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TWO-WEIGHTED INEQUALITIES FOR RIESZ POTENTIAL AND ITS COMMUTATORS IN GENERALIZED WEIGHTED MORREY SPACES

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Abstract. In this paper we find the conditions for the boundedness of Riesz potential I^α and its commutators from the generalized weighted Morrey spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the generalized weighted Morrey spaces $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$, where $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$, $(\omega_1,\omega_2)\in A_{p,q}(\mathbb{R}^n)$, φ_1, φ_2 are generalized functions and $b\in BMO(\mathbb{R}^n)$. Furthermore, we give some applications of our results.

1. Introduction

Let $1\le p<\infty$, φ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and ω be a non-negative measurable function on $\mathbb{R}^n$. We denote by $\mathcal{M}^{p,\varphi}_\omega$ the generalized weighted Morrey space, the space of all functions $f\in L^{loc}_{p,\omega}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{\mathcal{M}^{p,\varphi}_\omega}=\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)\|\omega\|_{L_p(B(x,r))}}\|f\|_{L_{p,\omega}(B(x,r))},$$

where the supremum is taken over all balls $B(x,r)$ in $\mathbb{R}^n$ and $L_{p,\omega}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,\omega}(B(x,r))}\equiv\|f\chi_{B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)}=\left(\int_{B(x,r)}|f(y)|^p\omega(y)\,dy\right)^{\frac{1}{p}}.$$

Moreover, by $W\mathcal{M}^{p,\varphi}_\omega$ we denote the weak generalized weighted Morrey space of all functions $f\in WL^{loc}_{p,\omega}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{W\mathcal{M}^{p,\varphi}_\omega}=\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)\|\omega\|_{L_p(B(x,r))}}\|f\|_{WL_{p,\omega}(B(x,r))},$$

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where $WL_{p,\omega}(B(x,r))$ denotes the weak weighted L_p-space of measurable functions f for which

$$\|f\|_{WL_{p,\omega}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,\omega}(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{y \in B(x,r): |f(y)|>t} |f(y)|^p \omega(y)\, dy \right)^{\frac{1}{p}}.$$

Note that if $\omega(x) = \chi_{B(x,r)}$, then $\mathcal{M}^{p,\varphi}_{\omega}(\mathbb{R}^n) = \mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space and if $\varphi(x,r) = r^{\frac{n-\lambda}{p}}$, then $\mathcal{M}^{p,\varphi}_{\omega}(\mathbb{R}^n) = L_{p,\lambda}(\omega)$ is the weighted Morrey space.

Let f be a locally integrable function on $\mathbb{R}^n$. The so-called fractional maximal function is defined by the formula

$$M^{\alpha} f(x) = \sup_{r>0} |B(x,r)|^{-1+\alpha/n} \int_{B(x,r)} |f(y)|\, dy, \quad 0 \le \alpha < n,$$

where $|B(x,r)| = \omega_n r^n$ is the Lebesgue measure of the ball $B(x,r)$. It coincides with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$. Maximal operators play an important role in the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more simplified approach to understanding problems in these areas than other methods.

Fractional maximal operator is intimately related to the Riesz potential

$$I^{\alpha} f(x) = \int_{\mathbb{R}^n} \frac{f(y)\, dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n.$$

The aim of this paper is to find the conditions for the boundedness of Riesz potential I^{α} and its commutators $[b, I^{\alpha}]$ and $|b, I^{\alpha}|$ from the generalized weighted Morrey spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the spaces $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$, where $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$, ω_1, ω_2 are generalized functions and $b \in BMO(\mathbb{R}^n)$. Furthermore, we give some applications of our main results.

In 2012, Guliyev gave a concept of generalized weighted Morrey space, which could be viewed as extension of both generalized Morrey space and weighted Morrey space [9]. In [9] Guliyev also obtained the boundedness of sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in generalized weighted Morrey spaces (see also [12, 15]).

In this paper we aim to give a characterization of two-weighted inequalities for Riesz potential and its commutators in generalized weighted Morrey spaces. Two-weight norm inequalities for the operators of harmonic analysis on various function spaces were widely studied (see, for example [5, 8, 16, 17, 20]). The weighted norm inequalities with different types of weights on Morrey spaces were also studied (see, for example [13,22,24]). The two-weight norm inequality for the Hardy-Littlewood maximal function on Morrey spaces was obtained in [28]. Two-weight norm inequalities on weighted Morrey spaces for fractional maximal operators and fractional integral operators were obtained in [23]. Two-weight norm inequalities on generalized weighted Morrey spaces for maximal, Calderón-Zygmund operators and their commutators were obtained in [3].

In the sequel we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence. If $p \in [1, \infty]$, the conjugate number p' is defined by $pp' = p + p'$. $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of non-negative and non-negative non-decreasing functions, respectively.

2. Background material

Even though the A_p class is well-known, we offer the definition of A_p weight functions.

Definition 2.1. The weight function ω belongs to the class $A_p(\mathbb{R}^n)$, for $1 \le p < \infty$, if the following

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \Big(\int\limits_{B(x,r)} \omega^p(y)\, dy \Big)^{\frac{1}{p}} \Big(\int\limits_{B(x,r)} \omega^{-p'}(y)\, dy \Big)^{\frac{1}{p'}}$$

is finite and ω belongs to $A_1(\mathbb{R}^n)$, if there exists a positive constant C such that for any $x \in \mathbb{R}^n$ and $r > 0$

$$|B(x,r)|^{-1} \int\limits_{B(x,r)} \omega(y)\, dy \le C \operatorname*{ess\,sup}_{y\in B(x,r)} \frac{1}{\omega(y)}.$$

Definition 2.2. The weight functions (ω_1, ω_2) belong to the class $\widetilde{A}_p(\mathbb{R}^n)$, for $1 \le p < \infty$, if the following supremum is finite

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \Big(\int\limits_{B(x,r)} \omega_2^p(y)\, dy \Big)^{\frac{1}{p}} \Big(\int\limits_{B(x,r)} \omega_1^{-p'}(y)\, dy \Big)^{\frac{1}{p'}}.$$

Definition 2.3. The weight functions (ω_1, ω_2) belong to the class $A_{p,q}(\mathbb{R}^n)$, for $1 \le p, q < \infty$, if the following supremum is finite

$$\sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}-1} \Big(\int\limits_{B(x,r)} \omega_2^q(y)\, dy \Big)^{\frac{1}{q}} \Big(\int\limits_{B(x,r)} \omega_1^{-p'}(y)\, dy \Big)^{\frac{1}{p'}}.$$

The following theorem was proved in [21].

Theorem 2.4. *Let $1 \le p < \infty$, then*
(i) $M : L_{p,\varphi}(\mathbb{R}^n) \to L_{p,\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in A_p(\mathbb{R}^n)$,
(ii) $M : L_{1,\varphi}(\mathbb{R}^n) \to WL_{1,\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in A_1(\mathbb{R}^n)$.

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}|\, dy,$$

where $f_{B(x,r)}(x) = |B(x,r)|^{-1} \int_{B(x,r)} f(y)\, dy$.

Definition 2.5. We define the $BMO(\mathbb{R}^n)$ space as the set of all locally integrable functions f such that

$$\|f\|_{BMO} = \sup_{x\in\mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}|\, dy < \infty$$

or

$$\|f\|_{BMO} = \inf_C \sup_{x\in\mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - C|\, dy < \infty.$$

Definition 2.6. We define the $BMO_{p,\omega}(\mathbb{R}^n)$ $(1 \le p < \infty)$ space as the set of all locally integrable functions f such that

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{\|(f(\cdot)-f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)}}{\|\omega\|_{L_p(B(x,r))}} < \infty$$

or

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \|(f(\cdot)-f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)} \|\omega^{-1}\|_{L_{p'}(B(x,r))} < \infty.$$

Theorem 2.7 ([18]). *Let $1 \le p < \infty$ and ω be a Lebesgue measurable function. If $\omega \in A_p(\mathbb{R}^n)$, then the norms $\|\cdot\|_{BMO_{p,\omega}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

Before proving the main theorems, we need the following lemma.

Lemma 2.8 ([14]). *Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that $\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_{BMO} \ln \frac{t}{r}$ for $0 < 2r < t$, where C is independent of b, x, r and t.*

Let $L_{\infty,v}(\mathbb{R}_+)$ be the weighted L_∞-space with the norm $\|g\|_{L_{\infty,v}(\mathbb{R}_+)} = \operatorname{ess\,sup}_{t>0} v(t)g(t)$.

We denote $\mathbb{A} = \{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\}$. Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by $(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(0,t)}$, $t \in (0,\infty)$.

The following theorem was proved in [4].

Theorem 2.9 ([4]). *Suppose that v_1 and v_2 are non-negative measurable functions such that $0 < \|v_1\|_{L_\infty(0,t)} < \infty$ for every $t > 0$. Let u be a continuous non-negative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L^\infty_{v_1}(\mathbb{R}_+)$ to $L^\infty_{v_2}(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if $\left\|v_2 \overline{S}_u\left(\|v_1\|^{-1}_{L_\infty(0,\cdot)}\right)\right\|_{L_\infty(\mathbb{R}_+)} < \infty$.*

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_0^t g(s)w(s)ds, \quad H^*_w g(t) := \int_t^\infty g(s)w(s)ds, \qquad 0 < t < \infty,$$

where w is a weight.

Theorem 2.10 ([10]). *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$ess \sup_{t>0} v_2(t) H^*_w g(t) \le C ess \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0,\infty)$ *if and only if*

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}\limits_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 2.11 ([10,11]). *Let* v_1, v_2 *and* w *be weights on* $(0,\infty)$ *and* $v_1(t)$ *be bounded outside a neighborhood of the origin. The inequality*

$$ess\sup_{t>0} v_2(t) H_w g(t) \le C ess\sup_{t>0} v_1(t) g(t) \tag{1}$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0,\infty)$ *if and only if*

$$B := ess\sup_{t>0} v_2(t) \int_0^t \frac{w(s)ds}{ess\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ *is the best constant for* (1).

3. Two-weighted inequalities for Riesz potential and its commutators in generalized weighted Morrey spaces

In this section we prove two-weighted inequalities for Riesz potential and its commutators in generalized weighted Morrey spaces.

Theorem 3.1 ([25]). *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and the* $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. *Then the operator* I^α *is bounded from* $L_{p,\omega_1}(\mathbb{R}^n)$ *to* $L_{q,\omega_2}(\mathbb{R}^n)$.

From the inequality $M^\alpha f(x) \le \omega_n^{\frac{\alpha}{n}-1} (I^\alpha)|f|(x)$, we get the following corollary.

Corollary 3.2. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. *Then the operator* M^α *is bounded from* $L_{p,\omega_1}(\mathbb{R}^n)$ *to* $L_{q,\omega_2}(\mathbb{R}^n)$.

Theorem 3.3. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$. *Then there exists a constant* $C > 0$ *such that for an arbitrary* $f \in L_{p,\omega_1}(B(x,t))$ *the inequality*

$$\|I^\alpha f\|_{L_{q,\omega_2}(B(x,t))} \le C \|\omega_2\|_{L_q(B(x,t))} \int_t^\infty s^{\alpha-n-1} \frac{\|f\|_{L_{p,\omega}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} ds \tag{2}$$

holds.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\mathbb{R}^n \setminus B(x,2t)}(y), \quad t > 0, \tag{3}$$

and have $I^\alpha f(x) = I^\alpha f_1(x) + I^\alpha f_2(x)$.

By Theorem 3.1 we obtain

$$\|I^\alpha f_1\|_{L_{q,\omega_2}(B(x,t))} \le \|I^\alpha f_1\|_{L_{q,\omega_2}(\mathbb{R}^n)} \le C\|f_1\|_{L_{p,\omega_1}(\mathbb{R}^n)} = C\|f\|_{L_{p,\omega_1}(B(x,2t))}.$$

Then

$$\|I^\alpha f_1\|_{L_{q,\omega_2}(B(x,t))} \leq C\|f\|_{L_{p,\omega_1}(B(x,2t))},$$

where the constant C is independent of f. Taking into account that, we get

$$\|I^\alpha f_1\|_{L_{q,\omega_2}(B(x,t))} \leq C\|\omega_2\|_{L_q(B(x,t))} \int_t^\infty s^{\alpha-n-1} \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} ds. \quad (4)$$

When $|x-z| \leq t$, $|z-y| \geq 2t$, we have $\frac{1}{2}|z-y| \leq |x-y| \leq \frac{3}{2}|z-y|$, and therefore

$$|I^\alpha f_2(z)| \leq \int_{\mathbb{R}^n\setminus B(x,2t)} |z-y|^{\alpha-n}|f(y)|\,dy \leq C\int_{\mathbb{R}^n\setminus B(x,2t)} |x-y|^{\alpha-n}|f(y)|\,dy.$$

We choose $\beta > \frac{n}{q}$ and obtain

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(x,2t)} |x-y|^{\alpha-n}|f(y)|\,dy =&\beta\int_{\mathbb{R}^n\setminus B(x,2t)} |f(y)|\left(\int_{|x-y|}^\infty s^{\alpha-n-1}ds\right)dy\\ =&\beta\int_{2t}^\infty s^{\alpha-n-1}\left(\int_{\{y\in\mathbb{R}^n:2t\leq|x-y|\leq s\}}|f(y)|\,dy\right)ds\\ \leq&C\int_{2t}^\infty s^{\alpha-n-1}\|\chi_{B(x,s)}\omega_1^{-1}\|_{L_{p'}(\mathbb{R}^n)}\|f\|_{L_{p,\omega_1}(B(x,s))}ds\\ \leq&C\int_t^\infty s^{\alpha-n-1}\|f\|_{L_{p,\omega_1}(B(x,s))}\|\omega_1^{-1}\|_{L_{p'}(B(x,s))}ds.\end{aligned}$$

Hence

$$\|I^\alpha f_2\|_{L_{q,\omega_2}(B(x,t))} \leq C\int_t^\infty s^{\alpha-n-1}\|f\|_{L_{p,\omega_1}(B(x,s))}\|\omega_1^{-1}\|_{L_{p'}(B(x,s))}ds\|\chi_{B(x,t)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}.$$

Therefore we get

$$\|I^\alpha f_2\|_{L_{q,\omega_2}(B(x,t))}\leq C\|\omega_2\|_{L_q(B(x,t))}\int_t^\infty s^{\alpha-n-1}\|f\|_{L_{p,\omega_1}(B(x,s))}\|\omega_2\|^{-1}_{L_q(B(x,s))}ds \quad (5)$$

which together with (4) yields (2). □

THEOREM 3.4. *Let $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$ and $(\omega_1,\omega_2)\in A_{p,q}(\mathbb{R}^n)$. Let the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ fulfill the condition*

$$\int_t^\infty s^{\alpha-n-1}\frac{\operatorname*{ess\,inf}_{s<r<\infty}\varphi_1(x,r)\|\omega_1\|_{L^p(B(x,r))}}{\|\omega_2\|_{L^q(B(x,s))}}ds \leq C\varphi_2(x,t). \quad (6)$$

Then the operator I^α is bounded from $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

Proof. Let $f\in\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$. From the definition of the norm of generalized weighted Morrey spaces we write

$$\|I^\alpha f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\,t>0}\frac{1}{\varphi_2(x,t)\|\omega_2\|_{L_q(B(x,t))}}\|I^\alpha f\chi_{B(x,t)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}. \quad (7)$$

We estimate $\|I^\alpha f\chi_{B(x,t)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}$ in (7) by means of Theorems 3.3, 2.11 and obtain

$$\|I^\alpha f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)} \leq C\sup_{x\in\mathbb{R}^n,\,t>0}\frac{\|\omega_2\|_{L_q(B(x,t))}}{\varphi_2(x,t)\|\omega_2\|_{L_q(B(x,t))}}\int_t^\infty s^{\alpha-n-1}\frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega\|_{L_q(B(x,s))}}ds$$

$$\leq C \sup_{x\in\mathbb{R}^n,\, t>0} \frac{1}{\varphi_1(x,t)\|\omega_1\|_{L_p(B(x,t))}} \|f\|_{L_{p,\omega_1}(B(x,t))} = C\|f\|_{\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)}.$$

It remains to make use of condition (6). □

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In this section we consider commutators of the Riesz potential defined by the following equality

$$[b, I^\alpha]f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))|x-y|^{\alpha-n} f(y)\,dy, \quad 0<\alpha<n.$$

Given a measurable function b the operator $|b, I^\alpha|$ is defined by

$$|b, I^\alpha|f(x) = \int_{\mathbb{R}^n} |b(x) - b(y)|\,|x-y|^{\alpha-n}\,|f(y)|\,dy, \quad 0<\alpha<n.$$

The maximal commutator is defined by

$$M_b(f)(x) := \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x)-b(y)||f(y)|\,dy$$

for all $x \in \mathbb{R}^n$.

Lemma 3.5 ([6]). *Let $b \in BMO(\mathbb{R}^n)$, $1<s<\infty$. Then*

$$M^\sharp(|b, I^\alpha|f(x)) \leq C\|b\|_{BMO}\left[(M|I^\alpha f(x)|^s)^{\frac{1}{s}} + (M^{s\alpha}|f(x)|^s)^{\frac{1}{s}}\right],$$

where $C>0$ is independent of f and x.

Lemma 3.6 ([27]). *Let $1<p<\infty$ and $\omega \in A_p(\mathbb{R}^n)$. Then $\|f\|_{L_{p,\omega}} \leq C\|M^\sharp f\|_{L_{p,\omega}}$ with a constant $C>0$ not depending on f.*

Theorem 3.7 ([3]). *Let $1<p<\infty$, $(\omega_1,\omega_2) \in \widetilde{A}_p(\mathbb{R}^n)$ and the function $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition*

$$\sup_{t>r} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\|\omega_1\|_{L_p(B(x,s))}}{\|\omega_2\|_{L_p(B(x,t))}} \leq C\varphi_2(x,r), \tag{8}$$

where C does not depend on x and t.

Then the operator M is bounded from the space $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the space $\mathcal{M}^{p,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

Theorem 3.8 ([3]). *Let $1<p<\infty$, $b \in BMO(\mathbb{R}^n)$ and $(\omega_1,\omega_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\omega_1 \in A_p(\mathbb{R}^n)$, then the operator M_b is bounded from $L_{p,\omega_1}(\mathbb{R}^n)$ to $L_{p,\omega_2}(\mathbb{R}^n)$.*

Theorem 3.9 ([3]). *Let $1<p<\infty$, $b \in BMO(\mathbb{R}^n)$ and $(\omega_1,\omega_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\omega_1,\omega_2 \in A_p(\mathbb{R}^n)$. If the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition*

$$\sup_{t>r} \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\|\omega_1\|_{L_p(B(x,s))}}{\|\omega_2\|_{L_p(B(x,t))}} \leq C\varphi_2(x,r), \tag{9}$$

where C does not depend on x and t.

Then the operator M_b is bounded from the space $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the space $\mathcal{M}^{p,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

THEOREM 3.10. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$ *and* $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$, $\omega_1 \in A_p(\mathbb{R}^n)$, $\omega_2 \in A_q(\mathbb{R}^n)$. *The operator* $|b, I^\alpha|$ *is bounded from* $L_{p,\omega_1}(\mathbb{R}^n)$ *to* $L_{q,\omega_2}(\mathbb{R}^n)$.

Proof. Let $f \in L_{p,\omega_1}(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$. From Lemma 3.6 we have

$$\||b, I^\alpha|f\|_{L_{q,\omega_2}(\mathbb{R}^n)} \le C\|M^\sharp([b, I^\alpha]f)\|_{L_{q,\omega_2}(\mathbb{R}^n)}.$$

From Lemma 3.5, we have

$$\|M^\sharp(|b, I^\alpha|f)\|_{L_{q,\omega_2}(\mathbb{R}^n)} \le C\|b\|_{BMO} \left\|(M|I^\alpha f|^s)^{\frac{1}{s}} + (M^{\alpha s}|f|^s)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\mathbb{R}^n)}$$
$$\le C\|b\|_{BMO} \left[\left\|(M|I^\alpha f|^s)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\mathbb{R}^n)} + \left\|(M^{\alpha s}|f|^s)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\mathbb{R}^n)}\right].$$

From Theorem 2.4 and Theorem 3.1, we get

$$\left\|(M|I^\alpha f|^s)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\mathbb{R}^n)} \le C \left\|I^\alpha f\right\|_{L_{q,\omega_2}(\mathbb{R}^n)} \le C \left\|f\right\|_{L_{p,\omega_1}}.$$

From Corollary 3.2, we obtain

$$\left\|(M^{\alpha s}|f|^s)^{\frac{1}{s}}\right\|_{L_{q,\omega_2}(\mathbb{R}^n)} \le C \left\|f\right\|_{L_{p,\omega_1}}.$$

Therefore we get

$$\|[b, I^\alpha]f\|_{L_{q,\omega_2}(\mathbb{R}^n)} \le C\|b\|_{BMO} \left\|f\right\|_{L_{p,\omega_1}}.$$

Thus the theorem has been proved. □

THEOREM 3.11. *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$ *and* $(\omega_1, \omega_2) \in A_{p,q}(\mathbb{R}^n)$, $\omega_1 \in A_p(\mathbb{R}^n)$, $\omega_2 \in A_q(\mathbb{R}^n)$. *Then*

$$\||b, I^\alpha|f\|_{L_{q,\omega_2}(B(x,t))} \le$$
$$C \left\|b\right\|_{BMO} \|\omega_2\|_{L_q(B(x,t))} \times \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_{p,\omega_1}(B(x,s))} \|\omega_2\|^{-1}_{L_q(B(x,s))} \frac{ds}{s}, \tag{10}$$

where $t > 0$, C *does not depend on* f, x *and* t.

Proof. We represent f as (3) and have $|b, I^\alpha|f(x) \le |b, I^\alpha|f_1(x) + |b, I^\alpha|f_2(x)$.

By Theorem 3.10 we obtain

$$\||b, I^\alpha|f_1\|_{L_{q,\omega_2}(B(x,t))} \le \||b, I^\alpha|f_1\|_{L_{q,\omega_2}(\mathbb{R}^n)}$$
$$\le C \left\|b\right\|_{BMO} \left\|f_1\right\|_{L_{p,\omega_1}(\mathbb{R}^n)} = C \left\|b\right\|_{BMO} \left\|f\right\|_{L_{p,\omega_1}(B(x,2t))}.$$

Then

$$\||b, I^\alpha|f_1\|_{L_{q,\omega_2}(B(x,t))} \le C \left\|b\right\|_{BMO} \left\|f\right\|_{L_{p,\omega_1}(B(x,2t))},$$

where the constant C is independent of f. Taking into account that, we get

$$\||b, I^\alpha|f_1\|_{L_{q,\omega_2}(B(x,t))} \le$$
$$C \left\|b\right\|_{BMO} \|\omega_2\|_{L_q(B(x,t))} \times \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_q(B(x,s))}} \frac{ds}{s}. \tag{11}$$

When $|x-z|\le t$, $|z-y|\ge 2t$, we have $\frac{1}{2}|z-y|\le|x-y|\le\frac{3}{2}|z-y|$. Therefore we get

$$\begin{aligned}|b,I^{\alpha}|f_2(z) &\le \int_{\mathbb{R}^n\setminus B(x,2t)}|b(y)-b(z)||z-y|^{\alpha-n}|f(y)|\,dy\\ &\le C\int_{\mathbb{R}^n\setminus B(x,2t)}|b(y)-b(z)||x-y|^{\alpha-n}|f(y)|\,dy.\end{aligned}$$

We obtain

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B(x,2t)}|b(y)-b(z)||x-y|^{\alpha-n}|f(y)|\,dy\\ &=\int_{\mathbb{R}^n\setminus B(x,2t)}|b(y)-b(z)||f(y)|\left(\int_{|x-y|}^{\infty}s^{\alpha-n-1}ds\right)\,dy\\ &\le C\int_{2t}^{\infty}s^{\alpha-n-1}\left(\int_{\{y\in\mathbb{R}^n:2t\le|x-y|\le s\}}|b(y)-b(z)||f(y)|\,dy\right)ds\\ &\le C\int_{2t}^{\infty}s^{\alpha-n-1}\left(\int_{\{y\in\mathbb{R}^n:2t\le|x-y|\le s\}}|b(y)-b_{B(x,t)}||f(y)|\,dy\right)ds\\ &+C|b(z)-b_{B(x,t)}|\int_{2t}^{\infty}s^{\alpha-n-1}\left(\int_{\{y\in\mathbb{R}^n:2t\le|x-y|\le s\}}|f(y)|\,dy\right)ds=J_1+J_2.\end{aligned}$$

To estimate J_1:

$$\begin{aligned}J_1&=C\int_{2t}^{\infty}s^{\alpha-n-1}\left(\int_{\{y\in\mathbb{R}^n:2t\le|x-y|\le s\}}|b(y)-b_{B(x,t)}||f(y)|\,dy\right)ds\\ &\le C\int_{t}^{\infty}s^{\alpha-n-1}\|b(\cdot)-b_{B(x,s)}\|_{L_{p',\omega^{-1}}(B(x,s))}\|f\|_{L_{p,\omega}(B(x,s))}ds\\ &+C\int_{t}^{\infty}s^{\alpha-n-1}|b_{B(x,t)}-b_{B(x,s)}|\left(\int_{B(x,s)}|f(y)|\,dy\right)ds\\ &\le C\|b\|_{BMO}\int_{t}^{\infty}s^{\alpha-n-1}\|\omega^{-1}\|_{L_{p'}(B(x,s))}\|f\|_{L_{p,\omega}(B(x,s))}ds\\ &+C\|b\|_{BMO}\int_{t}^{\infty}s^{\alpha-n-1}\ln\frac{s}{t}\|\omega^{-1}\|_{L_{p'}(B(x,s))}\|f\|_{L_{p,\omega}(B(x,s))}ds\\ &\le C\|b\|_{BMO}\int_{t}^{\infty}\left(1+\ln\frac{s}{t}\right)\|\omega_2\|^{-1}_{L_q(B(x,s))}\|f\|_{L_{p,\omega}(B(x,s))}\frac{ds}{s}.\end{aligned}\tag{12}$$

To estimate J_2:

$$\begin{aligned}J_2=&C|b(z)-b_{B(x,t)}|\int_{2t}^{\infty}s^{\alpha-n-1}\left(\int_{\{y\in\mathbb{R}^n:2t\le|x-y|\le s\}}|f(y)|\,dy\right)ds\\ &\le C|B(x,t)|^{-1}\int_{B(x,t)}|b(z)-b(y)|\,dy\int_{t}^{\infty}s^{\alpha-n}\|\omega^{-1}\|_{L_{p'}(B(x,s))}\|f\|_{L_{p,\omega}(B(x,s))}\frac{ds}{s}\end{aligned}$$

$$\leq CM_b\chi_{B(x,t)}(z)\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\,\|\omega_2\|^{-1}_{L_q(B(x,s))}\|f\|_{L_{p,\omega}(B(x,s))}\frac{ds}{s}, \tag{13}$$

where C does not depend on x,t. Then by Theorem 3.8 and (12), (13) we have

$$\begin{aligned}
&\||b,I^\alpha|f_2\|_{L_{q,\omega_2}(B(x,t))} \leq \|J_1\|_{L_{q,\omega_2}(B(x,t))} + \|J_2\|_{L_{q,\omega_2}(B(x,t))}\\
\leq& C\|b\|_{BMO}\|\chi_{B(x,t)}\|_{L_{q,\omega_2}}\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\,\|\omega_2\|^{-1}_{L_q(B(x,s))}\|f\|_{L_{p,\omega_1}(B(x,s))}\frac{ds}{s}\\
&+C\|M_b\chi_{B(x,t)}\|_{L_{q,\omega_2}}\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\,\|\omega_2\|^{-1}_{L_q(B(x,s))}\|f\|_{L_{p,\omega_1}(B(x,s))}\frac{ds}{s}\\
\leq& C\|b\|_{BMO}\|\omega_2\|_{L_q(B(x,t))}\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\,\|\omega_2\|^{-1}_{L_q(B(x,s))}\|f\|_{L_{p,\omega_1}(B(x,s))}\frac{ds}{s}\\
&+C\|b\|_{BMO}\|\omega_2\|_{L_q(B(x,t))}\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\,\|\omega_2\|^{-1}_{L_q(B(x,s))}\|f\|_{L_{p,\omega_1}(B(x,s))}\frac{ds}{s}\\
\leq& C\|b\|_{BMO}\|\omega_2\|_{L_q(B(x,t))}\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\,\|\omega_2\|^{-1}_{L_q(B(x,s))}\|f\|_{L_{p,\omega_1}(B(x,s))}\frac{ds}{s}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\||b,I^\alpha|f_2\|_{L_{q,\omega_2}(B(x,t))} \leq\\
&C\|b\|_{BMO}\|\omega_2\|_{L_q(B(x,t))}\times\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\,\|\omega_2\|^{-1}_{L_q(B(x,s))}\|f\|_{L_{p,\omega_1}(B(x,s))}\frac{ds}{s},
\end{aligned}$$

which together with (11) yields (10). □

In the following two theorems we prove the boundedness of commutators of the Riesz potential operator $|b,I^\alpha|$ from the generalized weighted Morrey spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the generalized weighted Morrey spaces $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$. We find conditions on the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ for the boundedness of $|b,I^\alpha|$ from $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

Theorem 3.12. *Let $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$, $b\in BMO(\mathbb{R}^n)$, $(\omega_1,\omega_2)\in A_{p,q}(\mathbb{R}^n)$, $\omega_1\in A_p(\mathbb{R}^n)$, $\omega_2\in A_q(\mathbb{R}^n)$ and the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ fulfill the condition*

$$\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\frac{\operatorname*{ess\,inf}\limits_{s<t<1}\varphi_1(x,r)\|\omega_1\|_{L_p(B(x,r))}}{\|\omega_2\|_{L_q(B(x,s))}}\frac{ds}{s}\leq C\varphi_2(x,t). \tag{14}$$

Then the operator $|b,I^\alpha|$ is bounded from $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

Proof. Let $f\in\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$. From the definition of norm of generalized weighted Morrey spaces we write

$$\||b,I^\alpha|f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,\,r>0}\frac{1}{\varphi_2(x,t)\|\omega_2\|_{L_q(B(x,t))}}\||b,I^\alpha|f\chi_{B(x,r)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}. \tag{15}$$

We estimate $\||b,I^\alpha|f\chi_{B(x,r)}\|_{L_{q,\omega_2}(\mathbb{R}^n)}$ in (15) by means of Theorems 3.11, 2.11 and obtain

$\||b,I^\alpha|f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)}$

$$\leq C\,\|b\|_{BMO} \sup_{x\in\mathbb{R}^n,\, t>0} \frac{\|\omega_2\|_{L_q(B(x,t))}}{\varphi_2(x,t)\|\omega_2\|_{L_q(B(x,t))}} \int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\frac{\|f\|_{L_{p,\omega}(B(x,s))}}{\|\omega_2\|_{L_{p'}(B(x,s))}}\frac{ds}{s}$$

$$\leq C\,\|b\|_{BMO} \sup_{x\in\mathbb{R}^n,\, t>0} \frac{1}{\varphi_1(x,t)\|\omega_1\|_{L_p(B(x,t))}}\|f\|_{L_{p,\omega_1}(B(x,t))} = C\,\|b\|_{BMO}\,\|f\|_{\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)}.$$

It remains to make use of condition (14). □

4. Some applications

In this section we give some applications of our main results. We apply the theorems of Section 3 to the operators which are estimated from above by Riesz potentials. Now we give some examples.

Let $0<\alpha<n$. The fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2}f(x)=\frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-tL}f(x)\frac{dt}{t^{-\alpha/2+1}},$$

where L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying the Gaussian upper bound, that is,

$$|p_t(x,y)|\leq \frac{c_1}{t^{n/2}}e^{-c_2\frac{|x-y|^2}{t}} \tag{16}$$

for $x,y\in\mathbb{R}^n$ and all $t>0$, c_1, $c_2>0$ are independent of x, y and t.

If $L=-\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I^α (see [27]).

The following theorem states the boundedness of the operator $L^{-\alpha/2}$ from the spaces $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to the spaces $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.

Theorem 4.1. *Let $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$, $(\omega_1,\omega_2)\in A_{p,q}(\mathbb{R}^n)$. Let also (φ_1,φ_2) satisfy the condition (6). Then the operator $L^{-\alpha/2}$ is bounded from $\mathcal{M}^{p,\varphi_1}_{\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}_{\omega_2}(\mathbb{R}^n)$.*

Proof. From the condition (16), it follows that $|L^{-\alpha/2}f(x)|\leq CI^\alpha|f|(x)$ for all $x\in\mathbb{R}^n$ (see [7]).

Since the semigroup e^{-tL} has the kernel $p_t(x,y)$, from the above mentioned theorems we have $\|L^{-\alpha/2}f\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}}\leq C\|I^\alpha|f|\|_{\mathcal{M}^{q,\varphi_2}_{\omega_2}}\leq C\|f\|_{\mathcal{M}^{p,\varphi_1}_{\omega_1}}$, where the constant $C>0$ is independent of f. □

Various classes of differential operators also satisfy the inequality (16). Two of these operators are considered here:

(i) First we consider the magnetic potential $\vec{a}$, i.e., a real-valued vector potential $\vec{a}=(a_1,a_2,\dots,a_n)$, and an electric potential V. Let us assume that for any $k=1,2,\dots,n$, $a_k\in L_2^{loc}$ and $0\leq V\in L_1^{loc}$. The operator L, which is given by $L=-(\nabla-i\vec{a})^2+V(x)$ is called the magnetic Schrödinger operator.

From the well-known diamagnetic inequality (see [26]) we have the following pointwise estimate. For any $t>0$ and $f\in L_2$, the following inequality $|e^{-tL}f|\leq e^{-t\triangle}|f|$

holds, which illustrates that the semigroup e^{-tL} has the kernel $p_t(x,y)$ that satisfies upper bound (16).

Furthermore, note that under the appropriate assumptions (see [2, 19, 27]) we can obtain similar results with Theorem 4.1 for a homogeneous elliptic operator L in L_2 of order $2m$ in the divergence form $Lf = (-1)^m \sum\limits_{|\alpha|=|\beta|=m} D^\alpha \left(a_{\alpha\beta} D^\beta f\right)$. In this case estimate (16) should be replaced by $|p_t(x,y)| \leq \frac{c_3}{t^{n/2m}} e^{-c_4 \left(\frac{|x-y|}{t^{1/(2m)}}\right)^{2m/(2m-1)}}$ for all $t > 0$ and all $x,\, y \in \mathbb{R}^n$.

(ii) Now let $A = (a_{ij}(x))_{1 \leq i,j \leq n}$ be an $n \times n$ matrix with complex-valued entries $a_{ij} \in L_\infty$ satisfying $\mathrm{Re} \sum\limits_{i,j=1}^n a_{ij}(x)\zeta_i\zeta_j \geq \lambda|\zeta|^2$, for all $x \in \mathbb{R}^n, \zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{C}^n$ and some $\lambda > 0$. Consider the divergence form operator $Lf \equiv -\mathrm{div}(A\nabla f)$, which is interpreted in the usual weak sense via the appropriate sesquilinear form.

It is well known that the Gaussian bound (16) for the kernel of e^{-tL} holds when A has real-valued entries (see, for example, [1]), or when $n = 1,\, 2$ in the case of complex-valued entries (see [2, Chapter 1]). Therefore we can obtain similar results with Theorem 4.1.

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