \frac{f(u)}{\|u\|} = +\infty$.

For any $u \in \operatorname{intdom} f$ and $y \in E$, the derivative of f at u in the direction y is defined by $f^\circ(u,v) := \lim_{t\to 0} \frac{f(u+tv)-f(u)}{t}$.

The notions of Gâteaux differentiable and Fréchet differentiable functions and their properties are well-known (see, e.g., [2, 5]).

Let E be a real Banach space and $f : E \to (-\infty,+\infty]$ be a strictly convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}(\operatorname{dom}(f)) \to [0,+\infty)$, defined by

$$D_f(x,y) := f(x) - f(y) - \langle\nabla f(y), x-y\rangle, \tag{3}$$

is called the Bregman distance with respect to f (see [6]).

Remark 1.1. If E is a smooth Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have $\nabla f(x) = 2Jx$ for all $x \in E$ where $J : E \to E^*$ is the normalized duality mapping. Hence $D_f(x,y) = \phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \forall\ x,y \in E$. Also if E is a Hilbert space, then $D_f(x,y) = \|x-y\|^2,\ \forall x,y \in E$.

Observe that from (3), we have for any $x \in \operatorname{dom} f$ and $y,z \in \operatorname{int}(\operatorname{dom}(f))$,

$$D_f(x,z) = D_f(x,y) + D_f(y,z) + \langle x-y, \nabla f(y) - \nabla f(z)\rangle, \tag{4}$$

which is called the three point identity. Recall that a mapping $T : C \to C$ is said to be:

– ϕ-asymptotically nonexpansive if there exists $\{k_n\} \subset [0,\infty)$ such that $\phi(T^n x, T^n y) \le (1+k_n)\phi(x,y)$, $\forall x,y \in C$, and $k_n \to 0$ as $n \to \infty$.

– quasi-ϕ-asymptotically nonexpansive if there exists $\{k_n\} \subset [0,\infty)$ such that $\phi(p, T^n x) \le (1+k_n)\phi(p,x)$, $\forall x \in C,\ p \in Fix(T)$ and $k_n \to 0$ as $n \to \infty$.

– Bregman asymptotically nonexpansive if there exists $\{k_n\} \subset [0,\infty)$ such that $D_f(T^n x, T^n y) \le (1+k_n)D_f(x,y)$, $\forall x,y \in C$ and $k_n \to 0$ as $n \to \infty$.

– Bregman quasi asymptotically nonexpansive if there exists $\{k_n\} \subset [0,\infty)$ such that $D_f(p, T^n x) \le (1+k_n)D_f(p,x)$, $\forall x \in C,\ p \in Fix(T)$ and $k_n \to 0$ as $n \to \infty$.

– uniformly L-Lipschitzian if there is exists a constant $L > 0$ such that $\forall x,\ y \in C$, $\|T^n x - T^n y\| \le L\|x-y\|,\ \forall\ n \le 1$.

– closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$ then $Tx = y$.

Many researchers have proposed and studied various iterative schemes for approximating solutions of variational inequality problems, fixed points of nonexpansive maps and their generalizations (see, for example Chidume [13] and the references therein).

In 1976, Korpelevič [16] introduced the following *extragradient method* in a finite dimensional Euclidean space $\mathbb{R}^n$:

$$x_1 = x \in C; \qquad x_{n+1} = P_C(x_n - \lambda A[P_C(x_n - \lambda Ax_n)]), \quad \forall\ n \in \mathbb{N}, \tag{5}$$

where A is assumed to be monotone and Lipschitz. The extragradient method has since then been studied and improved on by many authors in various ways.

Several mathematicians worked on various improvements of extragradient method—we mention Censor et al. [12], Agarwal et al. [1], Ceng and Yao [9], Ceng et al. [10]. In particular, motivated by the results of [9, 10], Chidume et al. [14] introduced a hybrid extragradient like algorithm in a uniformly smooth and 2-uniformly convex real Banach space.

Polyak [19] was the first to propose inertial-type algorithm as an acceleration process in solving a smooth convex optimization problems. Putting an inertial term in an algorithm speeds up or accelerates the rate of convergence of the sequence generated by the algorithm, consequently a lot of research interest is now devoted to the inertial-type algorithm (see, e.g., [15] and the references therein).

In this paper, motivated by the result of Chidume et al. [14], we introduce an inertial Bregman extragradient-like S-iteration process for approximating a common element of the set of solutions of some variational inequality problem involving a monotone Lipschitz map and a fixed point of Bregman asymptotically nonexpansive mapping in a reflexive Banach space. Our theorem is an improvement of some recently announced results in relation to the space and the map used.

2. Preliminaries

Definition 2.1 ([4]). The function f is said to be:
(i) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;

(ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of $\operatorname{dom} f$;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 2.2. Let E be a reflexive Banach space. Then we have the following results:
(i) f is essentially smooth iff f^* is essentially strictly convex (see [4, Theorem 5.4]);

(ii) $(\partial f)^{-1} = \partial f^*$ (see [5])

(iii) f is Legendre if and only if f^* is Legendre (see [4, Corrolary 5.5])

(iv) If f is Legendre, then $\bigtriangledown f$ is a bijection satifying $\bigtriangledown f = (\bigtriangledown f^*)^{-1}$, $\operatorname{ran}\bigtriangledown f = \operatorname{dom}\bigtriangledown f^* = \operatorname{intdom} f^*$ and $\operatorname{ran}\bigtriangledown f^* = \operatorname{dom} f = \operatorname{intdom} f$ (see [4, Theorem 5.10]).

Examples of Legendre functions were given in [3, 4].

The Bregman projection (see [11]) with respect to f of $x \in \operatorname{intdom} f$ onto a nonempty, closed and convex set $C \subset \operatorname{intdom} f$ is defined as the necessarily unique vector $\operatorname{Proj}^f_C x \in C$ which satisfies $D_f(\operatorname{Proj}^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}$.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of f at $x \in \operatorname{intdom} f$ is the function $v_f(x, \cdot) : [0, +\infty] \to [0, +\infty]$ defined by $v_f(x, t) := \inf\{D_f(x, y) : y \in \operatorname{dom} f, \|y - x\| = t\}$.

Note that from (2) and (3) we get $\frac{\sigma}{2}\|x - y\|^2 \le D_f(x, y)$, $\forall x, y \in E$. The function f is called totally convex at x if $v_f(x, t) > 0$ whenever $t > 0$. The function f is called convex if it is totally convex at any point $x \in \operatorname{intdom} f$ and is said to be totally convex on bounded set if $v_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function v_f : $\operatorname{intdom} f \times [0, +\infty) \to [0, +\infty)$ defined by $v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap \operatorname{dom} f\}$.

Definition 2.3. Let E be a Banach space with the dual E^* and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a mapping. Then A is said to be *hemicontinuous* if for any fixed $x, y \in C$, the function $h : [0, 1] \to \mathbb{R}$ defined by $h(t) = \langle A((1 - t)x + ty, x - y\rangle$ is continuous at 0^+.

Lemma 2.4 ([8]). *Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*
(a) $z = \operatorname{Proj}^f_C(x)$ *if and only if* $\langle y - z, \nabla f(x) - \nabla f(z)\rangle \le 0, \quad \forall y \in C$;

(b) $D_f(y, \operatorname{Proj}^f_C(x)) + D_f(\operatorname{Proj}^f_C(x), x) \le D_f(y, x), \quad \forall x \in E,\ y \in C.$

Example 2.5. Let E be a reflexive Banach space, C be a nonempty, closed and convex subset of E, and if the Legendre function $f : E \to (-\infty, +\infty]$ if uniformly Frechet differentiable and bounded on bounded subset of E, then the Bregman projection P^f_C

is closed Bregman relatively nonexpansive from E to C, so it is a closed Bregman quasi asymptotically nonexpansive mapping.

Example 2.6. Let E be a reflexive Banach space, $A : E \to 2^E$ be a maximal monotone mapping and E, and if the Legendre function $f : E \to (-\infty, +\infty]$ if uniformly Frechet differentiable and bounded on bounded subset of E, such that $A^{-1}(0) \neq \emptyset$ then the resolvent operator $\mathrm{Res}_A^f(x) = (\nabla f + A)^{-1} \cdot \nabla f(x)$ is closed Bregman relatively nonexpansive from E to C, so it is a closed Bregman quasi asymptotically nonexpansive mapping.

Lemma 2.7 ([18]). *Let E be a Banach space and $f : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then $\lim_{n\to\infty} D_f(x_n, y_n) = 0$ iff $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.8 ([20]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded then the sequence $\{x_n\}$ is bounded, too.*

Recall that the function f is called sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded, $\lim_{n\to+\infty} D_f(y_n, x_n) = 0$ implies $\lim_{n\to+\infty} \|y_n - x_n\| = 0$.

Lemma 2.9 ([18]). *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E. Let $r > 0$ be a constant, $B_r := \{z \in E : \|z\| \leq r\}$, $B_r^* := \{z^* \in E^* : \|z^*\| \leq r\}$ let ρ_r and ρ_r^* be the gauges of uniform convexity of g and g^* respectively. Then,*
(i) for any $x, y \in B_r$ and $\alpha \in (0, 1)$, $g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_r(\|x - y\|)$;

(ii) for any $x, y \in B_r$, $\rho_r(\|x - y\|) \leq D_g(x, y)$;

(iii) if in addition g is bounded on bounded subsets of E and uniformly convex on bounded subsets of E, then for any $x \in E, y^*$, $z^* \in B_r^*$ and $\alpha \in (0, 1)$, $V_g(x, \alpha y^* + (1-\alpha)z^*) \leq \alpha V_g(x, y^*) + (1-\alpha)V_g(x, z^*) - \alpha(1-\alpha)\rho_r^*(\|y^* - z^*\|)$;

(iv) if in addition g is bounded on bounded subsets of E, uniformly convex and uniformly smooth on bounded subsets of E, then for any $x \in E, y^*, z^* \in B_r^*$, $\rho_r^*(\|x^* - y^*\|) \leq D_g(x^*, y^*)$.

Lemma 2.10 ([8]). *Let E be a reflexive Banach space, let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and let V be the function defined by $V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*)$, $x \in E$, $x^* \in E^*$. Then the following assertions hold:*
(i) $D_f(x, \nabla f(x^*)) = V_f(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.

(ii) $V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^*\rangle \leq V_f(x, x^* + y^*)$, for all $x \in E$ and $x^*, y^* \in E^*$.

LEMMA 2.11 ([22]). *Let C be a nonempty closed and convex subset of a reflexive Banach space E and $B : C \to E^*$ be a monotone, hemicontinuous map. Let $T : E \to 2^{E^*}$ be an operator defined by:*

$$Tu = \begin{cases} Bu + N_C(u), & u \in C, \\ \emptyset, & u \notin C, \end{cases}$$

where $N_C(u)$ is defined as follows: $N_C(u) = \{w^* \in E^* : \langle u - z, w^* \rangle \geq 0, \ \forall z \in C\}$. Then, T is maximal monotone and $T^{-1}0 = VI(C, B)$.

3. Main results

THEOREM 3.1. *Let C be a nonempty closed and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $A : C \to E^*$ be a monotone, k-Lipschitz map and let $S_1, S_2 : C \to C$ be closed Bregman quasi asymptotically nonexpansive maps with sequences $\{k_n^1\}$ and $\{k_n^2\}$, respectively, where S_1, S_2 are uniformly Lipschitzian, such that $\bigcap_{i=1}^2 Fix(S_i) \neq \emptyset$ and assume $\Omega = (\bigcap_{i=1}^2 Fix(S_i)) \cap VI(C, A) \neq \emptyset$. Define iteratively the sequence $\{x_n\}$ by*

$$\begin{cases} x_0, x_1 \in C_0 = C; \\ w_n = x_n + \alpha_n(x_n - x_{n-1}); \\ z_n = \mathrm{Proj}_C^f(\nabla f^*(\nabla f w_n - \lambda A w_n)); \\ y_n = \mathrm{Proj}_C^f[\nabla f^*[(1 - \beta_n)\nabla f w_n + \beta_n \nabla f(S_1^n w_n)]; \\ u_n = \nabla f^*[(1 - a_n - \gamma_n)\nabla f w_n + a_n \nabla f(S_1^n y_n) \\ \qquad + \gamma_n \nabla f S_2^n \,\mathrm{Proj}_C^f \nabla f^*(\nabla f w_n - \lambda A z_n)]; \\ C_{n+1} = \{u \in C_n : D_f(u, u_n) \leq D_f(u, w_n) + \eta_n\}; \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}}^f x_0, \quad \forall n \geq 0 \end{cases} \tag{6}$$

where $\eta_n = l_n(k_n - 1) \sup_{p \in \Omega} D_f(p, w_n)$ for all $w_n \in C$, with $l_n = \max_{n \geq 1}\{\beta_n, a_n + \gamma_n\}$ and $\lambda \in (0, b]$, with $b < \frac{\sigma}{k}$, $\sigma > 0$ and $\{a_n\}$, $\{a_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy:
(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\{\alpha_n\} \subset [0, \alpha]$ and $0 \leq \alpha < 1$;
(ii) $\{\beta_n\}, \{\gamma_n\} \subset (0, 1)$. (iii) $\{a_n\} \subset [0, 1]$ and $a_n \to 0$ as $n \to \infty$.
Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are well defined and converge strongly to $q = \mathrm{Proj}_{(\bigcap_{i=1}^2 Fix(S_i)) \cap VI(C,A)}^f x_0$.

Proof. **Step 1.** We show that $(\bigcap_{i=1}^2 Fix(S_i)) \cap VI(C, A) \subset C_n$. $\forall n \in \mathbb{N}$. Let us start by setting $k_n = \max_{n \geq 1}\{k_n^1, k_n^2\}$. Let $p \in (\bigcap_{i=1}^2 Fix(S_i)) \cap VI(C, A)$ be arbitrary. Clearly $p \in C_0 = C$. Now assume $p \in C_n$ for some $n \geq 0$. Next we show that $p \in C_{n+1}$. Set $v_n = \mathrm{Proj}_C^f \nabla f^*[\nabla f w_n - \lambda A z_n]$. Then by applying Lemma 2.4, Lemma 2.10 and using definition of variational inequality,we have

$$D_f(p, v_n) = D_f(p, \mathrm{Proj}_C^f \nabla f^*[\nabla f w_n - \lambda A z_n])$$

$$
\begin{aligned}
&\leq D_f(p, \nabla f^*(\nabla f w_n - \lambda A z_n)) - D_f(v_n, \nabla f^*(\nabla f w_n - \lambda A z_n))\\
&= V_f(p, \nabla f w_n - \lambda A z_n) - V_f(v_n, \nabla f w_n - \lambda A z_n)\\
&= f(p) - \langle p, \nabla f w_n - \lambda A z_n\rangle + f^*(\nabla f w_n - \lambda A z_n) - f(v_n)\\
&\quad + \langle v_n, \nabla f w_n - \lambda A z_n\rangle - f^*(\nabla f w_n - \lambda A z_n)\\
&= f(p) - \langle p, \nabla f w_n\rangle + f^*(\nabla f w_n) - [f(v_n) - \langle v_n, \nabla f w_n\rangle + f^*(\nabla f w_n)] + \lambda\langle p - v_n, A z_n\rangle\\
&= V_f(p, \nabla f w_n) - V_f(v_n, \nabla f w_n) + \lambda\langle p - v_n, A z_n\rangle\\
&= V_f(p, \nabla f w_n) - V_f(v_n, \nabla f w_n) + \lambda\langle p - z_n, A z_n\rangle + \lambda\langle z_n - v_n, A z_n\rangle\\
&\leq V_f(p, \nabla f w_n) - V_f(v_n, \nabla f w_n) + \lambda\langle z_n - v_n, A z_n\rangle\\
&= D_f(p, \nabla f^*(\nabla f w_n)) - D_f(v_n, \nabla f^*(\nabla f w_n)) + \lambda\langle z_n - v_n, A z_n\rangle\\
&= D_f(p, w_n) - D_f(v_n, w_n) + \lambda\langle z_n - v_n, A z_n\rangle\\
&= D_f(p, w_n) - D_f(v_n, z_n) - D_f(z_n, w_n) - \langle z_n - v_n, \nabla f z_n - \nabla f w_n\rangle + \lambda\langle z_n - v_n, A z_n\rangle\\
&= D_f(p, w_n) - D_f(v_n, z_n) - D_f(z_n, w_n) + \langle v_n - z_n, \nabla f z_n - \nabla f w_n - \lambda A z_n\rangle\\
&= D_f(p, w_n) - D_f(v_n, z_n) - D_f(z_n, w_n) + \langle v_n - z_n, \nabla f z_n - \nabla f w_n - \lambda A w_n\rangle\\
&\quad + \lambda\langle v_n - z_n, A w_n - A z_n\rangle\\
&\leq D_f(p, w_n) - D_f(v_n, z_n) - D_f(z_n, w_n) + \lambda\langle v_n - z_n, A w_n - A z_n\rangle\\
&\leq D_f(p, w_n) - D_f(v_n, z_n) - D_f(z_n, w_n) + \lambda\|v_n - z_n\|\|A w_n - A z_n\|\\
&\leq D_f(p, w_n) - D_f(v_n, z_n) - D_f(z_n, w_n) + \frac{\lambda k}{2}[\|v_n - z_n\|^2 + \|z_n - w_n\|^2]\\
&\leq D_f(p, w_n) - D_f(v_n, z_n) - D_f(z_n, w_n) + \frac{\lambda k}{\sigma}[D_f(v_n, z_n) + D_f(z_n, w_n)]\\
&\leq D_f(p, w_n) - \left(1 - \frac{\lambda k}{\sigma}\right) D_f(v_n, z_n) - \left(1 - \frac{\lambda k}{\sigma}\right) D_f(z_n, w_n)
\end{aligned}
$$

Hence
$$D_f(p, v_n) \leq D_f(p, w_n). \tag{7}$$

From algorithm (6) and definition of Bregman quasi asymptotically nonexpansivity of S_1 we have

$$
\begin{aligned}
D_f(p, y_n) &= D_f(p, \mathrm{Proj}_C^f[\nabla f^*[(1 - \beta_n)\nabla f w_n + \beta_n \nabla f(S_1^n w_n)]])\\
&\leq (1 - \beta_n) D_f(p, w_n) + \beta_n D_f(p, S_1^n w_n) \leq (1 - \beta_n) D_f(p, w_n) + \beta_n k_n D_f(p, w_n)\\
&= D_f(p, w_n) + \beta_n(k_n - 1) D_f(p, w_n) \leq D_f(p, w_n) + l_n(k_n - 1) D_f(p, w_n)\\
&\leq D_f(p, w_n) + \eta_n. \qquad (8)
\end{aligned}
$$

Also, using Lemma 2.4 and equation (4) we have

$$
\begin{aligned}
&D_f(p, z_n) = D_f(p, \mathrm{Proj}_C^f \nabla f^*[\nabla f w_n - \lambda A w_n]) \leq D_f(p, \nabla f^*[\nabla f w_n - \lambda A w_n])\\
&= f(p) - f(\nabla f^*[\nabla f w_n - \lambda A w_n]) - \langle p - \nabla f^*[\nabla f w_n - \lambda A w_n], \nabla f w_n - \lambda A w_n\rangle\\
&= f(p) - f(\nabla f^*[\nabla f w_n - \lambda A w_n]) + D_f(p, w_n) + D_f(\nabla f^*[\nabla f w_n - \lambda A w_n], \nabla f^*[\lambda A w_n])\\
&\quad - D_f(p, \nabla f^*[\lambda A w_n]) - D_f(\nabla f^*[\nabla f w_n - \lambda A w_n], w_n)\\
&= f(p) - f(\nabla f^*[\nabla f w_n - \lambda A w_n]) + D_f(p, w_n) + D_f(\nabla f^*[\nabla f w_n - \lambda A w_n], \nabla f^*[\lambda A w_n])\\
&\quad - f(p) + f(\nabla f^*(\lambda A w_n)) + \langle w_n - w_n, \lambda A w_n\rangle + \langle w_n - \nabla f^*(\lambda A w_n), \lambda A w_n\rangle
\end{aligned}
$$

$-D_f(\nabla f^*[\nabla f w_n - \lambda A w_n], w_n) = D_f(p, w_n)$.

Therefore we have

$$D_f(p, z_n) \leq D_f(p, w_n). \tag{9}$$

Thus, from the inequalities (7) and (8), we have

$$\begin{aligned} D_f(p, u_n) &= D_f(p, \nabla f^*[(1 - a_n - \gamma_n)\nabla f w_n + a_n \nabla f(S_1^n y_n) + \gamma_n \nabla f(S_2^n v_n)]) \\ &\leq (1 - a_n - \gamma_n) D_f(p, w_n) + a_n D_f(p, S_1^n y_n) + \gamma_n D_f(p, S_2^n v_n) \\ &\leq (1 - a_n - \gamma_n) D_f(p, w_n) + a_n k_n D_f(p, y_n) + \gamma_n k_n D_f(p, v_n) \\ &\leq (1 - a_n - \gamma_n) D_f(p, w_n) + a_n k_n D_f(p, w_n) + \gamma_n k_n D_f(p, w_n) \\ &= D_f(p, w_n) + a_n(k_n - 1) D_f(p, w_n) + \gamma_n(k_n - 1) D_f(p, w_n) \\ &= D_f(p, w_n) + (a_n + \gamma_n)(k_n - 1) D_f(p, w_n) \\ &\leq D_f(p, w_n) + l_n(k_n - 1) D_f(p, w_n) \leq D_f(p, w_n) + \eta_n \end{aligned} \tag{10}$$

Therefore $p \in C_{n+1}$. So we conclude that $(\bigcap_{i=1}^2 Fix(S_i)) \cap VI(C, A) \subset C_{n+1}$.

Step 2. We show that the sequence $\{x_n\}$ is well defined in $C_n, \forall n \geq 0$. It is enough to show that C_{n+1} is closed and convex, i.e. for $n \geq 1$:

$$C_{n+1} = \Big\{u \in C_n : \langle u, \nabla f w_n - \nabla f u_n\rangle - \langle \nabla f(w_n), w_n\rangle + \langle \nabla f(u_n), u_n\rangle \leq f(u_n) - f(w_n) + \eta_n \Big\}.$$

It is not difficult to see C_{n+1} is closed and convex which contain nonempty element. Therefore the iterative sequence is well defined.

Step 3. We show the following:

(a) $\{w_n\}$ is bounded; (b) $\|w_n - x_n\| \to 0$; (c) $\|S_1^n w_n - w_n\| \to 0$;
(d) $\|S_2^n v_n - v_n\| \to 0$; (e) $\lim_{n\to\infty} \|S_2^n w_n - w_n\| = 0 = \lim_{n\to\infty} \|S_1^n w_n - w_n\|$.

Since $x_{n+1} \in C_{n+1} \subset C_n$, then $D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0)$. Hence, $\{D_f(x_n, x_0)\}$ is nondecreasing:

$$D_f(x_n, x_0) = D_f(\text{Proj}_{C_n}^f x_0, x_0) \leq D_f(p, x_0) - D_f(p, x_n) \leq D_f(p, x_0),$$

which implies that $\{D_f(x_n, x_0)\}$ is bounded. Hence, by Lemma 2.8, $\{x_n\}$ is bounded. Consequently, $\{D_f(x_n, x_0)\}$ is convergent. Let $m, n \in \mathbb{N}$ with $m > n$; then

$$D_f(x_m, x_n) = D_f(x_m, \text{Proj}_{C_n}^f x_0) \leq D_f(x_m, x_0) - D_f(x_n, x_0) \to 0, \text{ as } n, m \to \infty.$$

Hence, $\|x_m - x_n\| \to 0$ as $n, m \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence and

$$\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty. \tag{11}$$

From the definition of w_n and (11), $\|w_n - x_n\| = \alpha_n \|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Thus,

$$\|w_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{12}$$

Hence $\{w_n\}$ is bounded. By (11) and (12) we have, $\|x_{n+1} - w_n\| \to 0$ as $n \to \infty$. By Lemma 2.7 we have $D_f(w_{n+1}, x_n) \to 0$ as $n \to \infty$. Also, since $x_{n+1} \subset C_{n+1}$ $D_f(x_{n+1}, u_n) \leq D_f(x_{n+1}, w_n) \to 0$ as $n \to \infty$, and using Lemma 2.7 again, we have

$$\|x_{n+1} - u_n\| \to 0 \text{ as } n \to \infty. \tag{13}$$

Therefore $\{u_n\}$ is bounded. Also, $D_f(x_{n+1}, z_n) \leq D_f(x_{n+1}, w_n) \to 0$ as $n \to \infty$ and

$$\|x_{n+1} - z_n\| \to 0 \; as \; n \to \infty. \tag{14}$$

So, $\{z_n\}$ is bounded and $\{y_n\}$, too. Since

$$D_f(x_{n+1}, y_n) \leq D_f(x_{n+1}, w_n) - D_f(y_n, w_n) \leq D_f(x_{n+1}, w_n)$$
$$D_f(x_{n+1}, y_n) \leq D_f(x_{n+1}, w_n) \to 0 \quad \text{as} \quad n \to \infty,$$

then $\quad \|x_{n+1} - y_n\| \to 0 \quad \text{as} \quad n \to \infty.$

By (4) and the uniform continuity of ∇f on bounded sets, as $n \to \infty$, we have

$$\begin{aligned} D_f(w_n, y_n) &= D_f(w_n, x_n) + D_f(x_n, y_n) + \langle x_n - w_n, \nabla f y_n - \nabla f x_n \rangle \\ &\leq D_f(w_n, x_n) + D_f(x_n, y_n) + \|x_n - w_n\| \|\nabla f y_n - \nabla f x_n\| \to 0. \end{aligned}$$

Then by Lemma 2.7 we have

$$\|w_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{15}$$

From the definition of $\{y_n\}$ and Lemmas 2.4, 2.9 and 2.10, we have

$$\begin{aligned} &D_f(p, y_n) = D_f(p, \mathrm{Proj}_C^f[\nabla f^*[(1-\beta_n)\nabla f w_n + \beta_n \nabla f(S_1^n w_n)]]) \\ &\leq D_f(p, \nabla f^*[(1-\beta_n)\nabla f w_n + \beta_n \nabla f(S_1^n w_n)]) \\ &= V_f(p, (1-\beta_n)\nabla f w_n + \beta_n \nabla f(S_1^n w_n)) \\ &\leq (1-\beta_n) V_f(p, \nabla f w_n) + \beta_n V_f(p, \nabla f(S_1^n w_n)) - (1-\beta_n)\beta_n \rho_r^*(\|\nabla f w_n - \nabla f(S_1^n w_n))\| \\ &= (1-\beta_n) D_f(p, w_n) + \beta_n D_f(p, S_1^n w_n) - (1-\beta_n)\beta_n \rho_r^*(\|\nabla f w_n - \nabla f(S_1^n w_n)\| \\ &\leq (1-\beta_n) D_f(p, w_n) + \beta_n D_f(p, w_n) - (1-\beta_n)\beta_n p_s^*\big(\|\nabla f(w_n) - \nabla f(S_1^n w_n)\|\big) \\ &= D_f(p, w_n) - (1-\beta_n)\beta_n p_r^*\big(\|\nabla f(w_n) - \nabla f(S_1^n w_n)\|\big). \end{aligned}$$

Thus,

$$(1 - \beta_n)\beta_n p_r^*\big(\|\nabla f(w_n) - \nabla f(S_1^n w_n)\|\big) \leq D_f(p, w_n) - D_f(p, y_n). \tag{16}$$

Observe that

$$\begin{aligned} &D_f(p, w_n) - D_f(p, y_n) = f(y_n) - f(w_n) - \langle \nabla f w_n, w_n - p \rangle + \langle \nabla f y_n, y_n - p \rangle \\ &= f(y_n) - f(w_n) + \langle \nabla f y_n, w_n - p \rangle + \langle \nabla f y_n, y_n - w_n \rangle - \langle \nabla f w_n, w_n - p \rangle \\ &= f(y_n) - f(w_n) + \langle \nabla f y_n - \nabla f w_n, w_n - p \rangle + \langle \nabla f y_n, y_n - w_n \rangle. \end{aligned}$$

So, $|D_f(p, w_n) - D_f(p, y_n)| \leq |f(y_n) - f(w_n)| + |\langle \nabla f y_n - \nabla f w_n, w_n - p \rangle| + |\langle \nabla f y_n, y_n - w_n \rangle|$. By (15), it follows $|D_f(p, w_n) - D_f(p, y_n)| \to 0$ as $n \to \infty$. This and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, by using (16), yields $\lim_{n\to\infty} p_r^*\big(\|\nabla f(w_n) - \nabla f(S_1^n w_n)\|\big) = 0$. From the property of p_r^* we deduce $\lim_{n\to\infty} p_r^*\big(\|\nabla f(w_n) - \nabla f(S_1^n w_n)\|\big) = 0$. By uniform continuity of ∇f^* on bounded subsets of E, we obtain $\lim_{n\to\infty} \|S_1^n w_n - w_n\| = 0$. From (11), (12) and (13) we have

$$\|u_n - w_n\| \to 0 \text{ as } n \to \infty. \tag{17}$$

Now from Lemma 2.4 we have

$$D_f(v_n, y_n) = D_f(w_n, y_n) - D_f(w_n, v_n) \leq D_f(w_n, y_n) \to 0 \;\; \text{as} \;\; n \to \infty.$$

By Lemma 2.7 and (15) we have
$$\|v_n - w_n\| \to 0 \text{ as } n \to \infty. \tag{18}$$
From (17) and (18) we have $\|v_n - u_n\| \to 0$ as $n \to \infty$. Now from the definition of u_n
$$\begin{aligned}\nabla f u_n &= (1 - a_n - \gamma_n)\nabla f w_n + a_n \nabla f(S_1^n y_n) + \gamma_n \nabla f(S_2^n v_n)\\ &= (1 - a_n - \gamma_n)\nabla f w_n + a_n \nabla f(S_1^n y_n) + \gamma_n \nabla f(S_2^n v_n) - \gamma_n \nabla f v_n + \gamma_n \nabla f v_n\end{aligned}$$
which gives
$$\nabla f u_n - \nabla f w_n = a_n(\nabla f(S_1 y_n) - \nabla f w_n) + \gamma_n(\nabla f v_n - \nabla f w_n) + \gamma_n(\nabla f(S_2 v_n) - \nabla f v_n)$$
Using the uniform continuity of ∇f and ∇f^* on bounded sets, we have that
$$\|S_2 v_n - v_n\| \to 0 \text{ as } n \to \infty$$
and
$$\begin{aligned}\|S_2^n w_n - w_n\| &= \|S_2^n w_n - S_2^n v_n + S_2^n v_n - v_n + v_n - w_n\|\\ &\le \|S_2^n w_n - S_2^n v_n\| + \|S_2^n v_n - v_n\| + \|v_n - w_n\|\\ &\le \|w_n - v_n\| + \|S_2^n v_n - v_n\| + \|v_n - w_n\| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$
So we have
$$\lim_{n\to\infty} \|S_2^n w_n - w_n\| = 0 = \lim_{n\to\infty} \|S_1^n w_n - w_n\|. \tag{19}$$

Step 4. Since $\{x_n\}$ is a Cauchy sequence and E is a reflexive Banach space, then there exists $x^* \in C$ such that $x_n \to x^*$. From (12) we obtain
$$\lim_{n\to\infty} \|w_n - x^*\| = 0. \tag{20}$$
Hence by (20) and (19) we have
$$\begin{aligned}\|S_1 w_n - x*\| &\le \|S_1 w_n - w_n\| + \|w_n - x^*\| \qquad (21)\\ &\le \|S_1 w_n - S_1^n w_{n-1}\| + \|S_1^n w_{n-1} - w_n\| + \|w_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$
Similarly we have
$$\begin{aligned}\|S_2 w_n - x*\| &\le \|S_2 w_n - w_n\| + \|w_n - x^*\| \qquad (22)\\ &\le \|S_2 w_n - S_2^n w_{n-1}\| + \|S_2^n w_{n-1} - w_n\| + \|w_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$
By (20), (21) and (22) together with closedness of S_1 and S_2 we get $S_2 x^* = x^*$ and $S_1 x^* = x^*$. Thus, we have $x^* \in \bigcap_{i=1}^2 Fix(S_i)$.

This implies that from (18) and (20) we have $v_n \to x^*$ as $n \to \infty$.

Next we show that $x^* \in VI(C, A)$. Let
$$Tv = \begin{cases} Av + N_C(v) & \text{if} \quad v \in C;\\ \emptyset & \text{if} \quad v \neq C.\end{cases}$$
Then, by Lemma 2.11, T is maximal monotone and $T^{-1}(0) = VI(C, A)$, i.e. $v \in T^{-1}(0)$ iff $v \in VI(C, A)$.

Claim: $(x^*, 0) \in G(T)$.

Let $(v, x) \in G(T)$; then it is enough to show that $\langle v - x^*, x\rangle \ge 0$. Now,
$$(v, x) \in G(T) \Rightarrow x \in Tv = Av + N_C(v) \Rightarrow x - Av \in N_C(v).$$

Therefore $\langle v-y, x-Av\rangle \geq 0 \ \forall y \in C$. Since $v_n = \text{Proj}_C^f(\nabla f^*(\nabla f w_n - \lambda A z_n))$ and $v \in C$ we have by generalised projection properties $\langle v_n - v, \nabla f w_n - \lambda A z_n - \nabla f v_n\rangle \geq 0$. Thus, $\langle v - v_n, \frac{\nabla f v_n - \nabla f w_n}{\lambda} + Az_n\rangle \geq 0$, $n \geq 0$. Using the fact that $v_n \in C$ and $x - Av \in N_C(v)$, we have

$$\begin{aligned}\langle v-v_n, x\rangle &\geq \langle v-v_n, Av\rangle \geq \langle v-v_n, Av\rangle - \langle v-v_n, \frac{\nabla f v_n - \nabla f w_n}{\lambda} + Az_n\rangle \\ &= \langle v-v_n, Av - Av_n\rangle + \langle v-v_n, Av_n - Az_n\rangle - \langle v-v_n, \frac{\nabla f v_n - \nabla f w_n}{\lambda}\rangle \\ &\geq \langle v-v_n, Av_n - Az_n\rangle - \langle v-v_n, \frac{\nabla f v_n - \nabla f w_n}{\lambda}\rangle.\end{aligned}$$

Hence, as $n \to \infty$, we have $\langle v - x^*, x\rangle \geq 0$. Therefore, $x^* \in \bigcap_{i=1}^2 Fix(S_i) \cap VI(C, A)$.

Step 5. We show that $\{x_n\}$ converges strongly to $q = \text{Proj}^f_{(\bigcap_{i=1}^2 Fix(S_i)) \cap VI(C,A)} x_0$.

As $x_n \to x^*$ and $x^* \in \bigcap_{i=1}^2 Fix(S_i) \cap VI(C, A)$, using the lower semi continuity of $D_f(\cdot, x_0)$ and the fact that $q = \text{Proj}^f_{(\bigcap_{i=1}^2 Fix(S_i)) \cap VI(C,A)} x_0$, we have

$$D_f(q, x_0) \leq D_f(x^*, x_0) = \lim_{n\to\infty} D_f(x_n, x_0) \leq \lim_{n\to\infty} \big(D_f(q, x_0) - D_f(q, x_n)\big) \leq D_f(q, x_0).$$

The last inequality follows from the fact that $x_n = \text{Proj}^f_{C_n} x_0$. This implies that $D_f(q, x_0) = D_f(x^*, x_0)$. But, since $x^* \in \bigcap_{i=1}^2 Fix(S_i) \cap VI(C, A)$, we have $D_f(x^*, q) \leq D_f(x^*, x_0) - D_f(q, x_0) = 0$. Hence $D_f(x^*, q) = 0 \Rightarrow x^* = q$.

Thus, $\{x_n\}$ converges strongly to x^*. □

Remark 3.2. Our theorem generalises the main theorem of Chideme et al [14] in the following senses:

– If the function f from our result is $f(x)=\|x\|^2$, then D_f coincide with ϕ and Bregman quasi asymptotically nonexpansive map reduces to asymptotically quasi ϕ nonexpansive map, thus a relatively nonexpansive map is a special case of Bregman quasi asymptotically nonexpansive ones. As every relatively nonexpansive map is quasi ϕ nonexpansive which is in turn asymptotically quasi ϕ nonexpansive, the operator studied in this paper is a generalization of those studied in [14].

– It is known that every 2-uniformly convex Banach space is reflexive; hence, the current result extends the result [14] to a more general setting of Banach space.

– Another advantage of this result over some in the literature is the involvement of the inertial term in the scheme which speed up the convergence rate of the sequences. This is illustrated and confirmed using a numerical example given below.

4. A numerical example

In order to justify Theorem 3.1, we give the following numerical example.

Example 4.1. Let $E = \mathbb{R}$, $C = [-1, 1]$ and let $S_i^n : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ be defined by $S_1^n x = S_2^n x = \frac{x}{2}$, $\forall x \in \mathbb{R}$. Let the mapping $A : C \to \mathbb{R}$ be defined by $Ax = 3x$,

$\forall x \in C$. Let $f(x) = \frac{2}{3}x^2$, $\nabla f(x) = \frac{4}{3}x$; since $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in \mathbb{R}\}$, then $f^*(z) = \frac{3}{8}z^2$ and $\nabla f^*(z) = \frac{3}{4}z$. Clearly A, S_i^n and $f(x)$ satisfy the conditions of Theorem 3.1. So from the scheme we obtain the following

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}) \\ z_n = \mathrm{Proj}^f_{[-1,1]}(\frac{w_n}{4}) := \begin{cases} -1, & \text{if } w_n < -1 \\ \frac{w_n}{4}, & \text{if } w_n \in [-1,1] \\ 1, & \text{if } w_n > 1 \end{cases} \\ y_n = \mathrm{Proj}^f_{[-1,1]}(\frac{3w_n}{4}) := \begin{cases} -1, & \text{if } w_n < -1 \\ \frac{3w_n}{4} & \text{if } w_n \in [-1,1] \\ 1, & \text{if } w_n > 1 \end{cases} \\ t_n = \mathrm{Proj}^f_{[-1,1]}(w_n - \frac{3z_n}{4}) := \begin{cases} -1, & \text{if } w_n < -1 \\ w_n - \frac{3z_n}{4}, & \text{if } w_n \in [-1,1] \\ 1, & \text{if } w_n > 1 \end{cases} \\ u_n = \frac{2(n-1)w_n + 2y_n + nt_n}{4n} \\ C_{n+1} := [-1, \frac{3w_n^2 - u_n^2}{2(w_n+u_n)}] \\ x_{n+1} = \mathrm{Proj}^f_{C_{n+1}}(x_0) \end{cases}$$

where $\beta_n = \gamma_n = \frac{1}{2}$, $\alpha_n = 0.6$, $a_n = \frac{1}{n}$ and $\lambda = \frac{1}{2}$. Then $\{x_n\}$ converges to $0 \in \Omega = \{0\}$.

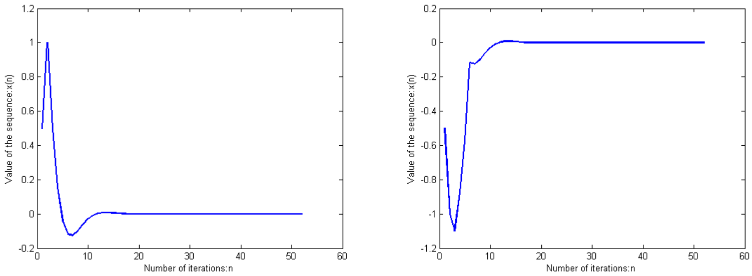

Figure 1: The inertial term speed the convergence rate of the iterative algorithm (6)

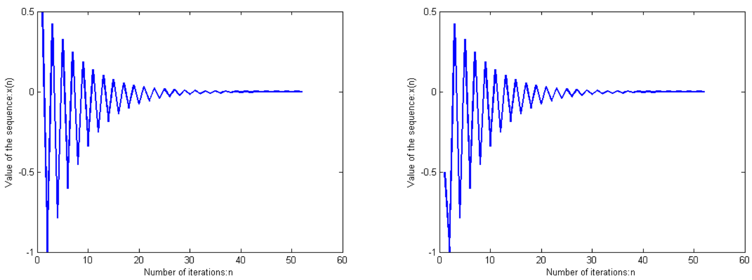

Figure 2: The iterative algorithm (6) without inertial term

The previous two figures were generated by Matlab software and they illustrate the convergence of the sequence $\{x_n\}$.

5. Corollaries

We present some corollaries of our theorem.

COROLLARY 5.1. *Let C be a nonempty, closed and convex subset of L_p (or l_p), $1 < p \leq 2$ such that $J(C)$ is convex. Let $A : C \to L_p^*$ (or l_q) be a monotone, k-Lipschitz map and let $S_1, S_2 : C \to C$ be asymptotically nonexpansive maps with sequences $\{k_n^1\}$ and $\{k_n^2\}$, respectively, such that $\bigcap_{i=1}^2 Fix(S_i) \neq \emptyset$ and assume $(\bigcap_{i=1}^2 Fix(S_i)) \cap VI(C, A) \neq \emptyset$. Define inductively the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in C_0 = C; \\ w_n = x_n - \alpha_n(x_n - x_{n-1}); \\ z_n = \Pi_C J^{-1}(Jw_n - \lambda A w_n); \\ y_n = J^{-1}[(1-\beta_n)Jw_n + \beta_n J S_1^n w_n]; \\ u_n = J^{-1}[(1-a_n-\gamma_n)Jw_n + a_n J S_1^n y_n + \gamma_n J S_2^n \Pi_C J^{-1}(Jw_n - \lambda A z_n)]; \\ C_{n+1} = \{u \in C_n : \phi(u, u_n) \leq \phi(u, w_n) + \eta_n\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $\eta_n = \beta_0(k_n - 1)\sup_{p\in\Omega} \phi(p, w_n)$ for all $w_n \in C$, with $l_n = \max_{n\geq1}\{\beta_n, a_n + \gamma_n\}$ and $\lambda \in (0, b]$, with $b < \frac{\alpha}{2k}$ and $\{a_n\}$, $\{a_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy:
(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\{\alpha_n\} \subset [0, \alpha]$ and $0 \leq \alpha < 1$.
(ii) $\{\beta_n\}, \{\gamma_n\} \subset (0, 1)$. (iii) $\{a_n\} \subset [0, 1]$ and $a_n \to 0$ as $n \to \infty$.
Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are well defined and converge strongly to some point $q = \Pi_{(\bigcap_{i=1}^2 Fix(S_i))\cap VI(C,A)} x_0$.

Proof. L_p (and l_p), $1 < p \leq 2$, are uniformly smooth and 2-uniformly convex Banach spaces. Hence the conclusion follows from Theorem 3.1. □

COROLLARY 5.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz map and let $S_1, S_2 : C \to C$ be asymptotically nonexpansive maps with sequences $\{k_n^1\}$ and $\{k_n^2\}$, respectively, such that $\bigcap_{i=1}^2 Fix(S_i) \neq \emptyset$, and assume $(\bigcap_{i=1}^2 Fix(S_i)) \cap VI(C, A) \neq \emptyset$. Define inductively the sequence $\{x_n\}$ by*

$$\begin{cases} x_0 \in C_0 = C; \\ w_n = x_n - \alpha_n(x_n - x_{n-1}); \\ z_n = P_C(w_n - \lambda A w_n); \\ y_n = (1-\beta_n)w_n + \beta_n S_1^n w_n; \\ u_n = (1-a_n-\gamma_n)w_n + a_n S_1^n y_n + \gamma_n S_2^n P_C(w_n - \lambda A z_n); \\ C_{n+1} = \{u \in C_n : \|u - u_n\| \leq \|u - w_n\| + \eta_n\}; \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $\eta_n = \beta_0(k_n - 1)\sup_{p\in\Omega}\|p - w_n\|$ *for all* $w_n \in C$, *with* $l_n = \max_{n\geq 1}\{\beta_n, a_n + \gamma_n\}$ *and* $\lambda \in (0, b]$, *with* $b < \frac{\alpha}{2k}$ *and* $\{a_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfy:*
(i) $\sum_{n=1}^{\infty}\alpha_n < \infty$, $\{\alpha_n\} \subset [0, \alpha]$ *and* $0 \leq \alpha < 1$.

(ii) $\{\beta_n\}, \{\gamma_n\} \subset (0, 1)$. *(iii)* $\{a_n\} \subset [0, 1]$ *and* $a_n \to 0$ *as* $n \to \infty$
Then the sequence $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *and* $\{u_n\}$ *are well defined and converge strongly to some point* $q = P_{(\bigcap_{i=1}^{2} Fix(S_i))\cap VI(C,A)}x_0$.

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