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NORMALIZED LAPLACIAN ENERGY AND NORMALIZED LAPLACIAN-ENERGY-LIKE INVARIANT OF SOME DERIVED GRAPHS

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Abstract. For a connected graph G, the smallest normalized Laplacian eigenvalue is 0 while all others are positive and the largest cannot exceed the value 2. The sum of absolute deviations of the eigenvalues from 1 is called the normalized Laplacian energy, denoted by $\mathbb{LE}(G)$. In analogy with Laplacian-energy-like invariant of G, we define here the normalized Laplacian-energy-like as the sum of square roots of normalized Laplacian eigenvalues of G, denoted by $\mathbb{LEL}(G)$.

In this paper, we obtain upper and lower bounds of $\mathbb{LE}(G)$ and upper bound of $\mathbb{LEL}(G)$ of some derived graphs, such as double graph, extended double cover and Mycielskian of a regular graph, and we show that the bounds obtained here are better than some existing bounds.

1. Introduction

For a connected graph G with vertices $\{v_1, v_2, \ldots, v_n\}$, let $A(G)$ be the adjacency matrix and $D(G) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, where d_k is the degree of the vertex $v_k, k = 1, 2, \ldots, n$. Then $L(G) = D(G) - A(G)$ is called the Laplacian matrix and $\mathcal{L}(G) = D^{-\frac{1}{2}} L(G) D^{-\frac{1}{2}}$ is called the normalized Laplacian matrix of G, respectively. It is well known that $L(G)$ and $\mathcal{L}(G)$ are both positive semi-definite.

Sum of absolute values of the eigenvalues of $A(G)$ is known as the energy of G. After motivation of the success of graph energy [11], Laplacian energy of G was put forward by Gutman and Zhou [6]. Let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n = 0$ be the eigenvalues of $L(G)$. Laplacian energy of G is defined as $LE(G) = \sum_1^n |\delta_i - \bar{d}|$, where, $\bar{d} = \sum_1^n \frac{d_i}{n}$.

Cavers et al. [3] extended it to define the normalized Laplacian energy as $\mathbb{LE}(G) = \sum_{i=1}^n |\mu_i - 1|$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the normalized Laplacian eigenvalues

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of G. They have also obtained upper and lower bounds of $\mathbb{LE}(G)$ in terms of maximum vertex degree $d_{\max}$, minimum vertex degree $d_{\min}$, as follows:

$$\frac{n}{n-1} \leq \frac{n}{d_{\max}} \leq \mathbb{LE}(G) \leq \frac{n}{\sqrt{d_{\min}}} \leq n. \tag{1}$$

The normalized Laplacian energy of the line and para-line graphs of a graph have been obtained by Hakimi-Nezhaad and Ashrafi [7].

In connection with Randić index, one of the oldest molecular structure-descriptor, Bozkurt et al. [2] proposed Randić matrix as the symmetric square matrix $\mathbf{R} = (R_{ij})_{n\times n}$, where

$$R_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}.$$

They have also introduced the notion of Randić energy $RE(G)$ as the sum of the absolute values of the eigenvalues of $\mathbf{R}$ and they have shown that for a graph G with no isolated vertices, $\mathbb{LE}(G)$ coincides with $RE(G)$.

Das and Sorgun [4] have shown that

$$RE(G) \leq 1 + \sqrt{\frac{(n-1)(n-d_{\min})}{d_{\min}}} \tag{2}$$

with equality if and only if $G \cong K_n$ or G is a strongly regular graph.

In particular, for a r-regular graph G, Das and Sorgun bound gives

$$RE(G) \leq 1 + \sqrt{\frac{(n-1)(n-r)}{r}} \tag{3}$$

with equality if and only if $r = n-1$ or G is a strongly regular graph wth parameters $n, r, \frac{r(r-1)}{n-1}, \frac{r(r-1)}{n-1}$.

Liu and Liu [13] introduced Laplacian-energy-like (LEL) invariant of a connected graph G as $LEL(G) = \sum_{i=1}^{n-1} \sqrt{\delta_i}$. In analogy with Laplacian-energy-like invariant of G [15, 16], we define here the normalized Laplacian-energy-like as

$$\mathbb{LEL}(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}.$$

Li [12] obtained an upper bound of the sum of the βth power of the non zero normalized Laplacian eigenvalues as follows:

$$s^*_\beta \leq k^{1-\beta}\left(\frac{nk}{n-1}\right)^\beta + (n-1-k)\left(\frac{n}{n-1}\right)^\beta$$

with equality if and only if $G \cong K_n$.

Thus, for $\beta = 1/2$, we have,

$$\mathbb{LEL}(G) = s^*_{1/2} \leq \sqrt{n(n-1)} \text{ with equality if and only if } G \cong K_n. \tag{4}$$

Huang and Li [8] deduced a formula for the normalized Laplacian characteristic polynomial of some derived graph of a regular graph G and obtained a sharp lower bound for the degree-Kirchhoff index and a formula for the number of spanning trees of those derived graphs. Recently, Amin and Nayeem [1] have studied some derived graphs, such as double graph, extended double cover and Mycielskian of a r-regular graph G to obtain some upper bounds of Kirchhoff index and Laplacian-energy-like invariant of those derived graphs. In the present paper, we have futher obtained the normalized Laplacian eigenvalues of those derived graphs, and hence we have obtained some bounds of $\mathbb{LE}$ and $\mathbb{LEL}$ for those derived graphs.

Let $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$ be another set of vertices each of which corresponds to a vertex of V. The double graph of G is the graph G^* whose vertex set consists of the disjoint union of V and U, and edge set is the union of E and $\{(u_i, v_j) : (v_i, v_j) \in E,\ i = 1, 2, \ldots, n, j = 1, 2, \ldots, n\}$. The extended double cover of G, denoted by G^{**} is the graph with vertex set $V \cup U$, and edge set $\{(u_i, v_i), i = 1, 2, \ldots, n\} \cup \{(u_i, v_j) : (v_i, v_j) \in E, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n\}$. It is easy to follow that the extended double cover of a r-regular graph is $(r + 1)$-regular. The Mycielskian of G, denoted by $\mu(G)$ is the graph with vertex set consisting of disjoint union $V \cup U \cup \{u\}$, and edge set $E \cup \{(u_i, v_j) : (v_i, v_j) \in E, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n\} \cup \{(u, u_i) : u_i \in U, i = 1, 2, \ldots, n\}$.

In Section 2, we find the normalized Laplacian spectrum of double graph, extended double cover and Mycielskian of a regular graph G in terms of adjacency eigenvalues of G. Using those values, we obtain upper and lower bounds of $\mathbb{LE}$ and upper bounds of $\mathbb{LEL}$ for those graphs in Section 3 and Section 4, respectively. In Section 5, we compare the bounds with some existing bounds. In Section 6, we obtain the exact formulas for $\mathbb{EL}$, $\mathbb{LEL}$ and $\mathbb{K}f$ of double graph, extended double cover and Mycielskian of K_n and $K_{r,r}$.

2. Normalized Laplacian eigenvalues of derived graphs

Theorem 2.1. *Let G be an r-regular graph of order n. Then the normalized Laplacian eigenvalues of G^* are given by $\frac{(4r-\lambda_1)\pm 3\lambda_1}{4r}$, $\frac{(4r-\lambda_2)\pm 3\lambda_2}{4r}$, $\ldots$, $\frac{(4r-\lambda_n)\pm 3\lambda_n}{4r}$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the (adjacency) eigenvalues of G.*

Proof. For simplicity, let us denote the adjacency matrix of G by A. Since A is real symmetric, there is an orthogonal matrix P such that $A = P\text{Ł}P^T$, where $\text{Ł} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Now, from the definition of G^*, we have

$$A^* = \begin{pmatrix} A & A \\ A & 0 \end{pmatrix} \quad \text{and } D^* = \begin{pmatrix} 2rI & 0 \\ 0 & rI \end{pmatrix}, \quad \text{where } I \text{ is the identity matrix of order } n.$$

Therefore, $\mathcal{L}^* = D^* - A^* = \begin{pmatrix} 2rI-P\text{Ł}P^T & -P\text{Ł}P^T \\ -P\text{Ł}P^T & rI \end{pmatrix}$, and

$$\begin{aligned}\mathcal{L}^* = D^{*-\frac{1}{2}} L^* D^{*-\frac{1}{2}} &= \begin{pmatrix} \frac{1}{\sqrt{2r}}I & 0 \\ 0 & \frac{1}{\sqrt{r}}I \end{pmatrix} \begin{pmatrix} 2rI - P\text{Ł}P^T & -P\text{Ł}P^T \\ -P\text{Ł}P^T & rI \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2r}}I & 0 \\ 0 & \frac{1}{\sqrt{r}}I \end{pmatrix} \\ &= \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} I - \frac{1}{2r}\text{Ł} & -\frac{1}{\sqrt{2}r}\text{Ł} \\ -\frac{1}{\sqrt{2}r}\text{Ł} & I \end{pmatrix} \begin{pmatrix} P^T & 0 \\ 0 & P^T \end{pmatrix}.\end{aligned}$$

Since P is an orthogonal matrix, the spectrum of $\mathcal{L}^*$ is the same as the spectrum of

$$B = \begin{pmatrix} I - \frac{1}{2r}\text{Ł} & -\frac{1}{\sqrt{2r}}\text{Ł} \\ -\frac{1}{\sqrt{2r}}\text{Ł} & I \end{pmatrix}.$$

Now, the characteristic polynomial of B, i.e., $\det(B - xI)$ is given by

$$\left|\begin{array}{cccc:cccc}
1-\frac{1}{2r}\lambda_1-x & & 0 & & -\frac{1}{\sqrt{2r}}\lambda_1 & & & 0 \\
 & 1-\frac{1}{2r}\lambda_2-x & & & & -\frac{1}{\sqrt{2r}}\lambda_2 & & \\
 & & \ddots & & & & \ddots & \\
0 & & & 1-\frac{1}{2r}\lambda_n-x & 0 & & & -\frac{1}{\sqrt{2r}}\lambda_n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\frac{1}{\sqrt{2r}}\lambda_1 & & 0 & & 1-x & & & 0 \\
 & -\frac{1}{\sqrt{2r}}\lambda_2 & & & & 1-x & & \\
 & & \ddots & & & & \ddots & \\
0 & & & -\frac{1}{\sqrt{2r}}\lambda_n & 0 & & & 1-x
\end{array}\right|.$$

We expand $\det(B - xI)$ by Laplace's method with respect to first and $(n + 1)$th columns. Clearly, in this expansion, the only non zero minor of order 2 is the minor consisting of the above two columns, and the first and $(n+1)$th rows and it is given by

$$M = \begin{vmatrix} 1 - \frac{1}{2r}\lambda_1 - x & -\frac{1}{\sqrt{2r}}\lambda_1 \\ -\frac{1}{\sqrt{2r}}\lambda_1 & 1 - x \end{vmatrix}.$$

Next we expand the complementary minor of M with respect to 2nd and $(n + 2)$th columns, and then the resulting complementary minor with respect to 3rd and $(n+3)$th columns and so on. Therefore the eigenvalues of $\mathcal{L}^*$ are the roots of the factors $\begin{vmatrix} 1-\frac{1}{2r}\lambda_1-x & -\frac{1}{\sqrt{2r}}\lambda_1 \\ -\frac{1}{\sqrt{2r}}\lambda_1 & 1-x \end{vmatrix}$, $\begin{vmatrix} 1-\frac{1}{2r}\lambda_2-x & -\frac{1}{\sqrt{2r}}\lambda_2 \\ -\frac{1}{\sqrt{2r}}\lambda_2 & 1-x \end{vmatrix}, \dots, \begin{vmatrix} 1-\frac{1}{2r}\lambda_n-x & -\frac{1}{\sqrt{2r}}\lambda_n \\ -\frac{1}{\sqrt{22}}\lambda_n & 1-x \end{vmatrix}$. Therefore the spectrum of $\mathcal{L}^*$ is given by $\left\{\frac{(4r-\lambda_i)\pm 3\lambda_i}{4r}, i = 1, 2, \dots, n\right\}$. □

THEOREM 2.2. *Let G be an r-regular graph of order n. Then the normalized Laplacian eigenvalues of G^{**} are given by $\frac{(r+1)\pm(\lambda_1+1)}{r+1}, \frac{(r+1)\pm(\lambda_2+1)}{r+1}, \dots, \frac{(r+1)\pm(\lambda_1+1)}{r+1}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the (adjacency) eigenvalues of G.*

Proof. From the definition of G^{**}, we have $A^{**} = \left(\begin{smallmatrix} 0 & A+I \\ A+I & 0 \end{smallmatrix}\right)$ and thus the adjacency eigenvalues of G^{**} are $\pm(\lambda_i + 1), i = 1, 2, \dots, n$.

Since G^{**} is an $(r + 1)$-regular graph with $2n$ vertices, $\mathcal{L}^{**} = I_{2n} - \frac{1}{r+1}A^{**}$ and hence the result follows. □

THEOREM 2.3. *Let G be an r-regular graph of order n. Then the normalized Laplacian eigenvalues of $\mu(G)$ are given by $0, \alpha, \beta, \frac{(4r-\lambda_2)\pm\lambda_2\sqrt{\frac{9r+1}{r+1}}}{4r}, \frac{(4r-\lambda_3)\pm\lambda_3\sqrt{\frac{9r+1}{r+1}}}{4r}, \dots, \frac{(4r-\lambda_n)\pm\lambda_n\sqrt{\frac{9r+1}{r+1}}}{4r}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the (adjacency) eigenvalues of G and α, β are the roots of the equation $2(r + 1)x^2 - 5(r + 1)x + 3r + 2 = 0$.*

Proof. As in Theorem 2.1, if A is the adjacency matrix of G and P is the orthogonal matrix formed by the orthonormal eigenvectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$ then, $A = P Ł P^T$ where $Ł = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Since G is r-regular, $\lambda_1 = r$ and $p_1 = \frac{e}{\sqrt{n}}$, where e denotes the n-vector with all entries equal to 1. Hence $e^T p_i = 0$ for $i = 2, 3, \ldots, n$, and consequently $e^T P = [\sqrt{n}, 0, \ldots, 0]$.

Therefore, the adjacency matrix of the Mycielskian $\mu(G)$ of G is a matrix of order $(2n+1)$ and it is given by $\mu(A) = \begin{pmatrix} A & A & 0 \\ A & 0 & e \\ 0 & e^T & 0 \end{pmatrix} = \begin{pmatrix} PŁP^T & PŁP^T & 0 \\ PŁP^T & 0 & e \\ 0 & e^T & 0 \end{pmatrix}$ and $\mu(D) = \begin{pmatrix} 2rI & 0 & 0 \\ 0 & (r+1)I & 0 \\ 0 & 0 & n \end{pmatrix}$, where I is the identity matrix of order n.

Therefore, $\mu(L) = \begin{pmatrix} 2rI-PŁP^T & -PŁP^T & 0 \\ -PŁP^T & (r+1)I & -e \\ 0 & -e^T & n \end{pmatrix}$, and

$$\begin{aligned}
\mu(\mathcal{L}) =& \mu(D)^{-\frac{1}{2}}\mu(L)\mu(D)^{-\frac{1}{2}} \\
=& \begin{pmatrix} \frac{1}{\sqrt{2r}}I & 0 & 0 \\ 0 & \frac{1}{\sqrt{r+1}}I & 0 \\ 0 & 0 & \frac{1}{\sqrt{n}} \end{pmatrix} \begin{pmatrix} 2rI-PŁP^T & -PŁP^T & 0 \\ -PŁP^T & (r+1)I & -e \\ 0 & -e^T & n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2r}}I & 0 & 0 \\ 0 & \frac{1}{\sqrt{r+1}}I & 0 \\ 0 & 0 & \frac{1}{\sqrt{n}} \end{pmatrix} \\
=& \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I-\frac{1}{2r}Ł & -\frac{1}{\sqrt{2r(r+1)}}Ł & 0 \\ -\frac{1}{\sqrt{2r(r+1)}}Ł & I & -\frac{1}{\sqrt{n(r+1)}}P^Te \\ 0 & -\frac{1}{\sqrt{n(r+1)}}e^TP & 1 \end{pmatrix} \begin{pmatrix} P^T & 0 & 0 \\ 0 & P^T & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Thus the spectrum of $\mu(\mathcal{L})$ is the same as the spectrum of

$$B = \begin{pmatrix} I-\frac{1}{2r}Ł & -\frac{1}{\sqrt{2r(r+1)}}Ł & 0 \\ -\frac{1}{\sqrt{2r(r+1)}}Ł & I & -\frac{1}{\sqrt{n(r+1)}}P^Te \\ 0 & -\frac{1}{\sqrt{n(r+1)}}e^TP & 1 \end{pmatrix}.$$

Expanding as before, we get the eigenvalues of $\mu(L)$ as the roots of the factors

$$\begin{vmatrix} 1-\frac{1}{2r}\lambda_1 - x & -\frac{1}{\sqrt{2r(r+1)}}\lambda_1 & 0 \\ -\frac{1}{\sqrt{2r(r+1)}}\lambda_1 & 1-x & -\sqrt{n} \\ 0 & -\sqrt{n} & 1-x \end{vmatrix} \begin{vmatrix} 1-\frac{1}{2r}\lambda_2 - x & -\frac{1}{\sqrt{2r(r+1)}}\lambda_2 \\ -\frac{1}{\sqrt{2r(r+1)}}\lambda_2 & 1-x \end{vmatrix}, \ldots,$$

$$\begin{vmatrix} 1-\frac{1}{2r}\lambda_n - x & -\frac{1}{\sqrt{2r(r+1)}}\lambda_n \\ -\frac{1}{\sqrt{2r(r+1)}}\lambda_n & 1-x \end{vmatrix}.$$

Hence the spectrum of $\mu((L))$ is given by $\left\{0, \alpha, \beta, \frac{(4r-\lambda_2)\pm\lambda_2\sqrt{\frac{9r+1}{r+1}}}{4r}, \ldots, \frac{(4r-\lambda_n)\pm\lambda_n\sqrt{\frac{9r+1}{r+1}}}{4r}\right\}$, where α, β, are the roots of the equation $2(r+1)x^2 - 5(r+1)x + 3r + 2 = 0$.

Here we note that $\alpha + \beta = \frac{5}{2}, \alpha\beta = \frac{3r+2}{2r+2}$ and so, $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{5(r+1)}{3r+2}$. □

3. Upper and lower bounds of normalized Laplacian energy of derived graphs

An upper bound for energy of an r-regular graph of order n, known as Koolen-Moulton inequality [9, 10], is given by

$$E(G) \leq r + \sqrt{r(n-1)(n-r)}. \tag{5}$$

Gutman et al. [5] obtained a lower bound for energy as follows:

$$E(G) \geq n. \tag{6}$$

THEOREM 3.1. *Let G be a simple r-regular graph with n vertices. Then*

$$\mathbb{EL}(G^*) \leq \frac{3}{2}\left\{1 + \sqrt{\frac{(n-1)(n-r)}{r}}\right\}.$$

Proof.

$$\begin{aligned}\mathbb{EL}(G^*) &= \sum_{i=1}^{2n} |\mu_i - 1| = \sum_{i=1}^{n} \left|\frac{(4r-\lambda_i) \pm 3\lambda_i}{4r} - 1\right| = \sum_{i=1}^{n} \left|\frac{-\lambda_i \pm 3\lambda_i}{4r}\right| \\ &= \frac{1}{4r}\sum_{i=1}^{n} |2\lambda_i| + \frac{1}{4r}\sum_{i=1}^{n} |-4\lambda_i| = \frac{1}{2r}\sum_{i=1}^{n} |\lambda_i| + \frac{1}{r}\sum_{i=1}^{n} |\lambda_i| = \frac{3}{2r}E(G) \\ &\leq \frac{3}{2}\left\{1 + \sqrt{\frac{(n-1)(n-r)}{r}}\right\} \text{ (using (5)).}\end{aligned}$$

□

THEOREM 3.2. *Let G be a simple r-regular graph with n vertices. Then $\mathbb{EL}(G^*) \geq \frac{3n}{2r}$.*

Proof. $\mathbb{EL}(G^*) = \sum_{i=1}^{2n} |\mu_i - 1| = \frac{3}{2r}E(G) \geq \frac{3n}{2r}$ (using (6)). □

THEOREM 3.3. *Let G be a simple r-regular graph with n vertices. Then*

$$\frac{2n}{r+1} \leq \mathbb{EL}(G^{**}) \leq 1 + \sqrt{\frac{(2n-1)(2n-r-1)}{r+1}}.$$

Proof. Since G^{**} is $(r+1)$-regular, $\mathbb{EL}(G^{**}) = \frac{1}{r+1}E(G^{**})$. Hence the theorem follows from Gutman's bound and Koolen-Moulton inequality. □

REMARK 3.4. Upper bound of $\mathbb{EL}(G^{**})$ in Theorem 3.3 coincides with that given by Das-Sorgun bound [see (3)].

LEMMA 3.5. *If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of an r-regular graph G on n vertices, then $\sum_{i=1}^{n} |\lambda_i + 1| \leq r + 1 + \sqrt{(n-1)(r+1)(n-r-1)}$.*

Proof. Since G is r-regular, $\lambda_1 = r$. Also we have, $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i^2 =$ twice the number of edges of $G = nr$. Hence, $\sum_{i=2}^{n} \lambda_i = -r$ and $\sum_{i=2}^{n} \lambda_i^2 = nr - r^2$. Now, by Cauchy-Schwarz inequality,

$$\sum_{i=2}^{n} |\lambda_i + 1| \leq \sqrt{(n-1)\sum_{i=2}^{n}(\lambda_i + 1)^2} = \sqrt{(n-1)\left(\sum_{i=2}^{n}\lambda_i^2 + 2\sum_{i=2}^{n}\lambda_i + n - 1\right)}$$

$$=\sqrt{(n-1)\left(nr-r^2-2r+n-1\right)}=\sqrt{(n-1)(r+1)(n-r-1)}.$$

Hence the result follows. □

THEOREM 3.6. *Let G be a simple r-regular graph with n vertices. Then*

$$\mathbb{EL}(G^{**})\leq 2\left\{1+\sqrt{\frac{(n-1)(n-r-1)}{r+1}}\right\}.$$

Proof.

$$\begin{aligned}\mathbb{EL}(G^{**})=&\frac{1}{r+1}E(G^{**})=\frac{1}{r+1}\sum_{i=1}^{n}|\pm(\lambda_i+1)|=\frac{2}{r+1}\sum_{i=1}^{n}|\lambda_i+1|\\ \leq&\frac{2}{r+1}\left\{r+1+\sqrt{(n-1)(r+1)(n-r-1)}\right\}\ \text{(by Lemma 3.5)}\\ =&2\left\{1+\sqrt{\frac{(n-1)(n-r-1)}{r+1}}\right\}.\end{aligned}$$

□

LEMMA 3.7. *If α and β satisfy the relations $\alpha+\beta=\frac{5}{2}$ and $\alpha\beta=\frac{3r+2}{2r+2}$, then $|\alpha-1|+|\beta-1|\leq\frac{\sqrt{5}}{2}$.*

Proof. As $\alpha,\beta>0$ and $\alpha+\beta=\frac{5}{2}$, α and β both cannot be less than 1 simultaneously and so $|\alpha-\beta|\leq|\alpha-1|+|\beta-1|\leq\max\{(\alpha-\beta),(\beta-\alpha)\}=|\alpha-\beta|$. Thus,

$$|\alpha-1|+|\beta-1|=|\alpha-\beta|=\sqrt{(\alpha+\beta)^2-4\alpha\beta}=\sqrt{\frac{25}{4}-4\cdot\frac{3r+2}{2(r+1)}}=\sqrt{\frac{25}{4}-\frac{6r+4}{r+1}}.$$

Now, $\frac{6r+4}{r+1}\geq 5$ since $r\geq 1$, and thus $|\alpha-\beta|\leq\frac{\sqrt{5}}{2}$. □

THEOREM 3.8. *Let G be a simple r-regular graph with n vertices. Then*

$$\mathbb{EL}(\mu(G))\leq 1+\frac{\sqrt{5}}{2}+\frac{1}{2}\sqrt{\frac{(n-1)(n-r)(9r+1)}{r(r+1)}}.$$

Proof.

$$\begin{aligned}\mathbb{EL}(\mu(G))=&\sum_{i=1}^{2n}|\mu_i-1|\\ =&|0-1|+|\alpha-1|+|\beta-1|+\left|\frac{(4r-\lambda_2)+\lambda_2\sqrt{\frac{9r+1}{r+1}}}{4r}-1\right|+\left|\frac{(4r-\lambda_2)-\lambda_2\sqrt{\frac{9r+1}{r+1}}}{4r}-1\right|\\ &+\cdots+\left|\frac{(4r-\lambda_n)+\lambda_n\sqrt{\frac{9r+1}{r+1}}}{4r}-1\right|+\left|\frac{(4r-\lambda_n)-\lambda_n\sqrt{\frac{9r+1}{r+1}}}{4r}-1\right|\\ =&1+|\alpha-1|+|\beta-1|+\frac{1}{4r}\left(\left|\sqrt{\frac{9r+1}{r+1}}+1\right|+\left|\sqrt{\frac{9r+1}{r+1}}-1\right|\right)(|\lambda_2|+|\lambda_3|+\cdots+|\lambda_n|)\end{aligned}$$

$$\leq 1+\frac{\sqrt{5}}{2}+2\cdot\frac{1}{4r}\sqrt{\frac{9r+1}{r+1}}(E(G)-r)\,(\text{by Lemma 3.7})$$
$$\leq 1+\frac{\sqrt{5}}{2}+\frac{1}{2}\sqrt{\frac{(n-1)(n-r)(9r+1)}{r(r+1)}}\,(\text{by Lemma 3.5})\,. \qquad \square$$

THEOREM 3.9. *Let G be a simple r-regular graph with n vertices. Then $\mathbb{EL}(\mu(G)) \geq \frac{n+2r}{2r}$.*

Proof.

$$\mathbb{EL}(\mu(G)) = \sum_{i=1}^{2n}|\mu_i-1|$$
$$=|0-1|+|\alpha-1|+|\beta-1|+\left|\frac{(4r-\lambda_2)+\lambda_2\sqrt{\frac{9r+1}{r+1}}}{4r}-1\right|+\left|\frac{(4r-\lambda_2)-\lambda_2\sqrt{\frac{9r+1}{r+1}}}{4r}-1\right|$$
$$+\cdots+\left|\frac{(4r-\lambda_n)+\lambda_n\sqrt{\frac{9r+1}{r+1}}}{4r}-1\right|+\left|\frac{(4r-\lambda_n)-\lambda_n\sqrt{\frac{9r+1}{r+1}}}{4r}-1\right|$$
$$\geq 1+|\alpha+\beta-2|+\frac{2|\lambda_2|}{4r}+\frac{2|\lambda_3|}{4r}+\cdots+\frac{2|\lambda_n|}{4r}=\frac{3}{2}+\frac{2}{4r}(E(G)-r)$$
$$\geq\frac{3}{2}+\frac{1}{2r}(n-r)=\frac{n+2r}{2r}. \qquad \square$$

4. Upper bounds of normalized Laplacian-energy-like invariant of derived graphs

THEOREM 4.1. *Let G be a simple r-regular graph with n vertices. Then*

$$\mathbb{LEL}(G^*) \leq \sqrt{\frac{3}{2}}+\sqrt{(n-1)(4n-3)}.$$

Proof. Since $\lambda_1 = r$ and by Cauchy-Schwarz inequality we get:

$$\mathbb{LEL}(G^*) = \sum_{i=1}^{2n}\sqrt{\mu_i}$$
$$=\sqrt{\frac{(4r-\lambda_1)+3\lambda_1}{4r}}+\sqrt{\frac{(4r-\lambda_1)-3\lambda_1}{4r}}+\sqrt{\frac{(4r-\lambda_2)+3\lambda_2}{4r}}+\sqrt{\frac{(4r-\lambda_2)-3\lambda_2}{4r}}$$
$$+\cdots+\sqrt{\frac{(4r-\lambda_n)+3\lambda_n}{4r}}+\sqrt{\frac{(4r-\lambda_n)-3\lambda_n}{4r}}$$
$$\leq\sqrt{\frac{3}{2}}+0+\sqrt{(2n-2)\left\{\begin{array}{c}\frac{(4r-\lambda_2)+3\lambda_2}{4r}+\frac{(4r-\lambda_2)-3\lambda_2}{4r}+\cdots\\ +\frac{(4r-\lambda_n)+3\lambda_n}{4r}+\frac{(4r-\lambda_n)-3\lambda_n}{4r}\end{array}\right\}}=\sqrt{\frac{3}{2}}+\sqrt{(n-1)(4n-3)}. \qquad \square$$

THEOREM 4.2. *Let G be a simple r-regular graph with n vertices. Then*

$$\mathbb{LEL}(G^{**}) \leq \sqrt{2} + 2n - 2.$$

Proof. Since $\lambda_1 = r$ and by Cauchy-Schwarz inequality we get:

$$\begin{aligned}
\mathbb{LEL}(G^{**}) &= \sum_{i=1}^{2n} \sqrt{\mu_i} \\
&= \sqrt{\frac{(r+1)+(\lambda_1+1)}{r+1}} + \sqrt{\frac{(r+1)-(\lambda_1+1)}{r+1}} + \sqrt{\frac{(r+1)+(\lambda_2+1)}{r+1}} \\
&\quad + \sqrt{\frac{(r+1)-(\lambda_2+1)}{r+1}} + \cdots + \sqrt{\frac{(r+1)+(\lambda_n+1)}{r+1}} + \sqrt{\frac{(r+1)-(\lambda_n+1)}{r+1}} \\
&\leq \sqrt{2} + 0 + \sqrt{(2n-2)\left\{\begin{array}{c} \frac{(r+1)+(\lambda_2+1)}{r+1} + \frac{(r+1)-(\lambda_2+1)}{r+1} + \cdots \\ + \frac{(r+1)+(\lambda_n+1)}{r+1} + \frac{(r+1)-(\lambda_n+1)}{r+1} \end{array}\right\}} = \sqrt{2} + 2n - 2. \qquad \square
\end{aligned}$$

THEOREM 4.3. *Let G be a simple r-regular graph with n vertices. Then*

$$\mathbb{LEL}(\mu(G)) \leq \sqrt{5} + \sqrt{(n-1)(4n-3)}.$$

Proof. As before we have,

$$\begin{aligned}
\mathbb{LEL}(\mu(G)) &= \sum_{i=1}^{2n+1} \sqrt{\mu_i} \\
&= 0 + \sqrt{\alpha} + \sqrt{\beta} + \sqrt{\frac{(4r-\lambda_2) + \lambda_2\sqrt{\frac{9r+1}{r+1}}}{4r}} + \sqrt{\frac{(4r-\lambda_2) - \lambda_2\sqrt{\frac{9r+1}{r+1}}}{4r}} \\
&\quad + \cdots + \sqrt{\frac{(4r-\lambda_n) + \lambda_n\sqrt{\frac{9r+1}{r+1}}}{4r}} + \sqrt{\frac{(4r-\lambda_n) - \lambda_n\sqrt{\frac{9r+1}{r+1}}}{4r}} \\
&\leq \sqrt{2(\alpha+\beta)} + \sqrt{(2n-2)\left\{\begin{array}{c} \frac{(4r-\lambda_2)+\lambda_2\sqrt{\frac{9r+1}{r+1}}}{4r} + \frac{(4r-\lambda_2)-\lambda_2\sqrt{\frac{9r+1}{r+1}}}{4r} + \cdots \\ + \frac{(4r-\lambda_n)+\lambda_n\sqrt{\frac{9r+1}{r+1}}}{4r} + \frac{(4r-\lambda_n)-\lambda_n\sqrt{\frac{9r+1}{r+1}}}{4r} \end{array}\right\}} \\
&= \sqrt{5} + \sqrt{(2n-2)\left\{\frac{(4r-\lambda_2)}{2r} + \cdots + \frac{(4r-\lambda_n)}{2r}\right\}} = \sqrt{5} + \sqrt{(n-1)(4n-3)}. \qquad \square
\end{aligned}$$

5. Comparison of our bounds with some existing bounds

Since for G^*, $d_{\max} = 2r$, $d_{\min} = r$, for G^{**}, $d_{\max} = d_{\min} = r+1$ and for $\mu(G), d_{\max} = 2r, d_{\min} = r+1$, Cavers' bounds (see (1)) for G^*, G^{**} and $\mu(G)$ are respectively

$$\frac{n}{r} \leq \mathbb{EL}(G^*) \leq \frac{2n}{\sqrt{r}}; \tag{7}$$

$$\frac{2n}{r+1} \leq \mathbb{EL}(G^{**}) \leq \frac{2n}{\sqrt{r+1}}; \tag{8}$$

$$\frac{2n+1}{2r} \leq \mathbb{EL}(\mu(G)) \leq \frac{2n+1}{\sqrt{r+1}}.. \tag{9}$$

Similarly, Das-Sorgun bounds (see (2), (3)) for G^*, G^{**} and $\mu(G)$ are respectively

$$\mathbb{EL}(G^*) \leq 1 + \sqrt{\frac{(2n-1)(2n-r)}{r}}; \tag{10}$$

$$\mathbb{EL}(G^{**}) \leq 1 + \sqrt{\frac{(2n-1)(2n-r-1)}{r+1}}; \tag{11}$$

$$\mathbb{EL}(\mu(G)) \leq 1 + \sqrt{\frac{2n(2n-r)}{r+1}}. \tag{12}$$

Again, by Theorem 3.1 to Theorem 3.9, we have

$$\frac{3n}{2r} \leq \mathbb{EL}(G^*) \leq \frac{3}{2}\left\{1 + \sqrt{\frac{(n-1)(n-r)}{r}}\right\}; \tag{13}$$

$$\frac{2n}{r+1} \leq \mathbb{EL}(G^{**}) \leq 2\left\{1 + \sqrt{\frac{(n-1)(n-r-1)}{r+1}}\right\}; \tag{14}$$

$$\frac{n+2r}{2r} \leq \mathbb{EL}(\mu(G)) \leq 1 + \frac{\sqrt{5}}{2} + \frac{1}{2}\sqrt{\frac{(n-1)(n-r)(9r+1)}{r(r+1)}}. \tag{15}$$

Now,

$$\frac{n}{r} < \frac{3n}{2r} \text{ and } \frac{3}{2}\left\{1 + \sqrt{\frac{(n-1)(n-r)}{r}}\right\} < \frac{2n}{\sqrt{r}}, \quad \text{since } \frac{n}{\sqrt{r}} > \frac{3}{2} \text{ as } r \leq n-1,$$

$$2\left\{1 + \sqrt{\frac{(n-1)(n-r-1)}{r+1}}\right\} < \frac{2n}{\sqrt{r+1}} \quad \text{for } n > 4, \text{ and}$$

$$\frac{n+2r}{2r} < \frac{2n+1}{2r} \text{ and } 1 + \frac{\sqrt{5}}{2} + \frac{1}{2}\sqrt{\frac{(n-1)(n-r)(9r+1)}{r(r+1)}} < \frac{2n+1}{\sqrt{r+1}} \quad \text{for } r > \frac{n+1}{2}.$$

Therefore the bounds (13), (14) and (15) are better than the bounds (7), (8) and (9), respectively.

Again since $1 \leq r \leq n-1$, observing the coefficients of n^2 within the square roots in the upper bounds given in (13) and (15) and those in the bounds (10) and (12), it is easy to conclude that for sufficiently large values of n, the upper bounds given in (13) and (15) are tighter than the bounds (10) and (12), respectively.

It can also be shown that $2\left\{1 + \sqrt{\frac{(n-1)(n-r-1)}{r+1}}\right\} < 1 + \sqrt{\frac{(2n-1)(2n-r-1)}{r+1}}$, because if it is not so, we reach to the conclusion $r^2(n-2)^2 + 4r(n-1)^2 \leq 0$, which is impossible.

Thus the upper bounds given in (13), (14) and (15) are also better than the bounds (10), (11) and (12), respectively.

Li's bound (see (4)) gives the following:

$$s^*_{\frac{1}{2}}(G^*) \le \sqrt{2n(2n-1)}; \tag{16}$$

$$s^*_{\frac{1}{2}}(G^{**}) \le \sqrt{2n(2n-1)}; \tag{17}$$

$$s^*_{\frac{1}{2}}(\mu(G)) \le \sqrt{2n(2n+1)}. \tag{18}$$

From Theorems 4.1, 4.2 and 4.3, we have

$$\mathbb{LEL}(G^*) \le \sqrt{\frac{3}{2}} + \sqrt{(n-1)(4n-3)}; \tag{19}$$

$$\mathbb{LEL}(G^{**}) \le \sqrt{2} + 2n - 2; \tag{20}$$

$$\mathbb{LEL}(\mu(G)) \le \sqrt{5} + \sqrt{(n-1)(4n-3)}. \tag{21}$$

Now, we shall show that $\sqrt{\frac{3}{2}} + \sqrt{(n-1)(4n-3)} \le \sqrt{2n(2n-1)}$. If possible, let $\sqrt{\frac{3}{2}} + \sqrt{(n-1)(4n-3)} > \sqrt{2n(2n-1)}$.

Then, $$\frac{3}{2} + (n-1)(4n-3) + \sqrt{6(4n-3)(n-1)} > 4n^2 - 2n$$

$$\Rightarrow \sqrt{6(4n-3)(n-1)} > 5n - \frac{9}{2} \Rightarrow 6(4n-3)(n-1) > \left(5n - \frac{9}{2}\right)^2$$

$$\Rightarrow n^2 - 3n + \frac{9}{4} < 0 \Rightarrow \left(n - \frac{3}{2}\right)^2 < 0, \text{ which is impossible.}$$

Hence, $$\sqrt{\frac{3}{2}} + \sqrt{(n-1)(4n-3)} \le \sqrt{2n(2n-1)}.$$

In a similar manner, we can show that $\sqrt{2} + 2n - 2 \le \sqrt{2n(2n-1)}$ for $n \ge 2$ and $\sqrt{5} + \sqrt{(n-1)(4n-3)} \le \sqrt{2n(2n+1)}$ for $n \ne 2$. Therefore, for $n \ge 3$, the bounds in (19), (20) and (21) are better than the bounds (16), (17) and (18), respectively.

6. Exact formulas of normalized Laplacian energy and normalized Laplacian-energy-like invariant of derived graphs of K_n and $K_{r,r}$

THEOREM 6.1. *(i)* $\mathbb{EL}(K_n^*) = 3$, *and* *(ii)* $\mathbb{LEL}(K_n^*) = \sqrt{\frac{3}{2}} + (n-1)\left\{\frac{\sqrt{2n-2}+\sqrt{2n}}{\sqrt{2n-2}}\right\}$.

Proof. The eigenvalues of K_n are $\{-1^{n-1}, n-1^1\}$, where superscripts denote the multiplicity of the eigenvalues. Therefore normalized Laplacian eigenvalues of K_n^* are $\left\{0^1, \left(\frac{3}{2}\right)^1, \left(\frac{2n-3}{2n-2}\right)^{n-1}, \left(\frac{n}{n-1}\right)^{n-1}\right\}$. So, $\mathbb{EL}(K_n^*) = \sum_{i=1}^{2n} |\mu_i - 1| = |0-1| + \left|\frac{3}{2} - 1\right| + (n-1)\left|\frac{2n-3}{2n-2} - 1\right| + (n-1)\left|\frac{n}{n-1} - 1\right| = 3$ and

$$\mathbb{LEL}(K_n^*) = \sum_{i=1}^{2n} \sqrt{\mu_i} = \sqrt{\frac{3}{2}} + (n-1)\sqrt{\frac{2n-3}{2n-2}} + (n-1)\sqrt{\frac{n}{n-1}}$$

$$=\sqrt{\frac{3}{2}}+(n-1)\left\{\frac{\sqrt{2n-2}+\sqrt{2n}}{\sqrt{2n-2}}\right\}=\sqrt{\frac{3}{2}}+n-1+\sqrt{n(n-1)}. \qquad \square$$

THEOREM 6.2. *(i)* $\mathbb{EL}(K_n^{**})=2$, *and* *(ii)* $\mathbb{LEL}(K_n^{**})=\sqrt{2}+2n-2$.

Proof. The eigenvalues of K_n are $\{-1^{n-1}, n-1^1\}$, where superscripts denote the multiplicity of the eigenvalues. Therefore the normalized Laplacian eigenvalues of K_n^{**} are $\{0^1, 2^1, 1^{2n-2}\}$.

So, $$\mathbb{EL}(K_n^{**})=\sum_{i=1}^{2n}|\mu_i-1|=|0-1|+|2-1|+(2n-2)|1-1|=2,$$

and $$\mathbb{LEL}(K_n^{**})=\sum_{i=1}^{2n}\sqrt{\mu_i}=\sqrt{2}+2n-2. \qquad \square$$

THEOREM 6.3. *(i)* $\mathbb{EL}(K_{r,r}^{*})=3$, *and* *(ii)* $\mathbb{LEL}(K_{r,r}^{*})=\frac{\sqrt{3}+3}{\sqrt{2}}+2r-4$.

Proof. The eigenvalues of $K_{r,r}$ are $\{0^{2r-2}, r^1, (-r)^1\}$, where superscripts denote the multiplicity of the eigenvalues. Therefore the normalized Laplacian eigenvalues of $K_{r,r}^{*}$ are $\{0^1, (\frac{3}{2})^1, (\frac{1}{2})^1, 2^1, 1^{2r-4}\}$.

So, $$\mathbb{EL}(K_{r,r}^{*})=\sum_{i=1}^{2n}|\mu_i-1|=|0-1|+\left|\frac{3}{2}-1\right|+\left|\frac{1}{2}-1\right|+|2-1|+(2r-4)|1-1|=3,$$

and $$\mathbb{LEL}(K_{r,r}^{*})=\sum_{i=1}^{2n}\sqrt{\mu_i}=\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}+\sqrt{2}+(2r-4)=\frac{\sqrt{3}+3}{\sqrt{2}}+2r-4. \qquad \square$$

THEOREM 6.4. *(i)* $\mathbb{EL}(K_{r,r}^{**})=\frac{6r-4}{r+1}$, *and*

(ii) $\mathbb{LEL}(K_{r,r}^{**})=\sqrt{2}+\sqrt{\frac{2}{r+1}}+\sqrt{\frac{2r}{r+1}}+(r-2)\sqrt{\frac{r}{r+1}}+(r-2)\sqrt{\frac{r+2}{r+1}}$.

Proof. The eigenvalues of $K_{r,r}$ are $\{0^{2r-2}, r^1, (-r)^1\}$, where superscripts denote the multiplicity of the eigenvalues. Therefore the normalized Laplacian eigenvalues of $K_{r,r}^{**}$ are $\{0^1, 2^1, \left(\frac{2}{r+1}\right)^1, \left(\frac{2r}{r+1}\right)^1, \left(\frac{r}{r+1}\right)^{r-2}, \left(\frac{r+2}{r+1}\right)^{r-2}\}$. So,

$$\mathbb{EL}(K_{r,r}^{**})=\sum_{i=1}^{2n}|\mu_i-1|$$
$$=|0-1|+|2-1|+\left|\frac{2}{r+1}-1\right|+\left|\frac{2r}{r+1}-1\right|+(r-2)\left|\frac{r}{r+1}-1\right|+(r-2)\left|\frac{r+2}{r+1}-1\right|$$
$$=\frac{6r-4}{r+1},$$

and $\mathbb{LEL}(K_{r,r}^{**})=\sum_{i=1}^{2n}\sqrt{\mu_i}=\sqrt{2}+\sqrt{\frac{2}{r+1}}+\sqrt{\frac{2r}{r+1}}+(r-2)\sqrt{\frac{r}{r+1}}+(r-2)\sqrt{\frac{r+2}{r+1}}$. $\square$

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