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UNIQUENESS OF THE SOLUTION OF A NONLINEAR ALGEBRAIC SYSTEM

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Abstract. In this article we give a sufficient condition for a nonlinear algebraic system of some classes of hypersurfaces to intersect in a unique point and we express the corresponding unique solution in exact form, as well as for the corresponding nonlinear functional system of equations. We conclude extending our results for the functional case in a Banach space of Bochner measurable functions.

1. Introduction

The study of nonlinear systems is generally a very difficult problem and fundamental to geometrical calculations amongst others. Applications that reduce to finding the roots of nonlinear polynomial equations include, for example: the problem of intersection of curves; the collision detection; the calculation of the distance from a point to a curve; and in particular applications are of importance in Celectial Mechanics (see for example Arnold in [1], and Borisevich, Potemkin, Strunkov and Wood in [3]).

Here we study the existence of a unique solution of a nonlinear system concerning geometric objects as well as its functional extension, and we express the corresponding unique solution in the exact form.

Many authors have studied the problem of unique solutions for nonlinear systems. For global methods for solving nonlinear systems and applications we refer to [3]. For recent results on existence and uniqueness of positive solutions of a system of nonlinear algebraic equations we refer to Györi, Hartung and Mohamady [7]. Moreover, applications to algebraic equations and to differential equations such as second order Dirichlet problems have been presented by Zhang and Feng [15], Yamamoto and Chen [13], Ciurte, Nedevschi and Rasa [5]. For standard notions in functional analysis we refer to the classic book of Yosida [14].

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2. Problem statement and preliminaries

Let $p \in \mathbb{R}$, $p \geqq 1$, $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ and consider the following nonlinear system of equations:

$$\begin{cases} \sum_{i=1}^{k} x_i = q \\ \sum_{i=1}^{k} (a_i - x_i^p)^{\frac{1}{p}} = r, \end{cases} \tag{1}$$

with $q^p + r^p \neq 0$, and $a_i > 0$ such that $0 \leqq x_i \leqq a_i^{\frac{1}{p}}$, $i = 1, 2, \ldots, k$.

The first equation of the system (1) is an algebraic hypersurface of order 1, i.e. a hyperplane, and the second one represents an algebraic hypersurface for $p \in \mathbb{N}$, $p \geqq 1$. We recall that for a k-dimensional Euclidean space the set of points that satisfy an equation of the form $\phi(x_1, x_2, \ldots, x_k) = 0$, where ϕ is a polynomial in the variables $x_1, x_2, \ldots, x_k$ is called an algebraic hypersurface.

The origin of the above system (1) can be seen from the natural generalization of a beautiful problem that had appeared in Quantum magazine [12], see also later Example 4.4 for $k = 4$, $p = 2$, $a_1 = 1$, $a_2 = 4$, $a_3 = 9$, $a_4 = 16$, $q = 6$ and $r = 8$. We would like to note that until now we have not been able to find out who had suggested this problem in [12], as both translated versions do not mention his/her name, while in the prototype version we did not find this particular problem though we searched.

Although we study the problem for much more general values of p, in the examples we will see at the end that the case where $p = 2$ is of particular importance, since then the system is algebraic.

A sufficient condition is given, in Section 3, such that equality is established in Minkowski's inequality concerning the nonlinear system (1). As is well-known, the map $\|\cdot\|_p : \mathbb{R}^k \to \mathbb{R}^+ : y = (y_1, y_2, \ldots, y_k) \to \|y\|_p$, with $\|y\|_p := \left(\sum_{i=1}^{k} |y_i|^p\right)^{\frac{1}{p}}$, is a norm in $\mathbb{R}^k$ (usually called the p-norm in $\mathbb{R}^k$), equivalent with the usual Euclidean norm, since for $p \geqq 2$ it is valid $\|y\|_p \leqq \|y\|_2 \leqq k^{\frac{p-2}{2p}} \|y\|_p$, for every $y \in \mathbb{R}^k$. Here the triangle inequality is the well-known Minkowski inequality, *i.e.* if $k \in \mathbb{N}$, $1 \leqq p < +\infty$, $x, y \in \mathbb{R}^k$, with $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$, then

$$\left(\sum_{i=1}^{k} |x_i + y_i|^p\right)^{\frac{1}{p}} \leqq \left(\sum_{i=1}^{k} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{k} |y_i|^p\right)^{\frac{1}{p}}.$$

The equality is validated in Minkowski's inequality if and only if $x = 0$ or $x_i = \xi y_i$, for some $\xi \geqq 0$, $i = 1, 2, \ldots, k$.

We further refer to Matkowski and Rätz's article [10] on equality in Minkowski inequality and a characterization of L^p-norm.

Also, in Section 3 we extend our results for the corresponding nonlinear functional system of equations under the p-norm in $\mathbb{R}^2$. Furthermore, we extend our results for the functional case concerning the Banach spaces $M^p(\mathbb{R}, \mathbb{R}^m)$ of Bochner measurable functions $\varphi : \mathbb{R} \to \mathbb{R}^m$, for which the integral $\int_t^{t+1} \|\varphi(s)\|_p^p \, d\lambda(s) < +\infty$, $t \in \mathbb{R}$ exists, under the norms $|||\cdot|||_p : M^p(\mathbb{R}, \mathbb{R}^m) \to \mathbb{R}$, for $m = 1, 2$ respectively, with

$$|||\varphi|||_p := \sup\left\{ \left(\int_t^{t+1} \|\varphi(s)\|_p^p \, d\lambda(s) \right)^{\frac{1}{p}} : t \in \mathbb{R} \right\}. \tag{2}$$

In the last most general functional case, the equality of functions is in the sense almost everywhere, since we have equivalence classes, *i.e.* $\varphi_1 = \varphi_2$ if and only if $\varphi_1(s) = \varphi_2(s)$, for λ-almost every $s \in \mathbb{R}$, *i.e.* the set in which the equality is not true has the measure of zero.

At this point we recall that, in general, a function $f(s)$ defined on a measure space $(S, \mathcal{B}, \mu)$ with values in a Banach space $(E, \|\cdot\|)$ is said to be Bochner μ-integrable, if there exists a sequence of finitely valued functions $(f_n(s))$ which strongly converges to $f(s)$ μ-almost everywhere in such a way that

$$\lim_{n\to+\infty} \int_S \|f(s) - f_n(s)\| \, d\mu(s) = 0.$$

Then, for any set $B \in \mathcal{B}$, the Bochner μ-integral of $f(s)$ over B is defined by

$$\int_B f(s) d\mu(s) := \text{strong-} \lim_{n\to+\infty} \int_S \chi_B(s) f_n(s) d\mu(s),$$

where χ_B is the defining function of the set B (see for example in Yosida [14]).

Consequently in the above spaces $M^p(\mathbb{R}, \mathbb{R}^m)$ that we have defined, since the Euclidean spaces are Banach spaces, we have that the corresponding measure λ is the classic Lebesgue measure.

On the other hand, the spaces M^p are more general than the classic L^p spaces. Also, these spaces are connected with the spaces of almost periodic functions and for the proofs of the basic properties we refer to the book of Levitan and Zhikov [8].

Concerning the further functional generalization we are studying, we could say that it is important for examples of generalized coordinates and the constraints (holonomic or nonholonomic) that could occur for example in mechanical systems and geometrical computations. Even the elements of the spaces M^p are very general as mentioned above and contain spaces of solutions of differential equations and especially almost periodic functions. Moreover, we refer to our previous work [2] on functional differential evolution equations, where we have presented results of existence, uniqueness and regularity in such general functional spaces.

The article is completed by listing some examples where the results can be applied to achieve a unique solution. In the case $k = 2$ and $p = 2$ (cf. Example 4.2), the system (1) reduces to the problem of intersection of two parabolas, of a parabola with two concrete intersecting real lines, or two pairs of two intersecting real lines. On the other hand, for $k = 3$ and $p = 2$ (cf. Example 4.3) we have a problem of finding the relative places of two surfaces of fourth degree.

3. Main results

3.1 Geometrical result

We begin with the next result on the existence of a unique solution for the nonlinear system (1).

PROPOSITION 3.1. *A sufficient condition for the nonlinear system* (1) *to have a unique solution is* $a_1^{\frac{1}{p}} + a_2^{\frac{1}{p}} + \cdots + a_k^{\frac{1}{p}} = (q^p + r^p)^{\frac{1}{p}}$.

Proof. If $q = r = 0$ then $x_i = a_i^{\frac{1}{p}}$ and $x_i = 0$, $i = 1, 2, \ldots, k$. Thus $x_i = a_i = 0$, $i = 1, 2, \ldots, k$ and in this case the problem is trivial.

In the case $r = 0$ and $q \neq 0$, all we can conclude from the second equation is that $x_i = a_i^{\frac{1}{p}}$, $i = 1, 2, \ldots, k$. But then the first equation is satisfied because of the assumption.

Let $k+1$ be the following number of vectors: $u_i := \left(x_i, (a_i - x_i^p)^{\frac{1}{p}}\right) \in \mathbb{R}^2$, $i = 1, 2, \ldots, k$ and $v := (q, r) \in \mathbb{R}^2$. Then, the nonlinear system (1) can be equivalently written in the vector form $u_1 + u_2 + \cdots + u_k = v$.

The p-norms in $\mathbb{R}^2$ of the above vectors are $\|u_i\|_p = (x_i^p + a_i - x_i^p)^{\frac{1}{p}} = a_i^{\frac{1}{p}}$ and $\|v\|_p = (q^p + r^p)^{\frac{1}{p}}$. Thus $\left\|\sum_{i=1}^k u_i\right\|_p = \sum_{i=1}^k \|u_i\|_p = \|v\|_p$, that is equality holds in Minkowski's inequality, so the above vectors are codirectional, i.e. $u_1 \upuparrows u_2 \upuparrows \cdots \upuparrows u_k \upuparrows v$.

Since $u_i \upuparrows v$, $i = 1, 2, \ldots, k$, there exists exactly one $\lambda_i \in \mathbb{R}_+^* := (0, +\infty)$ such that $u_i = \lambda_i v$, $i = 1, 2, \ldots, k$. Hence,

$$\|u_i\|_p = \lambda_i \|v\|_p \Leftrightarrow a_i^{\frac{1}{p}} = \lambda_i (q^p + r^p)^{\frac{1}{p}} \Leftrightarrow \lambda_i = \frac{a_i^{\frac{1}{p}}}{(q^p+r^p)^{\frac{1}{p}}},\ i = 1, 2, \ldots, k.$$

On top of that, $\left(x_i, (a_i - x_i^p)^{\frac{1}{p}}\right) = \lambda_i (q, r) \Leftrightarrow x_i = \lambda_i q$ and $(a_i - x_i^p)^{\frac{1}{p}} = \lambda_i r$, $i = 1, 2, \ldots, k$.

From the first equation we find $x_i = \frac{q}{(q^p+r^p)^{\frac{1}{p}}} a_i^{\frac{1}{p}}$, $i = 1, 2, \ldots, k$ and the second one is satisfied. Hence the nonlinear algebraic system (1) has the unique solution

$$(x_1, x_2, \ldots, x_k) = \frac{q}{(q^p + r^p)^{\frac{1}{p}}} \left(a_1^{\frac{1}{p}}, a_2^{\frac{1}{p}}, \ldots, a_k^{\frac{1}{p}}\right). \tag{3}$$

We observe that the special cases: ($r = 0$ and $q \neq 0$) and ($r \neq 0$ and $q = 0$) are included in the last general solution of the problem. Also, if the system (1) has a unique solution given by (3), then $\sum_{i=1}^k a_i^{\frac{1}{p}} = \frac{(q^p+r^p)^{\frac{1}{p}}}{q} \sum_{i=1}^k x_i = \frac{(q^p+r^p)^{\frac{1}{p}}}{q} q = (q^p + r^p)^{\frac{1}{p}}$. $\square$

3.2 Functional generalization

In more general terms, we can state the following nonlinear functional system, concerning the real functions $f_i, a_i, g, h : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} \sum_{i=1}^{k} f_i(t) = g(t) \\ \sum_{i=1}^{k} \left(a_i(t) - f_i^p(t)\right)^{\frac{1}{p}} = h(t), \end{cases} \tag{4}$$

with $g^p(t) + h^p(t) \neq 0$, $a_i(t) > 0$ and $0 \leqq f_i(t) \leqq a_i^{\frac{1}{p}}(t)$, for every $t \in \mathbb{R}$, $i = 1, 2, \ldots, k$ and

$$\sum_{i=1}^{k} a_i^{\frac{1}{p}}(t) = \left(g^p(t) + h^p(t)\right)^{\frac{1}{p}}, \text{ for every } t \in \mathbb{R}. \tag{5}$$

Proposition 3.2. *If the relation* (5) *is satisfied, then the functional nonlinear system* (4) *has a unique solution.*

Proof. By applying similar arguments as in the proof of Proposition 3.1 we conclude that the nonlinear functional system (4) has the unique solution $(f_1(t), f_2(t), \ldots, f_k(t))$, with $f_i(t) = \frac{g(t)}{(g^p(t)+h^p(t))^{\frac{1}{p}}} a_i^{\frac{1}{p}}(t)$, $i = 1, 2, \ldots k$ with p-norm in $\mathbb{R}^k$, $\|(f_1(t), \ldots, f_k(t))\|_p = \frac{g(t)}{(g^p(t)+h^p(t))^{\frac{1}{p}}} \left(\sum_{i=1}^{k} a_i(t)\right)^{\frac{1}{p}}$. □

Furthermore, let $M^p\left(\mathbb{R}, \mathbb{R}^2\right)$, $1 \leqq p < +\infty$, be the Banach space of Bochner measurable functions equipped with the norm defined by (2). We then consider for $f_i, a_i, g, h \in M^p(\mathbb{R}, \mathbb{R})$ the corresponding functional system of the form (4) for λ-almost every $t \in \mathbb{R}$, and the condition

$$\sum_{i=1}^{k} \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} a_i(s) d\lambda(s) \right)^{\frac{1}{p}} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \left(g^p(s) + h^p(s)\right) d\lambda(s) \right)^{\frac{1}{p}}. \tag{6}$$

Proposition 3.3. *If the condition* (6) *is satisfied, then the corresponding functional nonlinear system of the form* (4) *has a unique solution for λ-almost every $t \in \mathbb{R}$.*

Proof. Consider the functions $u_i : \mathbb{R} \to \mathbb{R}^2 : t \to u_i(t) := \left(f_i(t), (a_i(t) - f_i^p(t))^{\frac{1}{p}}\right)$, $i = 1, 2, \ldots, k$ and $v : \mathbb{R} \to \mathbb{R}^2 : t \to v(t) := (g(t), h(t))$. Calculating the norms we have

$$\begin{aligned} |||u_i|||_p &= \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|u_i(s)\|_p^p \, d\lambda(s) \right)^{\frac{1}{p}} \\ &= \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \left[\left(f_i^p(s) + \left[(a_i(s) - f_i^p(s))^{\frac{1}{p}} \right]^p \right)^{\frac{1}{p}} \right]^p d\lambda(s) \right)^{\frac{1}{p}} \\ &= \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} a_i(s) d\lambda(s) \right)^{\frac{1}{p}}, \text{ for } i = 1, 2, \ldots, k \end{aligned}$$

and

$$|||v|||_p = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|v(s)\|_p^p \, d\lambda(s) \right)^{\frac{1}{p}}$$

$$= \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left((g^p(s) + h^p(s))^{\frac{1}{p}} \right)^p d\lambda(s) \right)^{\frac{1}{p}}$$

$$= \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} (g^p(s) + h^p(s))\, d\lambda(s) \right)^{\frac{1}{p}}.$$

Therefore, by the condition (6) we establish equality in the triangle inequality for the norm $|||\cdot|||_p$, and so there are $\xi_i > 0$, $i = 1, 2, \dots k$ such that $u_i(t) = \xi_i v(t)$, for λ-almost every $t \in \mathbb{R}$. So $f_i(t) = \xi_i g(t)$ and $(a_i(t) - f_i^p(t))^{\frac{1}{p}} = \xi_i h(t)$, and then $|||u_i|||_p = \xi_i |||v(t)|||_p$, for $i = 1, 2, \dots, k$ we have:

$$\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} a_i(s) d\lambda(s) \right)^{\frac{1}{p}} = \xi_i \cdot \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} (g^p(s) + h^p(s))\, d\lambda(s) \right)^{\frac{1}{p}}.$$

Hence, the corresponding nonlinear functional system of the form (4) has a unique solution $(f_1(t), f_2(t), \dots, f_k(t))$, with

$$f_i(t) = \frac{g(t) \cdot \sup\left\{ \left(\int_t^{t+1} a_i(s) d\lambda(s) \right)^{\frac{1}{p}} : t \in \mathbb{R} \right\}}{\sup\left\{ \left(\int_t^{t+1} (g^p(s) + h^p(s))\, d\lambda(s) \right)^{\frac{1}{p}} : t \in \mathbb{R} \right\}},$$

$i = 1, 2, \dots, k$ for λ-almost every $t \in \mathbb{R}$. □

4. Examples

Here we present some examples of the application of our results.

Example 4.1. For $k = 1$ and $p = 2$, the system (1) becomes equivalent with our assumption $\sqrt{a_1} = \sqrt{q^2 + r^2}$. If $p = 2$ and $a_1 = a_2 = \cdots = a_k = 1$, then $\sqrt{1} + \sqrt{1} + \cdots + \sqrt{1} = \sqrt{q^2 + r^2} \Leftrightarrow k = \sqrt{q^2 + r^2} \Leftrightarrow k^2 = q^2 + r^2$. Hence in this case the condition we assume is actually the Pythagorean theorem, and in particular there exist Pythagorean triples, *i.e.* positive integers k, q, r such that the condition holds. In the more general case, $p \in [1, +\infty)$ if $a_i = 1$ for all $i = 1, 2, \dots, k$, the condition becomes $k = (q^p + r^p)^{\frac{1}{p}} \Leftrightarrow k^p = q^p + r^p$. Of course, for $p \in \mathbb{N}$, $p \geqq 3$ no three positive integers k, q and r can satisfy the equation $k^p = q^p + r^p$ for any integer value of p greater than two.

Example 4.2. For $k = 2$ and $p = 2$ we have the nonlinear algebraic system

$$\begin{cases} x_1 + x_2 = q \\ \sqrt{a_1 - x_1^2} + \sqrt{a_2 - x_2^2} = r. \end{cases}$$

Setting $x_i^2 = \Psi_i \geqq 0$, $i = 1, 2$ the system takes the form

$$\begin{cases} \sqrt{\Psi_1} + \sqrt{\Psi_2} = q \\ \sqrt{a_1 - \Psi_1} + \sqrt{a_2 - \Psi_2} = r. \end{cases}$$

Finally, we have the following system

$$\begin{cases} A'\Psi_1^2 + B'\Psi_2^2 + C'\Psi_1\Psi_2 + D'\Psi_1 + E'\Psi_2 + F' = 0 \\ A\Psi_1^2 + B\Psi_2^2 + C\Psi_1\Psi_2 + D\Psi_1 + E\Psi_2 + F = 0, \end{cases}$$

with $A' = 1$, $B' = 1$, $C' = -2$, $D' = -2q^2$, $E' = -2q^2$ and $F' = q^4$; $A = 1$, $B = 1$, $C = -2$, $D = 2\left(r^2 - a_1 + a_2\right)$, $E = 2\left(r^2 + a_1 - a_2\right)$ and $F = \left(r^2 - a_1 - a_2\right)^2 - 4a_1a_2$. Therefore, the problem is to find the relative places of the above two curves of second degree. For curves of second degree, i.e. $A\chi_1^2 + B\chi_2^2 + C\chi_1\chi_2 + D\chi_1 + E\chi_2 + F = 0$, the invariants (cf. Bush [4] who established the introduction of invariant theory into elementary analytic geometry for second degree curves and second degree surfaces, and Efimov [6]) are

$$J_1 = A + C, \quad J_2 = \begin{vmatrix} A & B \\ B & C \end{vmatrix} \quad \text{and} \quad J_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}. \tag{7}$$

We recall that Bush in [4, part I] established the introduction of invariant theory into elementary analytic geometry for second degree curves, while in [4, part II] he studied the second degree surfaces. Until then the difficulty encountered in presenting this subject in elementary analytic geometry was that a complete characterization could not be made in terms of invariant alone, but certain covariants are also needed (cf. [9,11]). However, as shown in [4, part II], the use of covariants can be avoided by the use of certain conditional invariants.

For the invariants of the first one by (7) we have $J_1' = -1$, $J_2' = -3 < 0$ and $J_3' = 9q^4$. Thus, since $J_2' < 0$, we have a parabola (if $q \neq 0$) or two intersecting real lines (if $q = 0$). Also by (7) for the second one we have the invariants $J_1 = 2$, $J_2 = -3 < 0$ and $J_3 = 9r^4 + (-18a_1 + 30a_2)\, r^2 - 7\left(a_1 - a_2\right)^2$. Hence, since $J_2 < 0$, the second equation represents a parabola (if $J_3 \neq 0$) or two intersecting real lines (if $J_3 = 0$). Setting $r^2 = \omega$ we have $J_3 = 9\omega^2 + (-18a_1 + 30a_2)\,\omega - 7\left(a_1 - a_2\right)^2$, with discriminant $36\left[\left(-3a_1 + 5a_2\right)^2 + 7\left(a_1 - a_2\right)^2\right] > 0$, for all $a_1, a_2 \in \mathbb{R}$, and thus two roots $\omega_{1,2} = \frac{1}{3}(3a_1 - 5a_2 \pm \sqrt{\left(-3a_1 + 5a_2\right)^2 + 7\left(a_1 - a_2\right)^2})$. In the special case $a_1 = a_2 = 1$ we have $J_3 = 9r^4 + 12r^2 = 3r^2\left(3r^2 + 4\right)$. Thus for $r = 0$ we have two intersecting real lines and for $r > 0$ we have a parabola. For $r = 0$ then $x_1^2 = x_2^2 = a_1 = a_2 = 1$ and the system is trivial. Hence, the interesting case is for $r > 0$. Finally, the condition $\sqrt{a_1} + \sqrt{a_2} = \sqrt{q^2 + r^2}$ that we assume is sufficient for the above curves to have a unique point of intersection.

Example 4.3. For $k = 3$ and $p = 2$, setting $x_i^2 = \Psi_i \geqq 0$, $i = 1, 2, 3$ the nonlinear system becomes

$$\begin{cases} \sqrt{\Psi_1} + \sqrt{\Psi_2} + \sqrt{\Psi_3} = q \\ \sqrt{a_1 - \Psi_1} + \sqrt{a_2 - \Psi_2} + \sqrt{a_3 - \Psi_3} = r, \end{cases}$$

and can be written in the form

$$\begin{cases} \sum\limits_{1 \leqq i \leqq 3} \left(A_i' \Psi_i^4 + B_i' \Psi_i^3 + C_i' \Psi_i^2 + D_i' \Psi_i \right) + \sum\limits_{1 \leqq i < j < l \leqq 3} E_{ijl}' \Psi_i \Psi_j \Psi_l + F' = 0 \\ \sum\limits_{1 \leqq i \leqq 3} \left(A_i \Psi_i^4 + B_i \Psi_i^3 + C_i \Psi_i^2 + D_i \Psi_i \right) + \sum\limits_{1 \leqq i < j < l \leqq 3} E_{ijl} \Psi_i \Psi_j \Psi_l + F = 0, \end{cases}$$

where the coefficients can easily be expressed as functions of q and r respectively. Evidently, the study of the above system is complicated, and thus the condition that we consider is helpful.

Example 4.4. In the special case for $k = 4$, $p = 2$, $a_1 = 1$, $a_2 = 4$, $a_3 = 9$, $a_4 = 16$, $q = 6$ and $r = 8$, the system is written in the form

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 6 \\ \sqrt{1 - x_1^2} + \sqrt{4 - x_2^2} + \sqrt{9 - x_3^2} + \sqrt{16 - x_4^2} = 8, \end{cases}$$

and has a unique solution provided by $(x_1, x_2, x_3, x_4) = \frac{3}{5}(1, 2, 3, 4)$. This very special case was a suggested problem by an Unknown Proposer in [12] as we have already mentioned in our preliminary comments. In general, therefore, having in mind the Proposition 3.1 we can pose the problem for $k \in \mathbb{N}$ and $p = 2$. Then, we have the system

$$\begin{cases} \sum_{i=1}^k x_i = q \\ \sum_{i=1}^k \sqrt{a_i - x_i^2} = r, \end{cases}$$

with $|q| + |r| \neq 0$, and $a_i > 0$ such that $0 \leqq x_i \leqq \sqrt{a_i}$, $i = 1, 2, \ldots, k$ and the condition for the existence of the unique solution is $\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_k} = \sqrt{q^2 + r^2}$. Then, the unique solution is $(x_1, x_2, \ldots, x_k) = \frac{q}{\sqrt{q^2 + r^2}} \left(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_k} \right)$.

Example 4.5. For $p = 2$ and $\sum_{i=1}^k \sqrt{a_i(t)} = 1$ we have $1 = g^2(t) + h^2(t)$, for every $t \in \mathbb{R}$. If in addition the functions g, h are differentiable with $g(0) = 0$, $h(0) = 1$, $g'(t) = h(t)$ and $h'(t) = -g(t)$, for every $t \in \mathbb{R}$, then $g(t) = \sin(t)$ and $h(t) = \cos(t)$, for every $t \in \mathbb{R}$. In this special case the solution of the system is $f_i(t) = \sqrt{a_i(t)} \sin(t)$, for every $t \in \mathbb{R}$, $i = 1, 2, \ldots, k$.

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