MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
74, 3 (2022), 197–204
September 2022

research paper
оригинални научни рад

CLAIRAUT ANTI-INVARIANT SUBMERSIONS FROM NEARLY KÄHLER MANIFOLDS

Punam Gupta and Amit Kumar Rai

Abstract. In the present paper, we investigate geometric properties of Clairaut anti-invariant submersions whose total spaces are nearly Kähler manifolds. We obtain a condition for a Clairaut anti-invariant submersion to be a totally geodesic map and also study Clairaut anti-invariant submersions with totally umbilical fibers.

1. Introduction

Riemannian submersion between two Riemannian manifolds was first introduced by O'Neill [14] and studied by many authors [7–9]. After that, Watson [26] introduced almost Hermitian submersions. Later, the notion of anti-invariant submersions and Lagrangian submersion from almost Hermitian manifolds onto Riemannian manifolds were introduced by Sahin [18], where the fibers of submersion are anti-invariant with respect to the almost complex structure of total manifold. After that, several new types of Riemannian submersions were defined and studied such as semi-invariant submersion [16, 19], slant submersion [20], generic submersion [5, 22], hemi-slant submersion [23], semi-slant submersion [17], pointwise slant submersion [12] and conformal semi-slant submersion [1]. Also, these kinds of submersions were considered in different kinds of structures such as nearly Kähler, Kähler, almost product, para-contact, Sasakian, Kenmotsu, cosymplectic and etc. In the book [21], we find the recent developments in this field.

In 1735, A. C. Clairaut [6] obtained a very important result in the theory of surfaces, now called Clairaut's theorem, stating that for any geodesic α on a surface of revolution S, the function $r \sin\theta$ is constant along α, where r is the distance from a point on the surface to the rotation axis and θ is the angle between α and the meridian through α. Bishop [4] introduced the idea of Riemannian submersions and gave a necessary and sufficient conditions for a Riemannian submersion to be Clairaut.

2020 Mathematics Subject Classification: 53C12, 53C15, 53C20, 53C55.

Keywords and phrases: Riemannian submersion; nearly Kähler manifolds; anti-invariant submersion; Clairaut submersion; totally geodesic maps.

Allison [2] considered Clairaut semi-Riemannian submersions and showed that such submersions have interesting applications in the static space-times.

In [24], Tastan and Gerdan gave new Clairaut conditions for anti-invariant submersions whose total manifolds are Sasakian and Kenmotsu and got many interesting results. In [25], Tastan and Aydin studied Clairaut anti-invariant submersions whose total manifolds are cosymplectic. Gündüzalp [11] introduced Clairaut anti-invariant submersions from a paracosymplectic manifold and gave characterization theorems. In [13], Lee et al. studied Clairaut anti-invariant submersions whose total manifolds are Kähler.

A geometrically interesting class of almost Hermitian manifolds is that of nearly Kähler manifolds, which is one of the sixteen classes of almost Hermitian manifolds that were obtained by Gray and Hervella in their remarkable paper [10]. The geometrical meaning of nearly Kähler condition is that the geodesics on the manifolds are holomorphically planar curves. Gray [9] studied nearly Kähler manifolds broadly and gave example of a non-Kählerian nearly Kähler manifold, which is 6-dimensional sphere.

Motivated by this, we study Clairaut anti-invariant submersions from nearly Kähler manifolds onto Riemannian manifolds. We also obtain conditions for a Clairaut Riemannian submersion to be a totally geodesic map. We investigate conditions for the Clairaut anti-invariant submersions to be a totally umbilical map.

2. Preliminaries

An almost complex structure on a smooth manifold M is a smooth tensor field φ of type $(1,1)$ such that $\varphi^2 = -I$. A smooth manifold equipped with such an almost complex structure is called an almost complex manifold. An almost complex manifold (M,φ) endowed with a chosen Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) \tag{1}$$

for all $X, Y \in TM$, is called an almost Hermitian manifold.

An almost Hermitian manifold M is called a nearly Kähler manifold [9] if

$$(\nabla_X \varphi) Y + (\nabla_Y \varphi) X = 0, \tag{2}$$

for all $X, Y \in TM$. If $(\nabla_X \varphi) Y = 0$ for all $X, Y \in TM$, then M is known as Kähler manifold. Every Kähler manifold is nearly Kähler but converse need not be true.

Definition 2.1 ([14,15]). Let (M, g_m) and (N, g_n) be Riemannian manifolds, where $\dim(M) = m$, $\dim(N) = n$ and $m > n$. A Riemannian submersion $\pi : M \to N$ is a map of M onto N satisfying the following axioms:

(i) π has maximal rank.

(ii) The differential π_* preserves the lengths of horizontal vectors.

For each $q \in N$, $\pi^{-1}(q)$ is an $(m-n)$-dimensional submanifold of M. The submanifolds $\pi^{-1}(q)$, $q \in N$, are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and π-related to a vector field X' on N, that is, $\pi_* X_p = X'_{\pi_*(p)}$ for all $p \in M$. We denote the projection morphisms on the distributions $\ker \pi_*$ and $(\ker \pi_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. The sections of $\mathcal{V}$ and $\mathcal{H}$ are called the vertical vector fields and horizontal vector fields, respectively. So $\mathcal{V}_p = T_p\left(\pi^{-1}(q)\right)$, $\mathcal{H}_p = T_p\left(\pi^{-1}(q)\right)^\perp$.

The second fundamental tensors of all fibers $\pi^{-1}(q)$, $q \in N$ gives rise to tensor field T and A in M defined by O'Neill [14] for arbitrary vector field E and F, which is

$$T_E F = \mathcal{H}\nabla^M_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{V}E}\mathcal{H}F, \quad A_E F = \mathcal{H}\nabla^M_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{H}E}\mathcal{H}F, \tag{3}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections.

On the other hand, from equations (3), we have

$$\begin{aligned} \nabla_V W &= T_V W + \widehat{\nabla}_V W, \quad \nabla_V X = \mathcal{H}\nabla_V X + T_V X, \\ \nabla_X V &= A_X V + \mathcal{V}\nabla_X V, \quad \nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y, \end{aligned} \tag{4}$$

for all $V, W \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma(\ker \pi_*)^\perp$, where $\mathcal{V}\nabla_V W = \widehat{\nabla}_V W$. If X is basic, then $A_X V = \mathcal{H}\nabla_V X$.

It is easily seen that for $p \in M$, $U \in \mathcal{V}_p$ and $X \in \mathcal{H}_p$, the linear operators T_U, $A_X : T_pM \to T_pM$ are skew-symmetric, that is, $g(A_X E, F) = -g(E, A_X F)$ and $g(T_U E, F) = -g(E, T_U F)$, for all $E, F \in T_pM$. We also see that the restriction of T to the vertical distribution $T|_{\ker \pi_* \times \ker \pi_*}$ is exactly the second fundamental form of the fibres of π. Since T_U is skew-symmetric, therefore π has totally geodesic fibres if and only if $T \equiv 0$.

Let $\pi : (M, g_m) \rightarrow (N, g_n)$ be a smooth map between Riemannian manifolds. Then the differential π_* of π can be observed as a section of the bundle $\mathrm{Hom}(TM, \pi^{-1}TN) \rightarrow M$, where $\pi^{-1}TN$ is the bundle which has fibres $\left(\pi^{-1}TN\right)_x = T_{f(x)}N$, has a connection ∇ induced from the Riemannian connection ∇^M and the pullback connection. Then the second fundamental form of π is given by $(\nabla\pi_*)(E, F) = \nabla^N_E \pi_* F - \pi_*(\nabla^M_E F)$, for all $E, F \in \Gamma(TM)$, where ∇^N is the pullback connection [3]. We also know that π is said to be totally geodesic map [3] if $(\nabla\pi_*)(E, F) = 0$, for all $E, F \in TM$.

Let π be an anti-invariant Riemannian submersion from nearly Kähler manifold (M, φ, g_m) onto Riemannian manifold (N, g_n). For any arbitrary tangent vector fields U and V on M, we set

$$(\nabla_U \varphi)V = P_U V + Q_U V \tag{5}$$

where $P_U V, Q_U V$ denote the horizontal and vertical part of $(\nabla_U \varphi)V$, respectively. Clearly, if M is a Kähler manifold then $P = Q = 0$.

If M is a nearly Kähler manifold then P and Q satisfy $P_U V = -P_V U$, $Q_U V = -Q_V U$. Consider $(\ker \pi_*)^\perp = \varphi \ker \pi_* \oplus \mu$, where μ is the complementary distribution to $\varphi \ker \pi_*$ in $(ker\pi_*)^\perp$ and $\varphi\mu \subset \mu$.

For $X \in \Gamma(ker\pi_*)^\perp$, we have $\varphi X = \alpha X + \beta X$, where $\alpha X \in \Gamma(\ker \pi_*)$ and $\beta X \in \Gamma(\mu)$. If $\mu = 0$, then an anti-invariant submersion is known as Lagrangian submersion.

Definition 2.2 ([18]). Let (M, φ, g) be an almost Hermitian manifold and N be a Riemannian manifold with Riemannian metric g_n. Suppose that there exists a Riemannian submersion $\pi : M \to N$, such that the vertical distribution $\ker \pi_*$ is anti-invariant with respect to φ, i.e., $\varphi \ker \pi_* \subseteq \ker \pi_*^{\perp}$. Then, the Riemannian submersion π is called an anti-invariant Riemannian submersion. We will briefly call such submersions as anti-invariant submersions.

Let S be a revolution surface in $\mathbb{R}^3$ with rotation axis L. For any $p \in S$, we denote by $r(p)$ the distance from p to L. Given a geodesic $\alpha : J \subset \mathbb{R} \to S$ on S, let $\theta(t)$ be the angle between $\alpha(t)$ and the meridian curve through $\alpha(t), t \in I$. A well-known Clairaut's theorem says that for any geodesic on S, the product $r \sin\theta$ is constant along α, i.e., it is independent of t. In the theory of Riemannian submersions, Bishop [4] introduced the notion of Clairaut submersion in the following way.

Definition 2.3 ([4]). A Riemannian submersion $\pi : (M, g) \to (N, g_n)$ is called a Clairaut submersion if there exists a positive function r on M, which is known as the girth of the submersion, such that, for any geodesic α on M, the function $(r \circ \alpha)\sin\theta$ is constant, where $\theta(t)$ is the angle between $\dot{\alpha}(t)$ and the horizontal space at $\alpha(t)$, for any t.

He also gave the following necessary and sufficient condition for a Riemannian submersion to be a Clairaut submersion:

Theorem 2.4 ([4]). *Let $\pi : (M, g) \to (N, g_n)$ be a Riemannian submersion with connected fibers. Then, π is a Clairaut submersion with $r = e^f$ if and only if each fiber is totally umbilical and has the mean curvature vector field $H = -\operatorname{grad} f$, where $\operatorname{grad} f$ is the gradient of the function f with respect to g.*

3. Anti-invariant Clairaut submersions from nearly Kähler manifolds

In this section, we give new Clairaut conditions for anti-invariant submersions from nearly Kähler manifolds after giving some auxiliary results.

Theorem 3.1. *Let π be an anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n). If $h : J \subset \mathbb{R} \to M$ is a regular curve and $U(s)$ and $X(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{h}(s) = W$ of $h(s)$, respectively, then h is a geodesic if and only if along h*

$$A_X \varphi U + A_X \beta X + T_U \beta X + \mathcal{V} \nabla_X \alpha X + T_U \varphi U + \hat{\nabla}_U \alpha X = 0, \tag{6}$$

$$\mathcal{H}\left(\nabla_{\dot{h}} \varphi U + \nabla_{\dot{h}} \beta X\right) + A_X \alpha X + T_U \alpha X = 0. \tag{7}$$

Proof. Since $\varphi^2 \dot{h} = -\dot{h}$, taking the covariant derivative of this and using (2), we have

$$\left(\nabla_{\dot{h}} \varphi\right) \varphi \dot{h} + \varphi\left(\nabla_{\dot{h}} \varphi \dot{h}\right) = -\nabla_{\dot{h}} \dot{h}. \tag{8}$$

Since $U(s)$ and $X(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{h}(s) = W$ of $h(s)$, that is, $\dot{h} = U + X$. So (8) becomes

$$\begin{aligned} -\nabla_{\dot{h}}\dot{h} =& \varphi\left(\nabla_{U+X}\varphi(U+X)\right) + P_{\dot{h}}\varphi\dot{h} + Q_{\dot{h}}\varphi\dot{h} \\ =& \varphi\left(\nabla_U\varphi U + \nabla_X\varphi U + \nabla_U\varphi X + \nabla_X\varphi X\right) + P_{\dot{h}}\varphi\dot{h} + Q_{\dot{h}}\varphi\dot{h} \\ =& \varphi\left(\nabla_U\varphi U + \nabla_X\varphi U + \nabla_U\left(\alpha X + \beta X\right) + \nabla_X\left(\alpha X + \beta X\right)\right) + P_{\dot{h}}\varphi\dot{h} + Q_{\dot{h}}\varphi\dot{h}. \quad (9) \end{aligned}$$

Using (4) in (9), we get

$$\begin{aligned} -\nabla_{\dot{h}}\dot{h} =& \varphi\big(\mathcal{H}\left(\nabla_{\dot{h}}\varphi U + \nabla_{\dot{h}}\beta X\right) + A_X\alpha X + A_X\beta X + A_X\varphi U \\ & + T_U\beta X + T_U\alpha X + \mathcal{V}\nabla_X\alpha X + T_U\varphi U + \hat{\nabla}_U\alpha X\big) + P_{\dot{h}}\varphi\dot{h} + Q_{\dot{h}}\varphi\dot{h}. \quad (10) \end{aligned}$$

Let $Y, Z \in TM$. Since $\varphi^2 Z = -Z$, on differentiation, we have

$$\varphi\left(\nabla_Y\varphi Z\right) + \left(\nabla_Y\varphi\right)\varphi Z = -\nabla_Y Z, \quad \varphi^2\left(\nabla_Y Z\right) + \varphi\left(\nabla_Y\varphi\right)Z + \left(\nabla_Y\varphi\right)\varphi Z = -\nabla_Y Z,$$

using (5) in above, we obtain $\varphi\left(P_Y Z + Q_Y Z\right) = -P_Y\varphi Z - Q_Y\varphi Z$. From here, it follows $\varphi\left(P_{\dot{h}}\varphi\dot{h} + Q_{\dot{h}}\varphi\dot{h}\right) = P_{\dot{h}}\dot{h} + Q_{\dot{h}}\dot{h}$, since P and Q are antisymmetric, so

$$\varphi\left(P_{\dot{h}}\varphi\dot{h} + Q_{\dot{h}}\varphi\dot{h}\right) = 0. \quad (11)$$

Using (11) and equating the vertical and horizontal part of (10), we obtain

$$\begin{aligned} \mathcal{V}\varphi\nabla_{\dot{h}}\dot{h} =& A_X\varphi U + A_X\beta X + T_U\beta X + \mathcal{V}\nabla_X\alpha X + T_U\varphi U + \hat{\nabla}_U\alpha X, \\ \mathcal{H}\varphi\nabla_{\dot{h}}\dot{h} =& \mathcal{H}\left(\nabla_{\dot{h}}\varphi U + \nabla_{\dot{h}}\beta X\right) + A_X\alpha X + T_U\alpha X. \end{aligned}$$

By using the above equations, we can say that h is geodesic if and only if (6) and (7) hold. $\square$

Theorem 3.2. *Let π be an anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n). Also, let $h : J \subset \mathbb{R} \to M$ be a regular curve and $U(s)$ and $X(s)$ be the vertical and horizontal parts of the tangent vector field $\dot{h}(s) = W$ of $h(s)$. Then π is a Clairaut submersion with $r = e^f$ if and only if along h: $g(grad\, f, X)g(U, U) = g(\mathcal{H}\nabla_{\dot{h}}\beta X + A_X\alpha X + T_U\alpha X + P_{\dot{h}(s)}U, \varphi U)$.*

Proof. Let $h : J \subset \mathbb{R} \to M$ be a geodesic on M and $\ell = \|\dot{h}(s)\|^2$. Let $\theta(s)$ be the angle between $\dot{h}(s)$ and the horizontal space at $h(s)$. Then

$$g(X(s), X(s)) = \ell\cos^2\theta(s), \quad g(U(s), U(s)) = \ell\sin^2\theta(s). \quad (12)$$

Differentiating the second term in (12), we get

$$2g(\nabla_{\dot{h}(s)}U(s), U(s)) = 2\ell\sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}. \quad (13)$$

Using (1) in (13), we have

$$g(\mathcal{H}\nabla_{\dot{h}(s)}\varphi U(s), \varphi U(s)) - g((\nabla_{\dot{h}(s)}\varphi)U(s), \varphi U(s)) = \ell\sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}.$$

Now, by use of (5), we have

$$g(\mathcal{H}\nabla_{\dot{h}(s)}\varphi U(s), \varphi U(s)) - g(P_{\dot{h}(s)}U + Q_{\dot{h}(s)}U, \varphi U(s)) = \ell\sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}.$$

Along the curve h, using Theorem 3.1, we obtain

$$-g(\mathcal{H}\nabla_{\dot{h}}\beta X + A_X\alpha X + T_U\alpha X + P_{\dot{h}(s)}U, \varphi U(s)) = \ell \sin\theta(s)\cos\theta(s)\frac{d\theta(s)}{ds}.$$

Now, π is a Clairaut submersion with $r = e^f$ if and only if $\frac{d}{ds}\left(e^f \sin\theta\right) = 0$. Therefore

$$e^f\left(\frac{df}{ds}\sin\theta + \cos\theta\frac{d\theta}{ds}\right) = 0, \quad e^f\left(\frac{df}{ds}\ell\sin^2\theta + \ell\sin\theta\cos\theta\frac{d\theta}{ds}\right) = 0.$$

So, we obtain

$$\frac{df}{ds}(h(s))g(U(s), U(s)) = g(\mathcal{H}\nabla_{\dot{h}}\beta X + A_X\alpha X + T_U\alpha X + P_{\dot{h}(s)}U, \varphi U(s)). \tag{14}$$

Since $\frac{df}{ds}(h(s)) = g(\operatorname{grad} f, \dot{h}(s)) = g(\operatorname{grad} f, X)$. Therefore by using (14), we get the result. □

Theorem 3.3. *Let π be a Clairaut anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n) with $r = e^f$. Then $A_{\varphi W}\varphi X + Q_W\varphi X = X(f)W$ for $X \in (\ker\pi_*)^{\perp}$, $W \in \ker\pi_*$ and φW is basic.*

Proof. We know that any fiber of Riemannian submersion π is totally umbilical if and only if

$$T_V W = g(V, W)H, \tag{15}$$

for all $V, W \in \Gamma(\ker\pi_*)$, where H denotes the mean curvature vector field of any fiber in M. By using Theorem 2.4 and (15), we have

$$T_V W = -g(V, W)\operatorname{grad} f. \tag{16}$$

Let $X \in \mu$ and $V, W \in \Gamma(\ker\pi_*)$, then by using (1) and (2), we have

$$g(\nabla_V\varphi W, \varphi X) = g(\varphi\nabla_V W + (\nabla_V\varphi)W, \varphi X) = g(\nabla_V W, X) + g(P_V W + Q_V W, \varphi X). \tag{17}$$

By using (1), we have $g(\varphi Y, Z) = -g(Y, \varphi Z)$, where $Y, Z \in TM$, taking covariant derivative of above equation, we get $g((\nabla_X\varphi)Y, Z) = -g(Y, (\nabla_X\varphi)Z)$, using (5), we get

$$g(P_X Y + Q_X Y, Z) = -g(Y, P_X Z + Q_X Z) = g(Y, P_Z X + Q_Z X). \tag{18}$$

Using (18), we have

$$g(P_W\varphi X + Q_W\varphi X, V) = g(\varphi X, P_V W + Q_V W). \tag{19}$$

Using (4), (16), (19) in (17), we have $g(\nabla_V\varphi W, \varphi X) = -g(V, W)g(\operatorname{grad} f, X) + g(V, Q_W\varphi X)$. Since φW is basic, so $\mathcal{H}\nabla_V\varphi W = A_{\varphi W}V$, therefore we have

$$\begin{aligned} &g(A_{\varphi W}V, \varphi X) = -g(V, W)g(\operatorname{grad} f, X) + g(V, Q_W\varphi X), \\ &g(V, A_{\varphi W}\varphi X) + g(V, Q_W\varphi X) = g(V, W)g(\operatorname{grad} f, X) \end{aligned} \tag{20}$$

because A is anti-symmetric. The result follows from (20). □

Theorem 3.4. *Let π be a Clairaut anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n) with $r = e^f$ and $\operatorname{grad} f \in \varphi\ker\pi_*$. Then either f is constant on $\varphi\ker\pi_*$ or the fibres of π are 1-dimensional.*

Proof. Using (4) and (16), we have $g(\nabla_V W, \varphi U) = -g(V, W)g(\operatorname{grad} f, \varphi U)$, where $U, V, W \in \Gamma(\ker \pi_*)$. Since $g(W, \varphi U) = 0$. Therefore we have

$$g(W, \nabla_V \varphi U) = g(V, W)g(\operatorname{grad} f, \varphi U). \tag{21}$$

By use of (1) and (5) in (21), we get $g(W, Q_V U) - g(\varphi W, \nabla_V U) = g(V, W)g(\operatorname{grad} f, \varphi U)$. By using (4), we obtain $g(W, Q_V U) - g(\varphi W, T_V U) = g(V, W)g(\operatorname{grad} f, \varphi U)$. Now, from (16), we get

$$g(W, Q_V U) + g(V, U)g(\operatorname{grad} f, \varphi W) = g(V, W)g(\operatorname{grad} f, \varphi U). \tag{22}$$

Taking $V = U$ in (22), we have

$$g(V, V)g(\operatorname{grad} f, \varphi W) = g(V, W)g(\operatorname{grad} f, \varphi V). \tag{23}$$

Interchanging V and W in (23), we obtain

$$g(W, W)g(\operatorname{grad} f, \varphi V) = g(V, W)g(\operatorname{grad} f, \varphi W). \tag{24}$$

By (23) and (24), we have $g^2(V, W)g(\operatorname{grad} f, \varphi V) = g(V, V)g(W, W)g(\operatorname{grad} f, \varphi V)$. Therefore either f is constant on $\varphi \ker \pi_*$ or $V = aW$, where a is constant (by using Schwarz's Inequality for equality case). □

Corollary 3.5. *Let π be a Clairaut anti-invariant submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n) with $r = e^f$ and grad $f \in \varphi \ker \pi_*$. If $\dim(\ker \pi_*) > 1$, then the fibres of π are totally geodesic if and only if $A_{\varphi W} \varphi X + Q_W \varphi X = 0$ for $W \in \ker \pi_*$ such that φW is basic and $X \in \mu$.*

Proof. By Theorem 3.3 and Theorem 3.4, we get the result. □

Corollary 3.6. *Let π be a Clairaut Lagrangian submersion from a nearly Kähler manifold (M, φ, g) onto a Riemannian manifold (N, g_n) with $r = e^f$. Then either the fibres of π are 1-dimensional or they are totally geodesic.*

Proof. In this case $\mu = \{0\}$, so $A_{\varphi W} \varphi X + Q_W \varphi X = 0$ holds. □

Acknowledgement. The authors are thankful to reviewers for their constructive suggestions and comments.

References

[1] M. A. Akyol, *Conformal semi-slant submersions*, Int. J. Geom. Methods Mod. Phys., **14(7)** (2017), 1750114.

[2] D. Allison, *Lorentzian Clairaut submersions*, Geom. Dedicata, **63(3)** (1996), 309–319.

[3] P. Baird, J. C. Wood, *Harmonic Morphisms Between Riemannian Manifolds, London Mathematical Society Monographs*, Vol. 29, Oxford University Press, The Clarendon Press, Oxford, 2003.

[4] R. L. Bishop, *Clairaut submersions, Differential Geometry (in Honor of Kentaro Yano)*, Kinokuniya, Tokyo, (1972), 21–31.

[5] B.Y. Chen, *Differential geometry of real submanifolds in Kähler manifold*, Monatsh. Math., **91** (1981), 257–274.

[6] A.C. Clairaut, Mem. Acad. Paris pour 1733, (1735), p. 86.

[7] M. Falcitelli, S. Ianus, A.M. Pastore, *Riemannian submersion and related topics*, World Scientific, River Edge 2004.

[8] A. Gray, *Pseudo-Riemannian almost product manifolds and submersion*, J. Math. Mech., **16** (1967), 715–737.

[9] A. Gray, *Nearly Kähler manifolds*, J. Differential Geom., **6** (1970), 283–309.

[10] A. Gray, M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., **123** (1980), 35–58.

[11] Y. Gündüzalp, *Anti-invariant pseudo-Riemannian submersions and Clairaut submersions from paracosymplectic manifolds*, Mediterr. J. Math. **16(4)**, paper no. 94 (2019), 18 pp.

[12] J.W. Lee, B. Sahin, *Pointwise slant submersions*, Bull. Korean Math. Soc., **51** (2014), 1115–1126.

[13] J. Lee, J.H. Park, B. Shahin, D.Y. Song, *Einstein conditions for the base of anti-invariant Riemannian submersions and Clairaut submersions*, Taiwanese J. Math., **19(4)** (2015), 1145–1160.

[14] B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J., **13** (1966), 458–469.

[15] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Academic Press, New York, 1983.

[16] F. Özdemir, C. Sayar, H.M. Tastan, *Semi-invariant submersions whose total manifolds are locally product Riemannian*, Quaest. Math., **40(7)** (2017), 909–926.

[17] K.S. Park, R. Prasad, *Semi-slant submersions*, Bull. Korean Math. Soc., **50(3)** (2013), 951–962.

[18] B. Sahin, *Anti-invariant Riemannian submersion from almost Hermitian manifolds*, Cent. Eur. J. Math., **8(3)** (2010), 437–447.

[19] B. Sahin, *Semi-invariant submersion from almost Hermitian manifolds*, Canad. Math. Bull., **56** (2011), 173–182.

[20] B. Sahin, *Slant submersions from almost Hermitian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie, **54(102)** (2011), 93–105.

[21] B. Sahin, *Riemannian Submersions, Riemannian Maps in Hermitian Geometry and their Applications*, Elsevier, Academic Press, Cambridge, 2017.

[22] C. Sayar, H.M. Taştan, F. Özdemir, M.M. Tripathi, *Generic submersions from Kähler manifolds*, Bull. Malaysian Math. Sci. Soc., **43** (2020), 809–831.

[23] H.M. Taştan, B. Sahin, S. Yanan, *Hemi-slant submersions*, Mediterr. J. Math., **13(4)** (2016), 2171–2184.

[24] H.M. Taştan, S. Gerdan, *Clairaut anti-invariant submersions from Sasakian and Kenmotsu manifolds*, Mediterr. J. Math., **14(6)**, paper no. 235 (2017), 17 pp.

[25] H.M. Taştan, S.G. Aydin, *Clairaut anti-invariant submersions from cosymplectic manifolds*, Honam Math. J., **41(4)** (2019), 707–724.

[26] B. Watson, *Almost Hermitian submersions*, J. Differential Geom., **11** (1976), 147–165.

(received 16.03.2021; in revised form 12.11.2021; available online 24.06.2022)

Department of Mathematics and Statistics, Dr. Harisingh Gour University, Sagar-470 003, Madhya Pradesh, India

E-mail: pgupta@dhsgsu.edu.in

University School of Basic and Applied Sciences, Guru Gobind Singh Indraprastha University, Delhi, India

E-mail: rai.amit08au@gmail.com