MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
74, 3 (2022), 205–213
September 2022

research paper
оригинални научни рад

SOME GENERALIZATIONS OF A THEOREM OF PAUL TURÀN CONCERNING POLYNOMIALS

Tawheeda Akhter, B. A. Zargar and M. H. Gulzar

Abstract. Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$. It was shown by Govil that $\max\limits_{|z|=1}|P'(z)| \ge \frac{n}{1+k^n}\max\limits_{|z|=1}|P(z)|$.

In this paper, we shall obtain some sharp estimates by involving the coefficients which not only refine the above result but also generalise some well-known results of this type.

1. Introduction

The classical Markov [12] and Bernstein [5] inequalities and corresponding extremal problems were generalised for various domains, various norms and for various subclasses for polynomials, both algebraic and trigonometric. These inequalities play a significant role in approximation theory. Inequalities of Markov and Bernstein are fundamental for the proofs of many inverse theorems in the approximation theory (see [11, 13, 18]). For instance, Telyakovskii [18] writes: Among those that are fundamental in approximation theory are extremal problems connected with inequalities for the derivatives of polynomials. The use of inequalities of this kind is a fundamental method in proofs of inverse problems of approximation theory (as can be seen in [6]). Further progress in inverse theorems depended on first obtained corresponding analogue or generalization of Markov's and Bernstein's inequalities and therefore it is of interest to obtain refinements and generalizations of polynomial inequalities.

If $P(z)$ is a polynomial of degree n and $P'(z)$ be its derivative, then it was shown by Turàn [19] that if $P(z)$ has all its zeros in $|z| \le 1$, then

$$\max_{|z|=1}|P'(z)| \ge \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1}$$

Equality in (1) holds for $P(z) = z^n + 1$.

2020 Mathematics Subject Classification: 26D10, 30C15, 41A17.
Keywords and phrases: Polynomials; inequalities in complex domain; derivative; s-fold zeros.

As an extension of inequality (1), Govil [9] showed that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k^n}\max_{|z|=1}|P(z)|. \tag{2}$$

The result is sharp and equality in (2) holds for $P(z) = z^n + k^n$.

In the literature there exists several extensions and generalizations of inequalities (1) and (2) (for reference see [1–3,14,15,17]). Dubinin [7] obtained the fascinating refinement of inequality (1) by introducing some of the coefficients of $P(z)$. In fact, he proved that if $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\max_{|z|=1}|P'(z)|. \tag{3}$$

Recently Rather et al. [16] have established the refinements and generalizations of the above inequalities by proving the following intriguing results.

Theorem 1.1. *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$, $k \geq 1$, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)| \geq & \frac{1}{1+k^n}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\max_{|z|=1}|P(z)| + \psi(k)|a_1| \\ & + \frac{|a_{n-1}|}{k(1+k^n)}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\phi(k),\end{aligned} \tag{4}$$

where $\phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = (1 - \frac{1}{k^2})$ or $(1 - \frac{1}{k})$, for $n > 2$ and $n = 2$ respectively.

The result is sharp and equality in (4) holds for $P(z) = z^n + k^n$.

Theorem 1.2. *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n \geq 2$, having all its zeros in $|z| \leq k$, $k \geq 1$ and $m = \min\limits_{|z|=k}|P(z)|$, then for $0 \leq l < 1$,*

$$\begin{aligned}\max_{|z|=1}|P'(z)| \geq & \frac{n}{1+k^n}\left(\max_{|z|=1}|P(z)| + lm\right) + \psi(k)|a_1| \\ & + \frac{1}{k^n(1+k^n)}\left\{\left(\frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\left(k^n\max_{|z|=1}|P(z)| - lm\right)\right. \\ & \left. + k^{n-1}|a_{n-1}|\phi(k)\left(n + \frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\right\}.\end{aligned} \tag{5}$$

where $\phi(k)$ and $\psi(k)$ are defined in Theorem 1.1.

The result is sharp and equality in (5) holds for the polynomial $P(z) = z^n + k^n$.

2. Main results

In this paper, we first present the following result which provides a refinement as well as a generalization of Theorem 1.1 by taking s-fold zeros at origin.

Theorem 2.1. *If $P(z) = z^s(a_0 + a_1 z + \ldots + a_{n-s} z^{n-s})$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$, $k \geq 1$ with s-fold zeros at origin, then*

$$\max_{|z|=1}|P'(z)| \geq \frac{1}{1+k^{n-s}}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\max_{|z|=1}|P(z)| + \psi(k)|a_1| + \frac{|a_{n-s-1}|}{k(1+k^{n-s})}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\phi(k), \tag{6}$$

where $\phi(k) = \left(\frac{k^{n-s}-1}{n-s} - \frac{k^{n-s-2}-1}{n-s-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = (1-\frac{1}{k^2})$ or $(1-\frac{1}{k})$, for $n > 2$ and $n = 2$ respectively.

The result is sharp and equality in (6) holds for $P(z) = z^n + k^n$.

Remark 2.2. By taking $s = 0$ in Theorem 2.1, it reduces to Theorem 1.1.

Remark 2.3. Since all the zeros of $P(z)$ lie in $|z| \leq k$, $k \geq 1$, therefore, $|a_0| \leq k^{n-s}|a_{n-s}|$. Hence it follows that inequality (6) gives a refinement of inequality (2).

By setting $k = 1$ in Theorem 2.1, we get the following result which provides a refinement as well as a generalization of inequality (3).

Corollary 2.4. *If $P(z) = z^s(a_0 + a_1 z + \ldots + a_{n-s} z^{n-s})$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq 1$, with s-fold zeros at origin, then*

$$\max_{|z|=1}|P'(z)| \geq \frac{1}{2}\left(n+s+\frac{|a_{n-s}|-|a_0|}{|a_{n-s}|+|a_0|}\right)\max_{|z|=1}|P(z)|. \tag{7}$$

The result is sharp and equality in (7) holds for $P(z) = z^n + 1$.

Next we prove the following result which produces a refinement as well as a generalization of Theorem 1.2.

Theorem 2.5. *If $P(z) = z^s(a_0 + a_1 z + \ldots + a_{n-s} z^{n-s})$ is a polynomial of degree $n \geq 2$, having all its zeros in $|z| \leq k$, $k \geq 1$ with s-fold zeros at origin and $m = \min_{|z|=k}|P(z)|$, then for $0 \leq l < 1$,*

$$\max_{|z|=1}|P'(z)| \geq \frac{(n+s)}{1+k^{n-s}}\left(\max_{|z|=1}|P(z)| + \frac{1}{k^s}lm\right) + \psi(k)|a_1| + \frac{1}{k^n(1+k^{n-s})}\left\{\left(\frac{k^{n-s}|a_{n-s}|-lm-|a_0|}{k^{n-s}|a_{n-s}|-lm+|a_0|}\right)\left(k^n\max_{|z|=1}|P(z)| - lm\right) + k^{n-1}|a_{n-s-1}|\phi(k)\left(n+s+\frac{k^{n-s}|a_{n-s}|-lm-|a_0|}{k^{n-s}|a_{n-s}|-lm+|a_0|}\right)\right\}, \tag{8}$$

where $\phi(k)$ and $\psi(k)$ are defined in Theorem 2.1.

The result is sharp and equality in (8) holds for polynomial $P(z) = z^n + k^n$.

Remark 2.6. By taking $s = 0$ in Theorem 2.5, it reduces to Theorem 1.2.

Remark 2.7. Since under the hypothesis of Theorem 2.5, $|a_0| \leq k^{n-s}|a_{n-s}|$, therefore it follows that Theorem 2.5 gives a refinement of Theorem 2.1.

For $k = 1$, we get the following result which constitutes a refinement of inequality (3) as a special case.

Corollary 2.8. *If $P(z) = z^s(a_0 + a_1 z + \ldots + a_{n-s}z^{n-s})$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq 1$ with s-fold zeros at origin and $m_1 = \min_{|z|=1}|P(z)|$, then for $0 \leq l < 1$, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)| \geq &\frac{(n+s)}{2}\left(\max_{|z|=1}|P(z)| + lm_1\right)\\ &+\frac{1}{2}\left(\frac{|a_{n-s}| - lm_1 - |a_0|}{|a_{n-s}| - lm_1 + |a_0|}\right)(\max_{|z|=1}|P(z)| - lm_1) \qquad (9)\end{aligned}$$

The result is sharp and equality in (9) holds for $P(z) = z^n + 1$.

3. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 3.1 ([10]). *If $F(z) = z^s(a_0 + a_1 z + \ldots + a_{n-s}z^{n-s})$, $0 \leq s \leq n$, is a polynomial of degree $n \geq 1$ having all its zeros in $|z| \leq 1$, then for all z on $|z| = 1$ for which $F(z) \neq 0$,*

$$Re\left(\frac{zF'(z)}{F(z)}\right) \geq \frac{1}{2}\left(n + s + \frac{|a_{n-s}| - |a_0|}{|a_{n-s}| + |a_0|}\right).$$

Lemma 3.2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then for $|z| = 1$: $|Q'(z)| \leq |P'(z)|$.*

The above lemma is a special case of a result due to Aziz and Rather [4].

Lemma 3.3 ([8]). *If $P(z)$ is a polynomial of degree $n \geq 1$, then for $R \geq 1$*

$$\max_{|z|=R}|P(z)| \leq R^n\max_{|z|=1}|P(z)| - (R^n - R^{n-2})|P(0)|, \quad \text{if } n > 1 \qquad (10)$$

and

$$\max_{|z|=R}|P(z)| \leq R\max_{|z|=1}|P(z)| - (R-1)|P(0)|, \qquad \text{if } n = 1. \qquad (11)$$

Lemma 3.4 ([17]). *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for $0 \leq l < 1$*

$$\max_{|z|=k}|P(z)| \geq \frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| - l\left(\frac{k^n - 1}{k^n + 1}\right)\min_{|z|=k}|P(z)|$$

$$+\frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right), \quad \textit{if } n>2 \tag{12}$$

and
$$\max_{|z|=k}|P(z)| \geq \frac{2k^2}{1+k^2}\max_{|z|=1}|P(z)| + l\left(\frac{k^2-1}{k^2+1}\right)\min_{|z|=k}|P(z)| + \frac{k(k-1)^2|a_1|}{k^2+1}, \quad \textit{if } n=2. \tag{13}$$

4. Proofs of the theorems

Proof of Theorem 2.1

Since, by hypothesis, $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, with s-fold zeros at origin, then $f(z) = P(kz)$ has all its zeros in $|z| \leq 1$ with s-fold zeros at origin. Hence by applying Lemma 3.1 to the polynomial $f(z)$, we obtain for all points z on $|z| = 1$ with $f(z) \neq 0$

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1}{2}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right),$$

which for $|z| = 1$ and $f(z) \neq 0$ implies

$$\left|\frac{zf'(z)}{f(z)}\right| \geq \frac{1}{2}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right),$$

$$|f'(z)| \geq \frac{1}{2}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)|f(z)|.$$

Replacing $f(z)$ by $P(kz)$, we obtain for $|z| = 1$

$$k|P'(kz)| \geq \frac{1}{2}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)|P(kz)|,$$

that is
$$k\max_{|z|=k}|P'(z)| \geq \frac{1}{2}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\max_{|z|=k}|P(z)|. \tag{14}$$

Since $P'(z)$ is a polynomial of degree $n-1$, we get by applying (10) of Lemma 3.3 with $R = k \geq 1$

$$k^{n-1}\max_{|z|=1}|P'(z)| - (k^{n-1}-k^{n-3})|a_1| \geq \max_{|z|=k}|P'(z)|, \quad \text{if } n>2.$$

Combining this with inequality (14), we obtain for $n > 2$

$$2k^n\max_{|z|=1}|P'(z)| - 2(k^n-k^{n-2})|a_1| \geq \left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\max_{|z|=k}|P(z)|. \tag{15}$$

Let $P(z) = z^s h(z)$, where $h(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \leq k$, $k \geq 1$, then by applying (12) of Lemma 3.4 with $l = 0$ and $R = k \geq 1$ to the polynomial $h(z)$ which is of degree $n-s$, we obtain for $n > 2$

$$\max_{|z|=k}|h(z)| \geq \frac{2k^{n-s}}{1+k^{n-s}}\max_{|z|=1}|h(z)| + \frac{2k^{n-s-1}|a_{n-s-1}|}{k^{n-s}+1}\left(\frac{k^{n-s}-1}{n-s}-\frac{k^{n-s-2}-1}{n-s-2}\right). \tag{16}$$

Also $\max_{|z|=k}|h(z)| = \frac{1}{k^s}\max_{|z|=k}|P(z)|$ and $\max_{|z|=1}|h(z)| = \max_{|z|=1}|P(z)|$. Using these in inequality (16), we have for $n > 2$

$$\max_{|z|=k}|P(z)| \geq \frac{2k^n}{k^{n-s}+1}\max_{|z|=1}|P(z)|+\frac{2|a_{n-s-1}|k^{n-1}}{k^{n-s}+1}\left(\frac{k^{n-s}-1}{n-s}-\frac{k^{n-s-2}-1}{n-s-2}\right). \quad (17)$$

Combining (17) and (15), we obtain for $n > 2$

$$2k^n\max_{|z|=1}|P'(z)|-2(k^n-k^{n-2})|a_1| \geq \frac{2k^n}{1+k^{n-s}}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\max_{|z|=1}|P(z)|$$
$$+\frac{2|a_{n-s-1}|k^{n-1}}{k^{n-s}+1}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\left(\frac{k^{n-s}-1}{n-s}-\frac{k^{n-s-2}-1}{n-s-2}\right).$$

This gives

$$\max_{|z|=1}|P'(z)| \geq \frac{1}{1+k^{n-s}}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\max_{|z|=1}|P(z)|+(1-1/k^2)|a_1|$$
$$+\frac{|a_{n-s-1}|}{k(1+k^{n-s})}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\left(\frac{k^{n-s}-1}{n-s}-\frac{k^{n-s-2}-1}{n-s-2}\right), \text{ if } n > 2.$$

This proves inequality (6) in case $n > 2$. The result follows similarly for the case $n = 2$, but instead of using the first part of Lemma 3.3 and Lemma 3.4, we use the second part of Lemma 3.3 and Lemma 3.4. This completes the proof of Theorem 2.1.

Proof of Theorem 2.5

By hypothesis, $P(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$ with s-fold zeros at origin. If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows from Theorem 2.1. Therefore we assume that all the zeros of $P(z)$ lie in $|z| < k$, with s-fold zeros at origin, so that $m > 0$. Hence the polynomial $g(z) = P(kz)$ has all its zeros in $|z| < 1$ with s-fold zeros at origin and $m = \min_{|z|=k}|P(z)| = \min_{|z|=1}|g(z)|$. Therefore we have $m < |g(z)|$ for $|z| = 1$. This gives, for every $\gamma \in \mathcal{C}$ with $|\gamma| < 1$, $m|\gamma z^n| < |g(z)|$ for $|z| = 1$. Hence it follows by applying Rouche's theorem that all the zeros of the polynomial $G(z) = g(z) + \gamma m z^n$ lie in $|z| < 1$ with s-fold zeros at origin. Therefore as before in Theorem 2.1 by applying Lemma 3.1 to the polynomial $G(z)$, we obtain for $|z| = 1$ and $G(z) \neq 0$ that

$$|G'(z)| \geq \frac{1}{2}\left(n+s+\frac{|k^{n-s}a_{n-s}+\gamma m|-|a_0|}{|k^{n-s}a_{n-s}+\gamma m|+|a_0|}\right)|G(z)|.$$

Since the function $t(x) = \frac{x-|a|}{x+|a|}$ ia non-decreasing function of x and $|k^{n-s}a_{n-s}+\gamma m| \geq k^{n-s}|a_{n-s}|-|\gamma m|$, we obtain for every $\gamma \in \mathcal{C}$ with $|\gamma| < 1$ and $|z| = 1$

$$|G'(z)| \geq \frac{1}{2}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|\gamma m|-|a_0|}{k^{n-s}|a_{n-s}|-|\gamma m|+|a_0|}\right)|G(z)|.$$

This gives for $|z| = 1$ and $|\gamma| < 1$

$$|g'(z)+nm\gamma z^{n-1}| \geq \frac{1}{2}\left(n+s+\frac{k^{n-s}|a_{n-s}|-|\gamma m|-|a_0|}{k^{n-s}|a_{n-s}|-|\gamma m|+|a_0|}\right)(|g(z)|-m|\lambda|). \quad (18)$$

Since all zeros of $G(z) = g(z)+\gamma m z^n$ lie in $|z| < 1$ with s-fold zeros at origin, therefore it follows from Gauss-Lucas theorem that all zeros of $G'(z) = g'(z)+\gamma n m z^{n-1}$ lie in

$|z| < 1$ with s-fold zeros at origin for every $\gamma \in \mathcal{C}$ with $|\gamma| < 1$. This gives

$$|g'(z)| \geq nm|z|^n \quad \text{for } |z| \geq 1. \tag{19}$$

Choosing argument of γ in the left hand side of (18) such that

$$|g'(z) + nm\gamma z^{n-1}| = |g'(z)| - nm|\gamma|, \quad \text{for } |z| = 1,$$

which is possible by (19), we get

$$|g'(z)| - nm|\gamma| \geq \frac{1}{2}\left(n + s + \frac{k^{n-s}|a_{n-s}| - |\gamma m| - |a_0|}{k^{n-s}|a_{n-s}| - |\gamma m| + |a_0|}\right)(|g(z)| - m|\gamma|),$$

which gives

$$\begin{aligned} |g'(z)| \geq & \frac{1}{2}\left(n + s + \frac{k^{n-s}|a_{n-s}| - |\gamma m| - |a_0|}{k^{n-s}|a_{n-s}| - |\gamma m| + |a_0|}\right)|g(z)| \\ & + \frac{1}{2}\left(n + s - \frac{k^{n-s}|a_{n-s}| - |\gamma m| - |a_0|}{k^{n-s}|a_{n-s}| - |\gamma m| + |a_0|}\right)|\gamma| m. \end{aligned}$$

Replacing $g(z)$ by $P(kz)$, we get

$$\begin{aligned} k \max_{|z|=k}|P'(z)| \geq & \frac{1}{2}\left(n + s + \frac{k^{n-s}|a_{n-s}| - |\gamma m| - |a_0|}{k^{n-s}|a_{n-s}| - |\gamma m| + |a_0|}\right)\max_{|z|=k}|P(z)| \\ & + \frac{1}{2}\left(n + s - \frac{k^{n-s}|a_{n-s}| - |\gamma m| - |a_0|}{k^{n-s}|a_{n-s}| - |\gamma m| + |a_0|}\right)|\gamma| m. \end{aligned}$$

Since $P'(z)$ is a polynomial of degree $n-1$, from above inequality we obtain by applying (10) of Lemma 3.3 with $R = k \geq 1$ that, if $n > 2$,

$$\begin{aligned} 2k^n \max_{|z|=1}|P'(z)| - 2(k^n - k^{n-2})|a_1| \geq & \left(n + s + \frac{k^{n-s}|a_{n-s}| - |\gamma m| - |a_0|}{k^{n-s}|a_{n-s}| - |\gamma m| + |a_0|}\right)\max_{|z|=k}|P(z)| \\ & + \left(n + s - \frac{k^{n-s}|a_{n-s}| - |\gamma m| - |a_0|}{k^{n-s}|a_{n-s}| - |\gamma m| + |a_0|}\right)|\gamma| m. \end{aligned} \tag{20}$$

Again as before, let $P(z) = z^s h(z)$, where $h(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \leq k$, $k \geq 1$. Therefore by using (12) of Lemma 3.4 with $R = k \geq 1$, we have for $0 \leq l < 1$ and $n > 2$

$$\begin{aligned} \max_{|z|=k}|h(z)| \geq & \frac{2k^{n-s}}{1 + k^{n-s}}\max_{|z|=1}|h(z)| - l\left(\frac{k^{n-s} - 1}{k^{n-s} + 1}\right)\min_{|z|=k}|h(z)| \\ & + \frac{2k^{n-s-1}|a_{n-s-1}|}{k^{n-s} + 1}\left(\frac{k^{n-s} - 1}{n - s} - \frac{k^{n-s-2} - 1}{n - s - 2}\right). \end{aligned} \tag{21}$$

Also $\max_{|z|=k}|h(z)| = \frac{1}{k^s}\max_{|z|=k}|P(z)|$, $\min_{|z|=k}|h(z)| = \frac{1}{k^s}\min_{|z|=k}|P(z)|$ and $\max_{|z|=1}|h(z)| = \max_{|z|=1}|P(z)|$. Using these in inequality (21), we have for $n > 2$ and $0 \leq l < 1$,

$$\begin{aligned} \max_{|z|=k}|P(z)| \geq & \frac{2k^n}{1 + k^{n-s}}\max_{|z|=1}|P(z)| + l\left(\frac{k^{n-s} - 1}{k^{n-s} + 1}\right)\min_{|z|=k|}|P(z)| \\ & + \frac{2k^{n-1}|a_{n-s-1}|}{k^{n-s} + 1}\left(\frac{k^{n-s} - 1}{n - s} - \frac{k^{n-s-2} - 1}{n - s - 2}\right). \end{aligned} \tag{22}$$

Using inequality (22) in inequality (20) and taking $\gamma = l$, we obtain for $n > 2$ and $0 \leq l < 1$

$$\begin{aligned}\max_{|z|=1}|P'(z)| \geq & \frac{1}{2k^n}\left(n+s+\frac{k^{n-s}|a_{n-s}|-lm-|a_0|}{k^{n-s}|a_{n-s}|-lm+|a_0|}\right)\left\{\frac{2k^n}{1+k^{n-s}}\max_{|z|=1}|P(z)|\right.\\ & \left.+l\left(\frac{k^{n-s}-1}{k^{n-s}+1}\right)\min_{|z|=k}|P(z)|+\frac{2k^{n-1}|a_{n-s-1}|}{k^{n-s}+1}\left(\frac{k^{n-s}-1}{n-s}-\frac{k^{n-s-2}-1}{n-s-2}\right)\right\}\\ & +(1-1/k^2)|a_1|+\left(n+s-\frac{k^{n-s}|a_{n-s}|-lm-|a_0|}{k^{n-s}|a_{n-s}|-lm+|a_0|}\right)lm.\end{aligned}$$

By simplifying this, for $0 \leq l < 1$ and $n > 2$, we get

$$\begin{aligned}\max_{|z|=1}|P'(z)| \geq & \frac{(n+s)}{1+k^{n-s}}\left(\max_{|z|=1}|P(z)|+\frac{1}{k^s}lm\right)\\ & +\frac{(n+s)|a_{n-s-1}|}{k(1+k^{n-s})}\left(\frac{k^{n-s}-1}{n-s}-\frac{k^{n-s-2}-1}{n-s-2}\right)+(1-1/k^2)|a_1|\\ & +\left(\frac{k^{n-s}|a_{n-s}|-lm-|a_0|}{k^{n-s}|a_{n-s}|-lm+|a_0|}\right)\left\{\frac{1}{k^n(1+k^{n-s})}\left(k^n\max_{|z|=1}|P(z)|-lm\right)\right.\\ & \left.+\frac{|a_{n-s-1}|}{k(1+k^{n-s})}\left(\frac{k^{n-s}-1}{n-s}-\frac{k^{n-s-2}-1}{n-s-2}\right)\right\}.\end{aligned}$$

This proves inequality (8) in the case $n > 2$. For the case $n = 2$, the result follows similarly but instead of using inequality (10) of Lemma 3.3 and inequality (12) of Lemma 3.4, we use inequality (11) of Lemma 3.3 and inequality (13) of Lemma 3.4. This completely proves Theorem 2.5.

References

[1] A. Aziz, *Inequalities for the derivative of a polynomial*, Proc. Amer. Math. Soc., **89(2)** (1983), 259–266.

[2] A. Aziz, Q. M. Dawood, *Inequalities for a polynomial and its derivative*, J. Approx. Theory, **54** (1998), 259–266.

[3] A. Aziz, N. A. Rather, *A refinement of a theorem of Paul Turan concerning polynomials*, Math. Ineq. Appl., **1** (1998), 231–238.

[4] A. Aziz, N. A. Rather, *Inequalities for the polar derivative of a polynomial with restricted zeros*, Math. Bulk., **17** (2003), 15–28.

[5] S. Bernstein, *Sur l'ordre da la meilleure approximation des functions continues pardes polynomials de degree donne*, Mem. Acad. R. Belg. **4** (1912), 1–103.

[6] P. Borwein, T. Erdélyi, *Polynmials and Polynomial Inequalities*, Springer-Verlag, New York, 1995.

[7] V. N. Dubinin, *Applications of the Schwarz lemma to inequalities for entire functions with constraints on zeros*, J. Math. Sci., **143** (2007), 3069–3076.

[8] C. Frappier, Q. I. Rahman and Rt. St. Ruscheweyh, *New inequalities for polynomials*, Trans. Amer. Math. Soc., **288** (1985), 69–99.

[9] N. K. Govil, *On the derivative of a polynomial*, Proc. Amer. Math. Soc. **41** (1973), 543–546.

[10] N. K. Govil and P. Kumar, *On sharpening of an inequality of Turan*, Appl. Anal. Discrete Math., **13** (2019), 711–720.

[11] G. G. Lorentz, M. V. Golitschek, Y.Makovoz, *Constructive approximation, Advanced Problems*, Springer Berlin Heidelberg, 1996.

[12] A. A. Markov, *On a problem of D. I. Mendeleev*, Zap. Imp. Akad. Nauk, St.Petersburg,**62** (1889), 1–24.

[13] G. V. Milovanović, M. Th. Rassias, *Analytic Number Theory, Approximation Theory and Special Functions*, Springer, New York (2014), 247–295.

[14] G. V. Milovanović, D. S. Mitrinović, M. Th. Rassias, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros*, World scientific Publishing Co., Singapore, 1994.

[15] N. A. Rather, I. Dar, A. Iqbal, *On a refinement of Turán's inequality*, Complex Anal Synerg., **6**, (2020).

[16] N. A. Rather, I. Dar, A. Iqbal, *Inequalities for the derivative of polynomials with restricted zeros*, Korean J. Math., **28** (2020), 931–942.

[17] N. A. Rather, I. Dar, A. Iqbal, *Some extensions of a theorem of Paul Turan concerning polynomials*, Kragujevac J. Math., **46** (2022), 969–979.

[18] S. A. Telyakovskii, *Research in the theory of Approximation of functions at the mathematical institute of the academy of sciences*, Trudi Mat. Inst. Steklov, 182(1988): English trans. in: Proc. Steklov Inst. Math., (1990), 141–197.

[19] P. Turán, *Über die Ableitung von polynomen*, Compos. Math. **7** (1939), 89–95.

(received 10.01.2021; in revised form 05.09.2021; available online 23.06.2022)

Department of Mathematics, University of Kashmir, Srinagar-190006, India

E-mail: takhter595@gmail.com

Department of Mathematics, University of Kashmir, Srinagar-190006, India

E-mail: bazargar@gmail.com

Department of Mathematics, University of Kashmir, Srinagar-190006, India

E-mail: gulzarmh@gmail.com