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MODIFIED INERTIAL HYBRID SUBGRADIENT EXTRAGRADIENT METHOD FOR SOLVING VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS FOR AN INFINITE FAMILY OF MULTIVALUED RELATIVELY NONEXPANSIVE MAPPINGS IN BANACH SPACES WITH APPLICATIONS

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Abstract. One of the most interesting and important problems in the theory of variational inequalities is the study of efficient iterative schemes for finding approximate solutions and the convergence analysis of algorithms. In this article, we introduce a new inertial hybrid subgradient extragradient method for approximating a common solution of monotone variational inequalities and fixed point problems for an infinite family of relatively nonexpansive multivalued mappings in Banach spaces. In our proposed method, the projection onto the feasible set is replaced with a projection onto certain half spaces, which makes the algorithm easy to implement. We incorporate inertial term into the algorithm, which helps to improve the rate of convergence of the proposed method. Moreover, we prove a strong convergence theorem and we apply our results to approximate common solutions of variational inequalities and zero point problems, and to finding a common solution of constrained convex minimization and fixed point problems in Banach spaces. Finally, we present a numerical example to demonstrate the efficiency and the advantages of the proposed method, and we compare it with some related methods. Our results extend and improve some recent works both in Hilbert spaces and Banach spaces in this direction.

1. Introduction

Throughout this article, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let E be a real Banach space with norm $\|\cdot\|$ and E^* be the dual of E. For $x \in E$ and $f \in E^*$, let $\langle x, f\rangle$ be the value of f at x. Suppose that C is a nonempty closed convex subset of E. The *Variational Inequality Problem* (VIP) is to find a point $x^* \in C$ such that

$$\langle x - x^*, Ax^*\rangle \geq 0, \quad \forall\, x \in C, \tag{1}$$

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where $A : E \to E^*$ is a single-valued mapping. We denote the solution set of VIP (1) by $VI(C, A)$. The VIP was first introduced by Lions and Stampacchia [17] in 1967. The problem is a very important tool in studying engineering mechanics, physics, economics, optimization theory and applied sciences in a unified and general framework [4].

Under appropriate conditions, there are two general approaches for solving the VIP, one is the regularized method and the other is the projection method. In this article, we consider the projection method. Several projection-type algorithms for solving VIP have been proposed and analyzed by many authors (see [6, 21] and the references therein). The gradient method (GM) is the simplest algorithm in which only one projection is made onto the feasible set, and the convergence of the method requires a strongly monotonicity condition. In order to avoid the hypothesis of the strongly monotonicity, Korpelevich [16] proposed an algorithm for solving the variational inequalities in Euclidean space, which is called the extragradient method (EgM). The EgM involves two projections onto the closed convex set C and two evaluations of the cost operator per iteration. Computing projection onto an arbitrary closed convex set is a difficult task, a drawback which may affect the efficiency of the EgM as mentioned in [7]. Hence, a major improvement on the EgM is to minimize the number of evaluations of P_C per iteration. Censor et al. [7] initiated an attempt in this direction, modifying the EgM by replacing the second projection with a projection onto a half-space. This new method involves only one projection onto the closed convex set and is called the subgradient extragradient method (SEgM).

In solving optimization problems, it is known that strong convergence results are more applicable than weak convergence results. Hence, the motivation for constructing iterative methods that converge strongly to the solution of optimization problems.

We note that all the above mentioned results on VIP are confined in Hilbert spaces. However, many important problems related to practical problems are generally defined in Banach spaces. Hence, it is more desirable to propose an iterative algorithm for finding a solution of VIP (1) in Banach spaces.

In order to increase the speed of convergence of iterative methods, researchers often employed the inertial technique. The inertial algorithm is a two-step iteration where the next iterate is defined by making use of the previous two iterates. Recently, several researchers have constructed some fast iterative algorithms by using inertial extrapolation (see, e.g., [1, 2, 9, 20, 28]).

Very recently, Tian and Jiang [27] proposed the inertial hybrid SEgM for solving VIP (1) and proved a strong convergence theorem in a Banach space E.

ALGORITHM 1.1.

$$
\begin{aligned}
&x_0, x_1 \in E,\\
&w_n = J^{-1}(Jx_n + \alpha_n(Jx_n - Jx_{n-1})),\\
&y_n = \Pi_C J^{-1}(Jw_n - \lambda_n A w_n),\\
&T_n = \{w \in E : \langle w - y_n, Jw_n - \lambda_n A w_n - Jy_n\rangle \le 0\},\\
&z_n = \Pi_{T_n} J^{-1}(Jw_n - \lambda_n A y_n),
\end{aligned}
$$

$$D_n = \{w \in E : \phi(w, z_n) \le \phi(w, x_n) - 2\alpha_n \langle w - x_n, Jx_n - Jx_{n-1}\rangle + \phi(x_n, w_n)\},$$
$$Q_n = \{w \in E : \langle w - x_n, Jx_1 - Jx_n\rangle \le 0\},$$
$$x_{n+1} = \Pi_{D_n \cap Q_n} x_1.$$

Here $A : E \to E^*$ is a monotone and L-Lipschitz continuous mapping with $L > 0$, $\{\alpha_n\}$ is a bounded sequence of real numbers, and $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{c^2}{2L})$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E.

Let $S : E \to E$ be a nonlinear mapping, a point $x^* \in E$ is called a fixed point of S if $Sx^* = x^*$. We denote by $F(S)$, the set of all fixed points of S, i.e. $F(S) = \{x^* \in E : Sx^* = x^*\}$. If S is a multivalued mapping, i.e. $S : E \to 2^E$, then $x^* \in E$ is called a fixed point of S if $x^* \in Sx^*$.

The fixed point theory for multivalued mappings find applications in various fields such as game theory, control theory, mathematical economics. On the other hand, the existence of common fixed points for a countable family of nonlinear mappings has been considered by many authors, for instance see [24, 29]. Many optimization problems can be formulated as finding a common fixed point of a countable family of nonlinear mappings. For instance, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a countable family of nonexpansive mappings (see [5]). The problem of finding an optimal point that minimizes a given cost function over the common set of fixed points of a countable family of nonlinear mappings is of wide interdisciplinary interest and practical importance (see [30]).

In this work, we are interested in studying the problem of finding a common solution of both the VIP (1) and the common fixed point problem for multivalued mappings in Banach spaces. The importance and motivation for studying the VIP and common fixed point problems lies in its potential application to mathematical models whose constraints can be expressed as fixed point problem and VIP. This arises in practical problems such as signal processing, network resource allocation, image recovery (see, for instance, [13] and the references therein).

We note that Algorithm 1.1 requires making a projection onto the closed convex set C per iteration, and as earlier pointed out computing projection onto a general closed convex set is a difficult task. Hence, in this article inspired and motivated by the cited works, we introduce a new inertial hybrid SEgM for approximating a common solution of VIP (1) and FPP for an infinite family of relatively nonexpansive multivalued mappings in the setting of Banach spaces. In our proposed algorithm, we replace the projection onto the closed convex set C in Algorithm 1.1 with a projection onto an half-space which makes our algorithm easier to implement. Furthermore, we prove a strong convergence theorem and apply our result to study other optimization problems.

The paper is organized as follows. In Section 2, we recall some basic definitions and results that will be needed in the sequel. Our hybrid SEgM is presented and analyzed in Section 3 and we also obtained some consequent results. Then, in Section 4, we apply our result to approximate the solutions of related optimization problems. In Section 5, we present some numerical experiments to demonstrate the efficiency of our proposed method in comparison with some recent works in the literature. Finally, concluding remarks are presented in Section 6.

2. Preliminaries

Here, we introduce some basic concepts and state some useful results that will be employed in our subsequent analysis. Let E be a Banach space, E^* the dual space of E, and $\langle \cdot, \cdot \rangle$ denote the duality pairing of E and E^*. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. An element $z \in E$ is called a weak cluster point of $\{x_n\}$ if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to z. We write $w_\omega(x_n)$ to indicate the set of all weak cluster points of $\{x_n\}$.

Next, we present some definitions and results which are employed in our subsequent analysis.

Definition 2.1. An operator $A : E \to E^*$ is said to be
(i) monotone if $\langle x - y, Ax - Ay \rangle \geq 0$, $\forall\, x, y \in E$;

(ii) α-inverse-strongly-monotone if there exists a positive real number α such that $\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$, $\forall\, x, y \in E$;

(iii) L-Lipschitz continuous if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$, $\forall\, x, y \in E$.
It is clear that an α-inverse-strongly-monotone mapping is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. However, the converse is not always true.

Let $g : E \to \mathbb{R}$ be a function. The subdifferential of g at x is defined by $\partial g(x) = \{w \in E^* : g(y) \geq g(x) + \langle y - x, w \rangle, \quad \forall\, y \in E\}$. If $\partial g(x) \neq \emptyset$, then we say that g is subdifferentiable at x.

A Banach space E is said to be strictly convex, if for all $x, y \in E$, $\|x\| = \|y\| = 1$ and $x \neq y$ implies $\|(x + y)/2\| < 1$. The modulus of convexity of E is defined as

$$\delta_E(\varepsilon) = \inf \Big\{ 1 - \Big\| \frac{x+y}{2} \Big\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \Big\}.$$

Then, E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. It is well known that a uniformly convex Banach space is strictly convex and reflexive. E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) > c\varepsilon^p$ for all $\varepsilon \in [0, 2]$ with $p \geq 2$. It is easy to see that a p-uniformly convex Banach space is uniformly convex. In particular, a Hilbert space is two-uniformly convex.

A Banach space E is said to be smooth, if the limit $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists for all $x, y \in S_E$, where $S_E = \{x \in E : \|x\| = 1\}$. Moreover, if this limit is attained uniformly for $x, y \in S_E$, then E is said to be uniformly smooth. It is obvious that a uniformly smooth space is smooth.

For $p > 1$, the generalized duality mapping $J_p : E \to 2^{E^*}$ is defined by

$$J_p x = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\} \quad \forall\, x \in E.$$

In particular, $J = J_2$ is called the normalized duality mapping. If $E = H$, where H is a Hilbert space, then $J = I$ (see [25]).

Let E be a smooth Banach space. The Lyapunov functional $\phi : E \times E \to \mathbb{R}$ (see [3]) is defined by $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \forall \ x, y \in E$.

From the definition, it is easy to see that $\phi(x, x) = 0$ for every $x \in E$. If E is strictly convex, then $\phi(x, y) = 0 \iff x = y$. If E is a Hilbert space, it is easy to see that $\phi(x, y) = \|x-y\|^2$ for all $x, y \in E$. Moreover, for every $x, y, z \in E$ and $\alpha \in (0, 1)$, the Lyapunov functional ϕ satisfies the following properties:

(P1) $0 \leq (\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$;

(P2) $\phi(x, J^{-1}(\alpha Jz + (1 - \alpha)Jy)) \leq \alpha\phi(x, z) + (1 - \alpha)\phi(x, y)$;

(P3) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle z - x, Jy - Jz\rangle$;

(P4) $\phi(x, y) \leq 2\langle y - x, Jy - Jx\rangle$;

(P5) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

Also, we define the functional $V : E \times E^* \to [0, +\infty)$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \quad \forall \ x \in E, \quad x^* \in E^*. \tag{2}$$

It can be deduced from (2) that V is non-negative and $V(x, x^*) = \phi(x, J^{-1}(x^*))$. We have the following result in a reflexive strictly convex and smooth Banach space.

Lemma 2.2 ([3]). *Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then, $V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*)$, for all $x \in E$ and $x^*, y^* \in E^*$.*

Definition 2.3. Let C be a nonempty closed convex subset of a real Banach space E. A point $p \in C$ is called an *asymptotic fixed point* (see [22]) of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to+\infty} \|x_n - Tx_n\| = 0$. We denote the set of asymptotic fixed points of T by $\hat{F}(T)$.

A mapping $T : C \to C$ is said to be:

(i) relatively nonexpansive if:

(a) $F(T) \neq \emptyset$; (b) $\phi(p, Tx) \leq \phi(p, x)$, $\forall p \in F(T)$, $x \in C$; (c) $\hat{F}(T) = F(T)$;

(ii) generalized nonspreading (see [12]) if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) + \gamma[\phi(Ty, Tx) - \phi(Ty, x)]\\ \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y) + \delta[\phi(y, Tx) - \phi(y, x)].\end{aligned}$$

The following result shows the relationship between generalized nonspreading mappings and relatively nonexpansive mappings.

Lemma 2.4 ([12]). *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a generalized nonspreading mapping of C into itself such that $F(T) \neq \emptyset$. Then, T is relatively nonexpansive.*

Let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subsets of C, respectively. The Hausdorff metric on $CB(C)$ is defined by

$H(A,B) := \max\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\}$, for all $A,B \in CB(C)$, where $\operatorname{dist}(a,B) := \inf\{\|a-b\| : b \in B\}$.

Let $T : C \to CB(C)$ be a multivalued mapping. An element $p \in C$ is called a fixed point of T if $p \in Tp$. A point $p \in C$ is called an asymptotic fixed point of T, if there exists a sequence $\{x_n\}$ in C which converges weakly to p such that $\lim_{n\to+\infty} \operatorname{dist}(x_n, Tx_n) = 0$.

A mapping $T : C \to CB(C)$ is said to be *relatively nonexpansive* if:
(a) $F(T) \neq \emptyset$; (b) $\phi(p,u) \leq \phi(p,x) \quad \forall\, u \in Tx, p \in F(T)$; (c) $\hat{F}(T) = F(T)$.

Remark 2.5 ([11]). Let E be a strictly convex and smooth Banach space, and C a nonempty closed convex subset of E. Suppose $T : C \to N(C)$ is a relatively nonexpansive multi-valued mapping. If $p \in F(T)$, then $Tp = \{p\}$.

Lemma 2.6 ([15]). *Let E be a smooth and uniformly convex Banach space, and $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) \to 0$ as $n \to +\infty$, then $\|x_n - y_n\| \to 0$ as $n \to +\infty$.*

Remark 2.7. From property (P4) of the Lyapunov functional, it follows that the converse of Lemma 2.6 also holds if the sequences $\{x_n\}$ and $\{y_n\}$ are bounded (see also, [3])

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. By Alber [3], for each $x \in E$, there exists a unique element $x_0 \in C$ (denoted by $\Pi_C(x)$) such that $\phi(x_0, x) = \min_{y\in C} \phi(y,x)$. The mapping $\Pi_C : E \to C$, defined by $\Pi_C(x) = x_0$, is called the generalized projection from E onto C. Moreover, x_0 is called the generalized projection of x. It is known that if E is a real Hilbert space, then Π_C coincides with the metric projection operator P_C. The following results relating to the generalized projection are well known.

Lemma 2.8 ([14]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E. Let $x \in E$ and $z \in C$ be given. Then, $z = \Pi_C x$ implies $\phi(y,z) + \phi(z,x) \leq \phi(y,x)$, $\forall\, y \in C$.*

Lemma 2.9 ([14]). *Let C be a nonempty closed and convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geq 0$, $\forall\, y \in C$.*

Lemma 2.10 ([14]). *Let p be a real number with $p \geq 2$. Then, E is p-uniformly convex if and only if there exists $c \in (0,1]$ such that $\frac{1}{2}(\|x+y\|^p + \|x-y\|^p) \geq \|x\|^p + c^p\|y\|^p \quad \forall\, x,y \in E$. Here, the best constant $1/c$ is called the p-uniformly convexity constant of E.*

Lemma 2.11 ([19]). *Let E be a 2-uniformly convex and smooth Banach space. Then, for every $x,y \in E$, $\phi(x,y) \geq \frac{c^2}{2}\|x-y\|^2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E.*

Lemma 2.12 ([14]). *Let E be a p-uniformly convex Banach space with $p \geq 2$. Then $\langle x - y, j_p(x) - j_p(y) \rangle \geq \frac{c^p}{2^{p-2}p}\|x - y\|^p$, $\forall\ x, y \in E$, $\forall j_p(x) \in J_p x, j_p(y) \in J_p y$, where $\frac{1}{c}$ is the p-uniformly convexity constant.*

An operator A of C into E^* is said to be hemicontinuous if for all $x, y \in C$, the mapping f of $[0, 1]$ into E^* defined by $f(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak*-topology of E^*.

Lemma 2.13 ([14]). *Let C be a nonempty, closed and convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Then $VI(C, A) = \{u \in C : \langle v - u, Av \rangle \geq 0 \quad \forall\ v \in C\}$.*

It is obvious from Lemma 2.13 that the set $VI(C, A)$ is a closed and convex subset of C.

3. Main results

In this section, we present our algorithm and prove some strong convergence results for the proposed algorithm. The strong convergence theorem of the algorithm is established under the following conditions:

Condition A

(A1) E is a 2-uniformly convex and uniformly smooth Banach space with 2-uniformly convexity constant $\frac{1}{c}$;

(A2) C is a nonempty closed convex set, which satisfies the following condition $C = \{x \in E : g(x) \leq 0\}$, where $g : E \to \mathbb{R}$ is a convex function;

(A3) $g(x)$ is weakly lower semicontinuous on E;

(A4) For any $x \in E$, at least one subgradient $\xi \in \partial g(x)$ can be calculated (i.e. g is subdifferentiable on E), where $\partial g(x) = \{z \in E^* : h(y) \geq h(x) + \langle y - x, z \rangle, \forall\ y \in E\}$. In addition, $\partial g(x)$ is bounded on bounded sets.

Condition B

(B1) The solution set denoted by $\Omega = VI(C, A) \cap \bigcap_{i=1}^{+\infty} F(S_i)$ is nonempty, where $S_i : E \to CB(E)$ is an infinite family of multivalued relatively nonexpnsive mappings;

(B2) The mapping $A : E \to E^*$ is monotone and Lipschitz continuous with Lipschitz constant $L > 0$.

Condition C

(C1) $\{\alpha_n\}$ is a bounded sequence of real numbers;

(C2) $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$ for some $a, b \in (0, 1)$, $\sum_{i=0}^{+\infty} \beta_{n,i} = 1$, and $\liminf_{n\to+\infty} \beta_{n,0}\beta_{n,i} > 0$ for all $i \geq 1$;

(C3) $\{\lambda_n\} \subset [d, k]$ for some $d, k \in (0, \frac{c^2}{2L})$.

Now, the algorithm is presented as follows.

ALGORITHM 3.1.
Step 0. *Select sequences* $\{\alpha_n\}, \{\beta_{n,i}\}$, *and* $\{\lambda_n\}$ *such that* ***Condition C*** *holds. Choose* $x_0, x_1 \in E$ *and set* $n = 1$.

Step 1. *Compute* $w_n = J^{-1}(Jx_n + \alpha_n(Jx_n - Jx_{n-1}))$.

Step 2. *Construct the half-space* $C_n = \{w \in E : g(w_n) + \langle w - w_n, \xi_n\rangle \le 0\}$, *where* $\xi_n \in \partial g(w_n)$, *and compute* $y_n = \Pi_{C_n} J^{-1}(Jw_n - \lambda_n Aw_n)$.

Step 3. *Construct the half-space* $T_n = \{w \in E : \langle w - y_n, Jw_n - \lambda_n Aw_n - Jy_n\rangle \le 0\}$, *and compute* $z_n = \Pi_{T_n} J^{-1}(Jw_n - \lambda_n Ay_n)$.

Step 4. *Compute* $v_n = J^{-1}(\beta_{n,0}Jx_n + \sum_{i=1}^{+\infty} \beta_{n,i}Ju_{n,i}), \quad u_{n,i} \in S_i z_n$.

Step 5. *Construct the half-spaces*

$$\begin{aligned} D_n = \{&w \in E : \phi(w, v_n) \le \phi(w, x_n) - 2\alpha_n(1 - \beta_{n,0})\langle w - x_n, Jx_n - Jx_{n-1}\rangle \\ &+ (1 - \beta_{n,0})\phi(x_n, w_n)\}, \\ Q_n = \{&w \in E : \langle w - x_n, Jx_1 - Jx_n\rangle \le 0\}, \end{aligned}$$

and compute $x_{n+1} = \Pi_{D_n \cap Q_n} x_1$, $\forall n \ge 1$.

REMARK 3.2. From the construction of the half-spaces C_n and T_n, it can easily be verified that $C \subseteq C_n$ and $C \subseteq T_n$.

Next, we prove the following lemma employed in establishing the convergence result for the proposed algorithm.

LEMMA 3.3. *Let* $\{x_n\}, \{w_n\}$ *and* $\{y_n\}$ *be sequences generated by Algorithm 3.1, and suppose* $\{x_n\}$ *is bounded and* $\lim_{n\to+\infty} \|w_n - y_n\| = 0$. *Let* $\{w_{n_k}\}$ *be a subsequence of* $\{w_n\}$, *which converges weakly to some* $\hat{x} \in E$ *as* $k \to +\infty$. *Then* $\hat{x} \in VI(C, A)$.

Proof. Since $w_{n_k} \rightharpoonup \hat{x}$, then by the hypothesis of the lemma, we have that $y_{n_k} \rightharpoonup \hat{x}$. Also, since $y_{n_k} \in C_{n_k}$, it follows from the definition of C_n that $g(w_{n_k}) + \langle y_{n_k} - w_{n_k}, \xi_{n_k}\rangle \le 0$. Since $\{x_n\}$ is bounded, it follows from the construction of the Algorithm that $\{w_n\}$ and $\{y_n\}$ are also bounded. Then, by Condition (A4) there exists a constant $M > 0$ such that $\|\xi_{n_k}\| \le M$ for all $k \ge 0$. Hence, $g(w_{n_k}) \le M\|w_{n_k} - y_{n_k}\| \to 0$, $k \to +\infty$.

By Condition (A3), we have that $g(\hat{x}) \le \liminf_{k\to+\infty} g(w_{n_k}) \le 0$. Hence, it follows from Condition (A2) that $\hat{x} \in C$. From Lemma 2.9, we obtain $\langle y_{n_k} - z, Jw_{n_k} - \lambda_{n_k} Aw_{n_k} - Jy_{n_k}\rangle \ge 0$, $\forall\, z \in C \subseteq C_{n_k}$. By the monotonicity of A, we have

$$\begin{aligned} 0 &\le \langle y_{n_k} - z, Jw_{n_k} - Jy_{n_k}\rangle - \lambda_{n_k}\langle y_{n_k} - z, Aw_{n_k}\rangle \\ &= \langle z - y_{n_k}, Jy_{n_k} - Jw_{n_k}\rangle + \lambda_{n_k}\langle z - y_{n_k}, Aw_{n_k}\rangle \\ &= \langle z - y_{n_k}, Jy_{n_k} - Jw_{n_k}\rangle + \lambda_{n_k}\langle z - w_{n_k}, Aw_{n_k}\rangle + \lambda_{n_k}\langle w_{n_k} - y_{n_k}, Aw_{n_k}\rangle \\ &\le \langle z - y_{n_k}, Jy_{n_k} - Jw_{n_k}\rangle + \lambda_{n_k}\langle z - w_{n_k}, Az\rangle + \lambda_{n_k}\langle w_{n_k} - y_{n_k}, Aw_{n_k}\rangle. \\ &\le \|z - y_{n_k}\|\|Jy_{n_k} - Jw_{n_k}\| + \lambda_{n_k}\langle z - w_{n_k}, Az\rangle + \lambda_{n_k}\|w_{n_k} - y_{n_k}\|\|Aw_{n_k}\|. \end{aligned}$$

Letting $k \to +\infty$, and since $\{Aw_n\}$ is bounded and $\lim_{n\to+\infty} \|w_n - y_n\| = 0$, we have $\langle z - \hat{x}, Az\rangle \geq 0$, $\forall\, z \in C$. From Lemma 2.13, it follows that $\hat{x} \in VI(C, A)$ as required. $\square$

Now, we state and prove strong convergence theorem for Algorithm 3.1.

THEOREM 3.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that conditions (A)-(C) are satisfied. Then the sequence $\{x_n\}$ converges strongly to $x^\dagger = \Pi_\Omega x_1$.*

Proof. We divide the proof into four steps as follows:
Step 1: We show that $\Omega \subset D_n \cap Q_n$ for each $n \in \mathbb{N}$.

Note that D_n and Q_n are half-spaces for each $n \in \mathbb{N}$. Let $p \in \Omega$, then by property (P3) of the Lyapunov functional we have

$$\begin{aligned}\phi(p, w_n) &= \phi(p, x_n) + \phi(x_n, w_n) + 2\langle p - x_n, Jx_n - Jx_n - \alpha_n(Jx_n - Jx_{n-1})\rangle \\ &= \phi(p, x_n) + \phi(x_n, w_n) - 2\alpha_n\langle p - x_n, Jx_n - Jx_{n-1}\rangle. \qquad (3)\end{aligned}$$

By applying Lemma 2.8 and the monotonicity of A, we obtain

$$\begin{aligned}&\phi(p, z_n) = \phi(p, \Pi_{T_n} J^{-1}(Jw_n - \lambda_n Ay_n)) \\ \leq&\phi(p, J^{-1}(Jw_n - \lambda_n Ay_n)) - \phi(z_n, J^{-1}(Jw_n - \lambda_n Ay_n)) \\ =&\phi(p, w_n) + \phi(w_n, J^{-1}(Jw_n - \lambda_n Ay_n)) + 2\lambda_n\langle p - w_n, Ay_n\rangle - \phi(z_n, w_n) \\ &- \phi(w_n, J^{-1}(Jw_n - \lambda_n Ay_n)) - 2\lambda_n\langle z_n - w_n, Ay_n\rangle \\ =&\phi(p, w_n) - \phi(z_n, w_n) + 2\lambda_n\langle p - z_n, Ay_n\rangle \\ =&\phi(p, w_n) - \phi(z_n, w_n) + 2\lambda_n\langle p - y_n, Ay_n - Ap\rangle + 2\lambda_n\langle p - y_n, Ap\rangle + 2\lambda_n\langle y_n - z_n, Ay_n\rangle \\ \leq&\phi(p, w_n) - \phi(z_n, w_n) + 2\lambda_n\langle y_n - z_n, Ay_n\rangle \\ =&\phi(p, w_n) - \phi(z_n, y_n) - \phi(y_n, w_n) - 2\langle z_n - y_n, Jy_n - Jw_n\rangle + 2\lambda_n\langle y_n - z_n, Ay_n\rangle \\ =&\phi(p, w_n) - \phi(z_n, y_n) - \phi(y_n, w_n) + 2\langle z_n - y_n, Jw_n - \lambda_n Ay_n - Jy_n\rangle. \qquad (4)\end{aligned}$$

By the definition of T_n, we have that $\langle z_n - y_n, Jw_n - \lambda_n Aw_n - Jy_n\rangle \leq 0$. Then, by the Lipschitz continuity of A, Lemma 2.11 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}&2\langle z_n - y_n, Jw_n - \lambda_n Ay_n - Jy_n\rangle = \langle z_n - y_n, Jw_n - \lambda_n Aw_n - Jy_n\rangle + 2\lambda_n\langle z_n - y_n, Aw_n - Ay_n\rangle \\ &\leq 2\lambda_n\langle z_n - y_n, Aw_n - Ay_n\rangle \leq 2\lambda_n L\|z_n - y_n\|\|w_n - y_n\| \\ &\leq 2\lambda_n L\frac{\sqrt{2\phi(z_n, y_n)}}{c}\frac{\sqrt{2\phi(y_n, w_n)}}{c} \leq \frac{2\lambda_n L}{c^2}\Big(\phi(z_n, y_n) + \phi(y_n, w_n)\Big) \qquad (5)\end{aligned}$$

By combining (3), (4) and (5) we obtain

$$\begin{aligned}\phi(p, z_n) &\leq \phi(p, w_n) - \phi(z_n, y_n) - \phi(y_n, w_n) + \frac{2\lambda_n L}{c^2}\Big(\phi(z_n, y_n) + \phi(y_n, w_n)\Big) \\ &= \phi(p, w_n) - \Big(1 - \frac{2\lambda_n L}{c^2}\Big)\Big(\phi(z_n, y_n) + \phi(y_n, w_n)\Big) \qquad (6) \\ &\leq \phi(p, w_n) \leq \phi(p, x_n) + \phi(x_n, w_n) - 2\alpha_n\langle p - x_n, Jx_n - Jx_{n-1}\rangle. \qquad (7)\end{aligned}$$

Next, we have that for each $p \in \Omega$

$$\begin{aligned}
\phi(p, v_n) =&\phi(p, J^{-1}(\beta_{n,0}Jx_n + \sum_{i=1}^{+\infty} \beta_{n,i}Ju_{n,i}))\\
=&\|p\|^2 - 2\langle p, \beta_{n,0}Jx_n + \sum_{i=1}^{+\infty} \beta_{n,i}Ju_{n,i}\rangle + \|\beta_{n,0}Jx_n + \sum_{i=1}^{+\infty} \beta_{n,i}Ju_{n,i}\|^2\\
\leq&\|p\|^2 - 2\beta_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{+\infty} \beta_{n,i}\langle p, Ju_{n,i}\rangle + \beta_{n,0}\|Jx_n\|^2\\
&+ \sum_{i=1}^{+\infty} \beta_{n,i}\|Ju_{n,i}\|^2 - \beta_{n,0}\beta_{n,j}g(\|Jx_n - Ju_{n,i}\|)\\
=&\beta_{n,0}\phi(p, x_n) + \sum_{i=1}^{+\infty} \beta_{n,i}\phi(p, u_{n,i}) - \beta_{n,0}\beta_{n,j}g(\|Jx_n - Ju_{n,i}\|)\\
\leq&\beta_{n,0}\phi(p, x_n) + \sum_{i=1}^{+\infty} \beta_{n,i}\phi(p, z_n) - \beta_{n,0}\beta_{n,j}g(\|Jx_n - Ju_{n,i}\|)\\
\leq&\beta_{n,0}\phi(p, x_n) + (1 - \beta_{n,0})\big(\phi(p, x_n) + \phi(x_n, w_n) - 2\alpha_n\langle p - x_n, Jx_n - Jx_{n-1}\rangle\big)\\
&- \beta_{n,0}\beta_{n,j}g(\|Jx_n - Ju_{n,i}\|)\\
=&\phi(p, x_n) + (1 - \beta_{n,0})\phi(x_n, w_n) - 2\alpha_n(1 - \beta_{n,0})\langle p - x_n, Jx_n - Jx_{n-1}\rangle\\
&- \beta_{n,0}\beta_{n,j}g(\|Jx_n - Ju_{n,i}\|) \qquad (8)\\
\leq&\phi(p, x_n) + (1 - \beta_{n,0})\phi(x_n, w_n) - 2\alpha_n(1 - \beta_{n,0})\langle p - x_n, Jx_n - Jx_{n-1}\rangle).
\end{aligned}$$

Hence, $p \in D_n$ for each $n \in \mathbb{N}$ and $\Omega \subset D_n$ for each $n \in \mathbb{N}$. For $n = 1$, we have that $Q_1 = E$ and it follows that $\Omega \subset D_1 \cap Q_1$. Suppose x_k is given and $\Omega \subset D_k \cap Q_k$ for some $k \in \mathbb{N}$. Then from $x_{k+1} = \Pi_{D_k \cap Q_k} x_1$ and Lemma 2.9, we have that $\langle y - x_{k+1}, Jx_1 - Jx_{k+1}\rangle \leq 0$, $\forall\ y \in D_k \cap Q_k$. Since $\Omega \subset D_k \cap Q_k$, we have $\langle y - x_{k+1}, Jx_1 - Jx_{k+1}\rangle \leq 0$, $\forall\ y \in \Omega$. It follows from the construction of Q_n that $\Omega \subset Q_{k+1}$. Therefore, $\Omega \subset D_{k+1} \cap Q_{k+1}$. By induction, we have that $\Omega \subset D_n \cap Q_n$ for each $n \in \mathbb{N}$.

Step 2: Next, we show that $\{x_n\}$ is bounded. From the construction $\langle y - x_n, Jx_1 - Jx_n\rangle \leq 0$, $\forall y \in Q_n$ and by Lemma 2.9, we have $x_n = \Pi_{Q_n} x_1$. From this, it follows that

$$\phi(x_n, x_1) \leq \phi(y, x_1) \quad \forall\ y \in Q_n. \tag{9}$$

Since $\Omega \subset Q_n$, we have

$$\phi(x_n, x_1) \leq \phi(y, x_1) \quad \forall\ y \in \Omega, \tag{10}$$

and this implies that $\{\phi(x_n, x_1)\}$ is bounded. Consequently, by property (P1) of the Lyapunov functional we have that $\{x_n\}$ is bounded.

Step 3: We next show that $w_\omega(x_n) \subset \Omega$. By $x_{n+1} \in Q_n$ and (9), we get $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. Hence, there exists $k = \lim_{n\to+\infty} \phi(x_n, x_1)$. Since $x_{n+1} \in Q_n$, then we obtain

$$\phi(x_{n+1},x_n)+\phi(x_n,x_1)=\phi(x_{n+1},x_1)+2\langle x_n-x_{n+1},Jx_n-Jx_1\rangle\leq\phi(x_{n+1},x_1).$$

It follows that $\phi(x_{n+1},x_n)\leq\phi(x_{n+1},x_1)-\phi(x_n,x_1)$. Hence,

$$\lim_{n\to+\infty}\phi(x_{n+1},x_n)=0. \tag{11}$$

Then it follows from Lemma 2.6 that

$$x_{n+1}-x_n\to 0 \quad \text{as} \quad n\to+\infty. \tag{12}$$

Since J is uniformly norm-to-norm continuous on each bounded subset of E, we have

$$Jx_{n+1}-Jx_n\to 0 \quad \text{as} \quad n\to+\infty. \tag{13}$$

By the definition of w_n, we have $Jw_n-Jx_n=\alpha_n(Jx_n-Jx_{n-1})$. We know that $\{\alpha_n\}$ is bounded, then we have

$$Jw_n-Jx_n\to 0 \quad \text{as} \quad n\to+\infty. \tag{14}$$

From Lemma 2.12, we have

$$\|w_n-x_n\|^2\leq\frac{2}{c^2}\langle w_n-x_n,Jw_n-Jx_n\rangle\leq\frac{2}{c^2}\|w_n-x_n\|\|Jw_n-Jx_n\|,$$

which gives $\quad \|w_n-x_n\|\leq\dfrac{2}{c^2}\|Jw_n-Jx_n\|\to 0, \quad n\to+\infty.$

Hence,

$$w_n-x_n\to 0, \quad n\to+\infty. \tag{15}$$

Since $\{x_n\}$ and $\{w_n\}$ are bounded, then it follows from Remark 2.7 that

$$\phi(x_n,w_n)\to 0, \quad n\to+\infty. \tag{16}$$

We know that $x_{n+1}\in D_n$, then we have $\phi(x_{n+1},v_n)\leq\phi(x_{n+1},x_n)-2\alpha_n(1-\beta_{n,0})\langle x_{n+1}-x_n,Jx_n-Jx_{n-1}\rangle+(1-\beta_{n,0})\phi(x_n,w_n)$, and it follows that

$$\phi(x_{n+1},v_n)\to 0, \quad n\to+\infty. \tag{17}$$

Then it follows from Lemma 2.6 that

$$x_{n+1}-v_n\to 0, \quad n\to+\infty. \tag{18}$$

By combining (12), (15) and (18), we obtain $v_n-w_n\to 0$, $n\to+\infty$. Since J is uniformly norm-to-norm continuous on each bounded subset of E, then we have $Jv_n-Jw_n\to 0, \quad n\to+\infty$, and by property (P5) of the Lyapunov functional we get $\phi(v_n,w_n)\to 0, \quad n\to+\infty$. Also, by property (P3) of the Lyapunov functional we have $\phi(p,w_n)-\phi(p,x_n)=\phi(x_n,w_n)+2\langle p-x_n,Jx_n-Jw_n\rangle$. By applying (14) and (16), we obtain

$$\phi(p,w_n)-\phi(p,x_n)\to 0, \quad n\to+\infty. \tag{19}$$

Similarly, $\phi(p,x_{n+1})-\phi(p,x_n)=\phi(x_n,x_{n+1})+2\langle p-x_n,Jx_n-Jx_{n+1}\rangle$. By applying (11) and (13), we have

$$\phi(p,x_{n+1})-\phi(p,x_n)\to 0, \quad n\to+\infty. \tag{20}$$

By using (6), (17), (19), (20) and the definition of v_n, we have

$$0=\lim_{n\to+\infty}\big(\phi(p,x_{n+1})-\phi(p,x_n)\big)$$

$$
\begin{aligned}
&= \lim_{n\to+\infty} \big(\phi(p,v_n)+\phi(v_n,x_{n+1})+2\langle v_n-p, Jx_{n+1}-Jv_n\rangle-\phi(p,x_n)\big)\\
&\le \lim_{n\to+\infty} \big(\beta_{n,0}\phi(p,x_n)+\sum_{i=1}^{+\infty}\beta_{n,i}\phi(p,S_iz_n)-\phi(p,x_n)\big)\\
&\le \lim_{n\to+\infty} \big(\beta_{n,0}\phi(p,x_n)+(1-\beta_{n,0})\phi(p,z_n)-\phi(p,x_n)\big)\\
&= \lim_{n\to+\infty} \big((1-\beta_{n,0})\big(\phi(p,z_n)-\phi(p,x_n)\big)\big)\\
&\le (1-a)\lim_{n\to+\infty} \big(\phi(p,z_n)-\phi(p,x_n)\big)\\
&\le (1-a)\lim_{n\to+\infty} \big(\phi(p,w_n)-\big(1-\frac{2\lambda_n L}{c^2}\big)\big(\phi(z_n,y_n)+\phi(y_n,w_n)\big)-\phi(p,x_n)\big)\\
&= -(1-a)\lim_{n\to+\infty} \big(\big(1-\frac{2\lambda_n L}{c^2}\big)\big(\phi(z_n,y_n)+\phi(y_n,w_n)\big)\big),
\end{aligned}
$$

which implies that $\phi(z_n,y_n)\to 0$, $n\to+\infty$ and $\phi(y_n,w_n)\to 0$, $n\to+\infty$. By Lemma 2.6, we obtain

$$z_n-y_n\to 0,\quad n\to+\infty \quad\text{and}\quad y_n-w_n\to 0,\quad n\to+\infty. \tag{21}$$

Combining (15) and (21), we have

$$x_n-y_n\to 0,\quad n\to+\infty. \tag{22}$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup z$. From (21) and (22), and by applying Lemma 3.3 we obtain $z\in VI(C,A)$. Hence, we have that

$$w_\omega(x_n)\subset VI(C,A). \tag{23}$$

Next, we show that $w_\omega(x_n)\subset\bigcap_{i=1}^{+\infty}F(S_i)$. By property (P3) of the Lyapunov functional we have $\phi(p,v_{n_k})-\phi(p,x_{n_k})=\phi(x_{n_k},v_{n_k})+2\langle p-x_{n_k},Jx_{n_k}-Jv_{n_k}\rangle$. Combining (12) and (18), and using property (P5) of the Lyapunov functional we obtain

$$\phi(p,v_{n_k})-\phi(p,x_{n_k})\to 0,\quad n\to+\infty. \tag{24}$$

By applying (8), (14), and (16) together with (24) we have

$$
\begin{aligned}
0 &= \lim_{k\to+\infty}\big(\phi(p,v_{n_k})-\phi(p,x_{n_k})\big)\\
&\le \lim_{k\to+\infty}\big(\phi(p,x_{n_k})+(1-\beta_{n_k,0})\phi(x_{n_k},w_{n_k})-2\alpha_{n_k}(1-\beta_{n_k,0})\langle p-x_{n_k},Jx_{n_k}-Jx_{n_k-1}\rangle\\
&\quad -\beta_{n_k,0}\beta_{n_k,j}g(\|Jx_{n_k}-Ju_{n_k,i}\|)-\phi(p,x_{n_k})\big)\\
&= \lim_{k\to+\infty}\big(-2(1-\beta_{n_k,0})\langle p-x_{n_k},\alpha_{n_k}(Jx_{n_k}-Jx_{n_k-1})\rangle-\beta_{n_k,0}\beta_{n_k,j}g(\|Jx_{n_k}-Ju_{n_k,i}\|)\big)\\
&= \lim_{k\to+\infty}\big(-2(1-\beta_{n_k,0})\langle p-x_{n_k},Jw_{n_k}-Jx_{n_k}\rangle-\beta_{n_k,0}\beta_{n_k,j}g(\|Jx_{n_k}-Ju_{n_k,i}\|)\big)\\
&= \lim_{k\to+\infty}\big(-\beta_{n_k,0}\beta_{n_k,j}g(\|Jx_{n_k}-Ju_{n_k,i}\|)\big),
\end{aligned}
$$

and this implies that $g(\|Jx_{n_k}-Ju_{n_k,i}\|)\to 0$, $k\to+\infty$. It follows from the property of g that $\lim_{k\to+\infty}\|Jx_{n_k}-Ju_{n_k,i}\|=0$, $\forall\, i\ge 1$. Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{k\to+\infty} \|x_{n_k} - u_{n_k,i}\| = 0, \quad \forall\, i \geq 1. \tag{25}$$

By combining (21) and (22), we get

$$\lim_{k\to+\infty} \|z_{n_k} - x_{n_k}\| = 0. \tag{26}$$

Using (25) and (26), we obtain $\|z_{n_k} - u_{n_k,i}\| \leq \|z_{n_k} - x_{n_k}\| + \|x_{n_k} - u_{n_k,i}\| \to 0$, $\forall\, i \geq 1$. Hence, $\lim_{k\to+\infty} \|z_{n_k} - u_{n_k,i}\| = 0, \quad \forall\, i \geq 1$. It follows that

$$\lim_{k\to+\infty} d(z_{n_k}, S_i z_{n_k}) \leq \lim_{k\to+\infty} \|z_{n_k} - u_{n_k,i}\| = 0, \quad \forall\, i \geq 1. \tag{27}$$

By (26) and (27), and the definition of S_i for all $i \geq 1$, we have $z \in S_i z$, $\forall\, i \geq 1$, and this implies that $z \in \bigcap_{i=1}^{+\infty} F(S_i)$. Thus, we have

$$w_\omega(x_n) \subset \bigcap_{i=1}^{+\infty} F(S_i). \tag{28}$$

From (23) and (28), we have $w_\omega(x_n) \subset \Omega$.

Step 4: Finally, we show that $x_n \to x^\dagger = \Pi_\Omega x_1$ as $n \to +\infty$.

By the convexity and lower semicontinuity of the norm, we have $\|z\| \leq \liminf_{k\to+\infty} \|x_{n_k}\|$. Therefore,

$$\begin{aligned}
\liminf_{k\to+\infty} \phi(x_{n_k}, x_1) &= \liminf_{k\to+\infty} (\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_1\rangle + \|x_1\|^2) \\
&= \liminf_{k\to+\infty} \|x_{n_k}\|^2 - 2\liminf_{k\to+\infty} \langle x_{n_k}, Jx_1\rangle + \|x_1\|^2 \\
&\geq \|z\|^2 - 2\langle z, Jx_1\rangle + \|x_1\|^2 = \phi(z, x_1).
\end{aligned} \tag{29}$$

From $x^\dagger = \Pi_\Omega x_1, z \in \Omega$, (10) and (29), we obtain

$$\phi(x^\dagger, x_1) \leq \phi(z, x_1) \leq \liminf_{k\to+\infty} \phi(x_{n_k}, x_1) \leq \limsup_{k\to+\infty} \phi(x_{n_k}, x_1) \leq \phi(x^\dagger, x_1).$$

It follows that $\lim_{k\to+\infty} \phi(x_{n_k}, x_1) = \phi(z, x_1) = \phi(x^\dagger, x_1)$. Since $x^\dagger = \Pi_\Omega x_1$, then it implies that $z = x^\dagger$, i.e. $w_\omega(x_n) = \{x^\dagger\}$. Hence, $x_n \rightharpoonup x^\dagger$, $n \to +\infty$, and $\lim_{n\to+\infty} \phi(x_n, x_1) = \phi(x^\dagger, x_1)$. Since

$$\begin{aligned}
\|x_n\|^2 - \|x^\dagger\|^2 &= \phi(x_n, x_1) + 2\langle x_n, Jx_1\rangle - \|x_1\|^2 - \phi(x^\dagger, x_1) - 2\langle x^\dagger, Jx_1\rangle + \|x_1\|^2 \\
&= \phi(x_n, x_1) - \phi(x^\dagger, x_1) + 2\langle x_n - x^\dagger, Jx_1\rangle,
\end{aligned}$$

then we have $\|x_n\|^2 - \|x^\dagger\|^2 \to 0$, $n \to +\infty$. Also, from

$$\phi(x_n, x^\dagger) = \|x_n\|^2 - 2\langle x_n, Jx^\dagger\rangle + \|x^\dagger\|^2 = \|x_n\|^2 - \|x^\dagger\|^2 - 2\langle x_n - x^\dagger, Jx^\dagger\rangle,$$

we get $\phi(x_n, x^\dagger) \to 0$, $n \to +\infty$.

By Lemma 2.6, we have $x_n \to x^\dagger$ as $n \to +\infty$, and this completes the proof. □

Remark 3.5. Observe that if we take the multivalued relatively nonexpansive mappings S_i, $i \in \mathbb{N}$ in Theorem 3.4 as single-valued relatively nonexpansive mappings and then apply Lemma 2.4, we obtain a consequent result for approximating the common solution of an infinite family of generalized nonspreading mappings and VIP (1).

4. Applications

In this section, we present some applications of our results to approximating solutions of other optimization problems in Banach spaces.

4.1 Variational inequality and zero point problems of maximal monotone mappings

Here, we apply our results to find a common solution of variational inequality and zero point problems for an infinite family of maximal monotone mappings in Banach spaces.

Let E be a real Banach space. We consider the following zero point problem: find $x \in E$ such that $0 \in Bx$, where $B : E \to 2^{E^*}$ is a maximal monotone operator. We denote the solution set of this problem by $B^{-1}0$.

Let E be a smooth, strictly convex, and reflexive Banach space. An operator $B : E \to 2^{E^*}$ is said to be *monotone*, if $\langle x - y, x^* - y^* \rangle \geq 0$, whenever $x, y \in E$, $x^* \in Bx$, $y^* \in By$. A monotone operator B is said to be *maximal*, if its graph $G(B) := \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone operator. If B is maximal monotone, then $B^{-1}0$ is closed and convex. Let B be a maximal monotone operator, then for each $r > 0$ and $x \in E$, there exists a unique $x_r \in D(B)$ such that $J(x) \in J(x_r) + rB(x_r)$ (see, for example, [3]), where $D(B) = \{x \in E : Bx \neq \emptyset\}$. We can define a single valued mapping $J_r : E \to D(B)$ by $J_r(x) = x_r$, that is, $J_r = (J + rB)^{-1}J$ and such J_r is called the *relative resolvent* of B. We know that $B^{-1}0 = F(J_r)$ for all $r > 0$ (see [26] for more details).

LEMMA 4.1 ([26]). *Let E be a uniformly convex and uniformly smooth Banach space and let $B : E \to 2^{E^*}$ be a maximal monotone operator. Let J_r be the relative resolvent of B, where $r > 0$. If $B^{-1}0$ is nonempty, then J_r is a relatively nonexpansive mapping on E.*

In Theorem 3.4, if we consider the case where $S_i,\ i \in \mathbb{N}$ are singled-valued mappings and we set $S_i = J_{r_i}$ for all $i \in \mathbb{N}$, then by Lemma 4.1 $S_i : E \to E,\ i \in \mathbb{N}$ is an infinite family of single-valued relatively nonexpansive mappings and $\bigcap_{i=1}^{+\infty} B_i^{-1}0 = \bigcap_{i=1}^{+\infty} F(S_i) = \bigcap_{i=1}^{+\infty} F(J_{r_i}),\ r_i > 0$ is a nonempty closed convex subset of E. Hence, we obtain from Theorem 3.4 a consequent result for approximating a common solution of zero point problem (30) and VIP (1).

4.2 Constrained convex minimization and fixed point problems

Next, we apply our result to approximate a common solution of constrained convex minimization problem and fixed point problem in Banach spaces.

Let E be a real Banach space and C be a nonempty closed convex subset of E. The constrained convex minimization problem is to find a point $x^* \in C$ such that

$$f(x^*) = \min_{x \in C} f(x), \tag{30}$$

where f is a real-valued convex function.

The following lemma will be employed in establishing our result in this section.

Lemma 4.2 ([27]). *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let f be a convex function of E into $\mathbb{R}$. If f is Fréchet differentiable, then z is a solution of problem* (30) *if and only if $z \in VI(C, \triangledown f)$.*

If $f : E \to \mathbb{R}$ is a Fréchet differentiable convex function such that $\triangledown f$ is L-Lipschitz continuous, then it is known that $\triangledown f$ is monotone [27]. Now, taking $A = \triangledown f$ in Theorem 3.4 and by applying Lemma 4.2, we obtain a consequent result for approximating a common solution of the constrained convex minimization problem (30) and fixed point problem for an infinite family of multivalued relatively nonexpansive mappings.

5. Numerical example

In this section, we present a numerical example to illustrate the performance of our Algorithm 3.1 as well as comparing it with [19, Algorithm 3.3] and [18, Algorithm 4.1]. All numerical computations are carried out using Matlab version R2019 (b).

We choose $\beta_{n,0} = \frac{9}{10}, \beta_{n,i} = \frac{9^{i-1}}{10^{i+1}}, \alpha_n = \frac{1}{n+1}$.

Example 5.1. Let $E = L^2([0, 2\pi])$ with inner product

$$\langle x, y\rangle := \int_0^{2\pi} x(t)y(t)\, dt, \quad \forall x, y \in E,$$

and induced norm

$$\|x\| := \left(\int_0^{2\pi} |x(t)|^2 dt\right)^{\frac{1}{2}}, \ \forall\ x \in E.$$

Let $C[0, 2\pi]$ denote the continuous function space defined on the interval $[0, 2\pi]$ and choose an arbitrary fixed $\varphi \in C[0, 2\pi]$. Let $C := \{x \in E : \|\varphi x\| \leq 1\}$. It can easily be verified that C is a nonempty closed convex subset of E. Define an operator $A : E \to E^*$ by $(Ax)(t) = \max(0, x(t))$, $\forall\ x \in E$. It can easily be checked that A is Lipschitz continuous and monotone. With these C and A given, the solution set to the VIP (1) is given by $VI(C, A) = \{0\} \neq \emptyset$. Define $g : E \to \mathbb{R}$ by $g(x) = \frac{1}{2}(\|\varphi x\|^2 - 1)$, $\quad \forall\ x \in E$, then g is a convex function and C is a level set of g, i.e. $C = \{x \in E : g(x) \leq 0\}$. Also, g is differentiable on E and $\partial g(x) = \varphi^2 x$, $\forall\ x \in E$ (see [10]). In this numerical example, we choose $\varphi(t) = e^{-t}$, $\forall\ t \in [0, 2\pi]$. For each $i \in \mathbb{N}$, let $S_i : L^2([0, 2\pi]) \to L^2([0, 2\pi])$ be defined by $(S_i x)(t) = \int_0^{2\pi} x(t)\, dt$, $t \in [0, 1]$.

Observe that $F(S_i) \neq \emptyset$ since $0 \in F(S_i)$ for each $i \in \mathbb{N}$. Moreover, S_i is quasi-nonexpansive (note that in a Hilbert space relatively nonexpansive mapping reduces to quasi-nonexpansive mapping). Therefore, the solution set of the problem is $x^\dagger(t) = \{0\} \neq \emptyset$. Taking $\lambda_n = 0.7$, we test the algorithms for four different starting points using $\|x_{n+1} - x_n\| < \epsilon$ as stopping criterion, where $\epsilon = 10^{-2}$. The numerical result is reported in Figure 1 and Table 1.

Case I $x_0(t) = \exp(5t)$, $x_1(t) = \sin(\pi t)$; Case II $x_0(t) = \exp(t)$, $x_1(t) = 9t^2 + \frac{1}{2}t$;

Case III $x_0(t) = \cos(\pi t)$, $x_1(t) = \exp(2t)$; Case IV $x_0(t) = t+1$, $x_1(t) = t^3 + 3t + 2$.

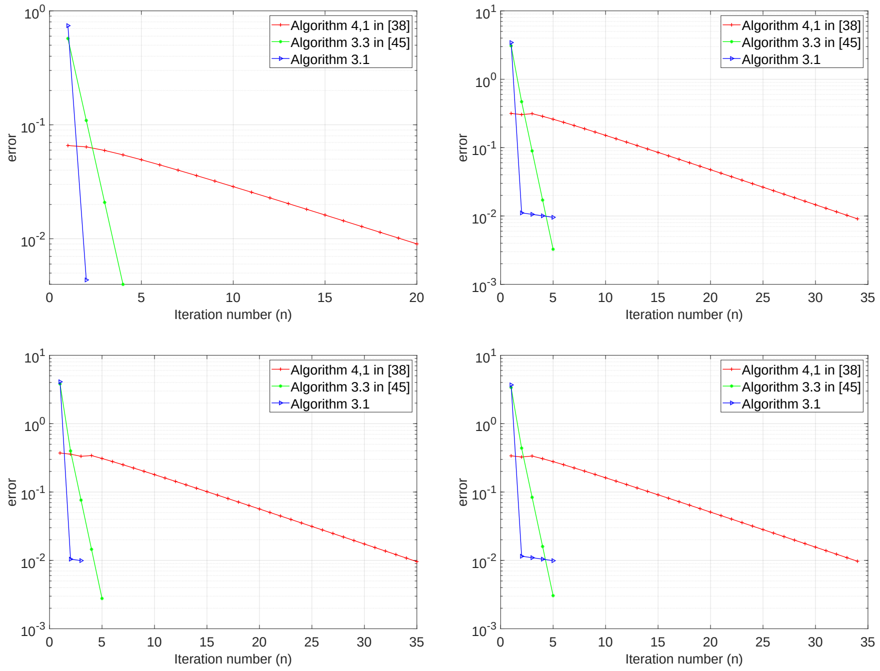

Figure 1: Top left: Case I; Top right: Case III; Bottom left: Case II; Bottom right: Case IV

		Alg. 4.1 in [18]	Alg. 3.3 in [19]	Alg. 3.1
Case I	CPU time (sec) No of Iter.	2.2198 20	0.3185 4	10.0611 2
Case III	CPU time (sec) No. of Iter.	47.1152 34	1.0815 5	8.6748 5
Case II	CPU time (sec) No of Iter.	2.5361 35	0.3575 5	3.2710 3
Case IV	CPU time (sec) No of Iter.	2.7027 34	0.3455 5	0.5916 5

Table 1: Numerical results for Example 5.1

6. Conclusion

In this paper, we considered the monotone variational inequality and fixed point problems for an infinite family of multivalued relatively nonexpansive mappings. We proposed a new inertial hybrid subgradient extragradient method for approximating a common solution of the problem considered in Banach spaces. We obtained strong convergence result for the proposed algorithm and applied our result to study related optimization problems.

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