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REFINING NUMERICAL RADIUS INEQUALITIES OF HILBERT SPACE OPERATORS

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Abstract. Several upper estimates for the numerical radius of Hilbert space operators are given. Among many other inequalities, it is shown that

$$\omega^2(A) \le \frac{1}{4}\left\||A|^2+|A^*|^2\right\| + \frac{1}{2}\omega\left(A^2\right) - \frac{1}{2}\inf_{\substack{x\in\mathscr{H}\\ \|x\|=1}}\left(\sqrt{\langle |A|^2x,x\rangle} - \sqrt{\langle |A^*|^2x,x\rangle}\right)^2.$$

1. Introduction

Let $(\mathscr{H};\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The numerical range of an operator A is the subset of the complex numbers $\mathbb{C}$ given by $W(A)=\{\langle Ax,x\rangle:\ x\in\mathscr{H}, \|x\|=1\}$. The numerical radius $\omega(A)$ of an operator A on $\mathscr{H}$ is given by $\omega(A)=\{|\langle Ax,x\rangle|:\ x\in\mathscr{H},\ \|x\|=1\}$. It is well known that $\omega(\cdot)$ is a norm on the Banach algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators $A:\mathscr{H}\to\mathscr{H}$. This norm is equivalent with the usual operator norm $\|A\|=\sup_{\|x\|=1,\ x\in\mathscr{H}}\|Ax\|$. In fact, the following more precise result holds: $\frac{1}{2}\|A\|\le\omega(A)\le\|A\|$.

Kittaneh has shown in [7], that if $A\in\mathscr{B}(\mathscr{H})$, $\omega(A)\le\frac{1}{2}\||A|+|A^*|\|$, where $|A|=(A^*A)^{1/2}$. In the same paper, and by using a refinement of triangle inequality for positive operators, namely,

$$\|A+B\|\le\frac{1}{2}\left(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^2}\right)$$

it has been shown that

$$\omega(A)\le\frac{1}{2}\left(\|A\|+\left\|A^2\right\|^{\frac{1}{2}}\right). \tag{1}$$

For an operator A, let $A=U|A|$ be the polar decomposition of A, where U is a partial isometry such that $\ker U=\ker A$. The Aluthge transform of A, denoted by $\widetilde{A}$,

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is defined as $\widetilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}}$. The generalized Aluthge transform, denoted by $\widetilde{A}_t$, is defined as $\widetilde{A}_t = |A|^t U |A|^{1-t}$, $0 \le t \le 1$. In particular, $\widetilde{A}_0 = U^* U U \, |A| = U \, |A| = A$, $\widetilde{A}_1 = |A| \, U U^* U = |A| \, U$, and $\widetilde{A}_{1/2} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}} = \widetilde{A}$. Here $|A|^0$ is defined as $U^* U$.

In [12], Yamazaki proved that $\omega(A) \le \frac{1}{2}\left(\|A\| + \omega\left(\widetilde{A}\right)\right)$. In fact, this is a refinement of the inequality (1).

Concerning the product of two operators, Dragomir [4] (see also [8, (17)]) has shown the following estimate of $\omega(B^* A)$,

$$\omega(B^* A) \le \frac{1}{2} \left\| |A|^2 + |B|^2 \right\|. \tag{2}$$

For some recent and interesting results concerning inequalities for the numerical radius, see [6, 9, 10].

2. Results

We start our work with the following result.

Theorem 2.1. *Let $A \in \mathscr{B}(\mathscr{H})$ and $0 \le t \le 1$. Then*

$$\omega(A) \le \frac{1}{4} \left\| |A|^{2t} + |A^*|^{2(1-t)} + A + A^* \right\| + \frac{1}{4} \left\| |A|^{2t} + |A^*|^{2(1-t)} - (A + A^*) \right\|.$$

Proof. We have

$$\begin{aligned}
\omega(B^* A) &\le \frac{1}{2} \left\| |A|^2 + |B|^2 \right\| \qquad \text{(by (2))} \\
&= \frac{1}{4} \left\| |A+B|^2 + |A-B|^2 \right\| \quad \text{(by the operator parallelogram law)} \\
&\le \frac{1}{4} \left\| |A+B|^2 \right\| + \frac{1}{4} \left\| |A-B|^2 \right\| \quad \text{(by the triangle inequality)} \\
&= \frac{1}{4} \left\| |A|^2 + |B|^2 + A^* B + B^* A \right\| + \frac{1}{4} \left\| |A|^2 + |B|^2 - A^* B - B^* A \right\|.
\end{aligned}$$

Namely,

$$\omega(B^* A) \le \frac{1}{4} \left\| |A|^2 + |B|^2 + A^* B + B^* A \right\| + \frac{1}{4} \left\| |A|^2 + |B|^2 - (A^* B + B^* A) \right\|. \tag{3}$$

Let $A = U \, |A|$ be the polar decomposition of A. By letting $A = |A|^t$ and $B = |A|^{1-t} U^*$, in (3), and by taking into account that

$$|B|^2 = B^* B = U |A|^{1-t} |A|^{1-t} U^* = U |A|^{2(1-t)} U^* = |A^*|^{2(1-t)} \quad \text{(by [2, (1)])}$$

we get

$$\omega(A) \le \frac{1}{4} \left\| |A|^{2t} + |A^*|^{2(1-t)} + A + A^* \right\| + \frac{1}{4} \left\| |A|^{2t} + |A^*|^{2(1-t)} - (A + A^*) \right\|. \qquad \square$$

We continue this section by establishing another upper estimate.

Theorem 2.2. *Let $A \in \mathscr{B}(\mathscr{H})$ and $0 \le t \le 1$. Then for any mean σ,*

$$\omega(A) \le \frac{1}{2} \left(\left\| |A|^{2(1-t)} + |A^*|^{2t} \right\| \sigma \left\| |A|^{2t} + |A^*|^{2(1-t)} \right\| \right).$$

Proof. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\langle Ax,x\rangle| &= |\langle U\,|A|\,x,x\rangle| = \left|\left\langle U|A|^{1-t}|A|^{t}x,x\right\rangle\right| \\ &= \left|\left\langle |A|^{t}x,|A|^{1-t}U^{*}x\right\rangle\right| \le \left\||A|^{t}x\right\| \left\||A|^{1-t}U^{*}x\right\|,\end{aligned}$$

and

$$\begin{aligned}|\langle Ax,x\rangle| &= |\langle U\,|A|\,x,x\rangle| = \left|\left\langle U|A|^{t}|A|^{1-t}x,x\right\rangle\right| \\ &= \left|\left\langle |A|^{1-t}x,|A|^{t}U^{*}x\right\rangle\right| \le \left\||A|^{1-t}x\right\| \left\||A|^{t}U^{*}x\right\|.\end{aligned}$$

It follows from the first relation in the above that

$$\begin{aligned}|\langle Ax,x\rangle| &\le \left\||A|^{t}x\right\| \left\||A|^{1-t}U^{*}x\right\| = \sqrt{\left\langle |A|^{t}x,|A|^{t}x\right\rangle \left\langle |A|^{1-t}U^{*}x,|A|^{1-t}U^{*}x\right\rangle} \\ &= \sqrt{\left\langle |A|^{2t}x,x\right\rangle \left\langle U|A|^{2(1-t)}U^{*}x,x\right\rangle} = \sqrt{\left\langle |A|^{2t}x,x\right\rangle \left\langle |A^{*}|^{2(1-t)}x,x\right\rangle} \quad \text{(by [5, (1)])} \\ &\le \frac{1}{2}\left(\left\langle |A|^{2t}x,x\right\rangle + \left\langle |A^{*}|^{2(1-t)}x,x\right\rangle\right) \qquad \text{(by arithmetic-geometric mean inequality)} \\ &= \frac{1}{2}\left\langle \left(|A|^{2t}+|A^{*}|^{2(1-t)}\right)x,x\right\rangle = \frac{1}{2}\left\||A|^{2t}+|A^{*}|^{2(1-t)}\right\|.\end{aligned}$$

Similarly, we can obtain $|\langle Ax,x\rangle| \le \frac{1}{2}\left\||A|^{2(1-t)}+|A^{*}|^{2t}\right\|$.

Now, by combining the inequalities 1 and 2 and employing the monotonicity property of the mean for positive numbers, we get,

$$\begin{aligned}|\langle Ax,x\rangle| &= |\langle Ax,x\rangle|\,\sigma\,|\langle Ax,x\rangle| \le \frac{1}{2}\left\||A|^{2(1-t)}+|A^{*}|^{2t}\right\|\sigma\frac{1}{2}\left\||A|^{2t}+|A^{*}|^{2(1-t)}\right\| \\ &= \frac{1}{2}\left(\left\||A|^{2(1-t)}+|A^{*}|^{2t}\right\|\sigma\left\||A|^{2t}+|A^{*}|^{2(1-t)}\right\|\right) \qquad \text{(by the homogeneity of } \sigma).\end{aligned}$$

The result follows by taking supremum over all unit vectors $x \in \mathscr{H}$. $\square$

It should be mentioned here that there is no ordering between $\left\||A|^{2(1-t)}+|A^{*}|^{2t}\right\|$ and $\left\||A^{*}|^{2(1-t)}+|A|^{2t}\right\|$, in general. The following example clarifies this statement.

Example 2.3. Let $A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix}$.

If $t = 0.3$, then
$$\left\||A|^{2(1-t)}+|A^{*}|^{2t}\right\| \approx 8.48,$$
$$\left\||A^{*}|^{2(1-t)}+|A|^{2t}\right\| \approx 8.89.$$

If $t = 0.6$, then
$$\left\||A|^{2(1-t)}+|A^{*}|^{2t}\right\| \approx 7.68,$$
$$\left\||A^{*}|^{2(1-t)}+|A|^{2t}\right\| \approx 7.01.$$

The next theorem provides an extension for the celebrated inequality $\omega^{2}(A) \le \frac{1}{2}\left\||A|^{2}+|A^{*}|^{2}\right\|$ (see [8, Theorem 1]).

Theorem 2.4. *Let $A \in \mathscr{B}(\mathscr{H})$ and $0 \le t \le 1$. Then*

$$\omega^2(A) \le \int_0^1 \left\| t|A|^2 + (1-t)|A^*|^2 \right\|^{1-v} \left\| (1-t)|A|^2 + t|A^*|^2 \right\|^v dv.$$

Proof. Applying the same procedure as in the proof of Theorem 2.2, one can write

$$\begin{aligned}
|\langle Ax, x\rangle|^2 &\le \left\| |A|^t x \right\| \left\| |A|^{1-t} U^* x \right\| \left\| |A|^{1-t} x \right\| \left\| |A|^t U^* x \right\| \\
&= \sqrt{\left\langle |A|^{2t} x, x \right\rangle \left\langle |A|^{2(1-t)} x, x \right\rangle} \sqrt{\left\langle U|A|^{2(1-t)} U^* x, x \right\rangle \left\langle U|A|^{2t} U^* x, x \right\rangle} \\
&= \sqrt{\left\langle |A|^{2t} x, x \right\rangle \left\langle |A|^{2(1-t)} x, x \right\rangle} \sqrt{\left\langle |A^*|^{2(1-t)} x, x \right\rangle \left\langle |A^*|^{2t} x, x \right\rangle} \\
&= \sqrt{\left\langle |A|^{2t} x, x \right\rangle \left\langle |A^*|^{2(1-t)} x, x \right\rangle} \sqrt{\left\langle |A|^{2(1-t)} x, x \right\rangle \left\langle |A^*|^{2t} x, x \right\rangle} \\
&\le \int_0^1 \left(\left\langle |A|^{2t} x, x \right\rangle \left\langle |A^*|^{2(1-t)} x, x \right\rangle \right)^{1-v} \left(\left\langle |A|^{2(1-t)} x, x \right\rangle \left\langle |A^*|^{2t} x, x \right\rangle \right)^v dv
\end{aligned}$$

(by logarithmic-geometric mean inequality)

$$\le \int_0^1 \left(\left\langle |A|^2 x, x \right\rangle^t \left\langle |A^*|^2 x, x \right\rangle^{1-t} \right)^{1-v} \left(\left\langle |A|^2 x, x \right\rangle^{1-t} \left\langle |A^*|^2 x, x \right\rangle^t \right)^v dv$$

(by the Hölder-McCarthy inequality [11, Theorem 1.4])

$$\le \int_0^1 \left(t \left\langle |A|^2 x, x \right\rangle + (1-t) \left\langle |A^*|^2 x, x \right\rangle \right)^{1-v} \left((1-t) \left\langle |A|^2 x, x \right\rangle + t \left\langle |A^*|^2 x, x \right\rangle \right)^v dv$$

(by the weighted arithmetic-geometric mean inequality)

$$\begin{aligned}
&= \int_0^1 \left\langle \left(t|A|^2 + (1-t)|A^*|^2 \right) x, x \right\rangle^{1-v} \left\langle \left((1-t)|A|^2 + t|A^*|^2 \right) x, x \right\rangle^v dv \\
&\le \int_0^1 \left\| t|A|^2 + (1-t)|A^*|^2 \right\|^{1-v} \left\| (1-t)|A|^2 + t|A^*|^2 \right\|^v dv.
\end{aligned}$$

Thus, $$|\langle Ax, x\rangle|^2 \le \int_0^1 \left\| t|A|^2 + (1-t)|A^*|^2 \right\|^{1-v} \left\| (1-t)|A|^2 + t|A^*|^2 \right\|^v dv.$$

Whence, $$\omega^2(A) \le \int_0^1 \left\| t|A|^2 + (1-t)|A^*|^2 \right\|^{1-v} \left\| (1-t)|A|^2 + t|A^*|^2 \right\|^v dv,$$

as required. □

The next result provides a refinement of the well-known inequality $\omega^2(A) \le \frac{1}{4}\left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{2}\omega(A^2)$ (see [1, Theorem 2.4]). Notice that our method is different from [1].

Theorem 2.5. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega^2(A) \le \frac{1}{4}\left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{2}\omega(A^2) - \frac{1}{2} \inf_{\substack{x \in \mathscr{H} \\ \|x\|=1}} \left(\sqrt{\left\langle |A|^2 x, x \right\rangle} - \sqrt{\left\langle |A^*|^2 x, x \right\rangle} \right)^2.$$

Proof. Buzano's inequality [3] asserts that $|\langle z,x\rangle|\,|\langle z,y\rangle| \le \frac{\|z\|^2}{2}\left(|\langle x,y\rangle| + \|x\|\,\|y\|\right)$ for any $x,y,z\in\mathscr{H}$. Put $x=Ax$, $y=A^*x$, and $z=x$ with $\|x\|=1$, then

$$\begin{aligned}
&|\langle Ax,x\rangle|^2 \le \frac{1}{2}\left(\sqrt{\left\langle |A|^2x,x\right\rangle\left\langle |A^*|^2x,x\right\rangle} + \left|\left\langle A^2x,x\right\rangle\right|\right)\\
=&\frac{1}{2}\left(\sqrt{\langle A^*Ax,x\rangle\,\langle AA^*,x\rangle} + \left|\left\langle A^2x,x\right\rangle\right|\right) = \frac{1}{2}\left(\sqrt{\langle Ax,Ax\rangle\,\langle A^*,A^*x\rangle} + \left|\left\langle A^2x,x\right\rangle\right|\right)\\
=&\frac{1}{2}\left(\|Ax\|\,\|A^*x\| + \left|\left\langle A^2x,x\right\rangle\right|\right)\\
=&\frac{1}{2}\left(\frac{1}{2}\left(\|Ax\|^2+\|A^*x\|^2 - (\|Ax\|-\|A^*x\|)^2\right) + \left|\left\langle A^2x,x\right\rangle\right|\right)\\
=&\frac{1}{2}\left(\frac{1}{2}\left(\left\langle |A|^2x,x\right\rangle + \left\langle |A^*|^2x,x\right\rangle - \left(\sqrt{\left\langle |A|^2x,x\right\rangle} - \sqrt{\left\langle |A^*|^2x,x\right\rangle}\right)^2\right) + \left|\left\langle A^2x,x\right\rangle\right|\right)\\
=&\frac{1}{2}\left(\frac{1}{2}\left(\left\langle \left(|A|^2+|A^*|^2\right)x,x\right\rangle - \left(\sqrt{\left\langle |A|^2x,x\right\rangle} - \sqrt{\left\langle |A^*|^2x,x\right\rangle}\right)^2\right) + \left|\left\langle A^2x,x\right\rangle\right|\right)\\
\le&\frac{1}{4}\left\||A|^2+|A^*|^2\right\| + \frac{1}{2}\omega\left(A^2\right) - \frac{1}{2}\inf_{\substack{x\in\mathscr{H}\\ \|x\|=1}}\left(\sqrt{\left\langle |A|^2x,x\right\rangle} - \sqrt{\left\langle |A^*|^2x,x\right\rangle}\right)^2.
\end{aligned}$$

This implies,

$$\omega^2(A) \le \frac{1}{4}\left\||A|^2+|A^*|^2\right\| + \frac{1}{2}\omega\left(A^2\right) - \frac{1}{2}\inf_{\substack{x\in\mathscr{H}\\ \|x\|=1}}\left(\sqrt{\left\langle |A|^2x,x\right\rangle} - \sqrt{\left\langle |A^*|^2x,x\right\rangle}\right)^2. \qquad \square$$

Before stating the next result, we recall the famous polarization identity, which says that

$$\langle x,y\rangle = \frac{1}{4}\sum_{k=0}^{3}\left\|x+i^ky\right\|^2 i^k \quad (x,y\in\mathscr{H}).$$

THEOREM 2.6. *Let $A\in\mathscr{B}(\mathscr{H})$ and $0\le t\le 1$. Then,*

$$\omega(A) \le \frac{1}{4}\sqrt{2\left\||A|^{4(1-t)}+|A^*|^{4t}\right\| + 4\omega^2(A) + 4\left\||A|^{2(1-t)}+|A^*|^{2t}\right\|\omega(A)}.$$

Proof. Let $A=U|A|$ be the polar decomposition of A. We have

$$\begin{aligned}
\operatorname{Re}\left\langle e^{i\theta}Ax,x\right\rangle &= \operatorname{Re}\left\langle e^{i\theta}U|A|x,x\right\rangle = \operatorname{Re}\left\langle e^{i\theta}U|A|^t|A|^{1-t}x,x\right\rangle = \operatorname{Re}\left\langle e^{i\theta}|A|^{1-t}x,|A|^tU^*x\right\rangle\\
&= \frac{1}{4}\left\|\left(e^{i\theta}|A|^{1-t}+|A|^tU^*\right)x\right\|^2 - \frac{1}{4}\left\|\left(e^{i\theta}|A|^{1-t}-|A|^tU^*\right)x\right\|^2\\
&\le \frac{1}{4}\left\|\left(e^{i\theta}|A|^{1-t}+|A|^tU^*\right)x\right\|^2 \le \frac{1}{4}\left\|e^{i\theta}|A|^{1-t}+|A|^tU^*\right\|^2\\
&= \frac{1}{4}\left\|\left(e^{i\theta}|A|^{1-t}+|A|^tU^*\right)^*\left(e^{i\theta}|A|^{1-t}+|A|^tU^*\right)\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left\| |A|^{2(1-t)} + U|A|^{2t}U^* + 2\operatorname{Re}\left(e^{i\theta}U\,|A|\right)\right\| \\
&= \frac{1}{4}\left\| |A|^{2(1-t)} + |A^*|^{2t} + 2\operatorname{Re}\left(e^{i\theta}U\,|A|\right)\right\| \\
&= \frac{1}{4}\left\| |A|^{2(1-t)} + |A^*|^{2t} + 2\operatorname{Re}\left(e^{i\theta}A\right)\right\| \\
&= \frac{1}{4}\left\|\left(\left(|A|^{2(1-t)} + |A^*|^{2t}\right) + \left(2\operatorname{Re}\left(e^{i\theta}A\right)\right)\right)^2\right\|^{\frac{1}{2}}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\left\|\left(\left(|A|^{2(1-t)} + |A^*|^{2t}\right) + \left(2\operatorname{Re}\left(e^{i\theta}A\right)\right)\right)^2\right\| \\
&= \Big\|\left(|A|^{2(1-t)} + |A^*|^{2t}\right)^2 + 4\left(\operatorname{Re}\left(e^{i\theta}A\right)\right)^2 \\
&\quad + \left(|A|^{2(1-t)} + |A^*|^{2t}\right)\left(2\operatorname{Re}\left(e^{i\theta}A\right)\right) + \left(2\operatorname{Re}\left(e^{i\theta}A\right)\right)\left(|A|^{2(1-t)} + |A^*|^{2t}\right)\Big\| \\
&\le \left\|\left(|A|^{2(1-t)} + |A^*|^{2t}\right)^2\right\| + 4\left\|\left(\operatorname{Re}\left(e^{i\theta}A\right)\right)^2\right\| \\
&\quad + 2\left\|\left(|A|^{2(1-t)} + |A^*|^{2t}\right)\left(\operatorname{Re}\left(e^{i\theta}A\right)\right)\right\| + 2\left\|\left(\operatorname{Re}\left(e^{i\theta}A\right)\right)\left(|A|^{2(1-t)} + |A^*|^{2t}\right)\right\| \\
&\le \left\|\left(|A|^{2(1-t)} + |A^*|^{2t}\right)^2\right\| + 4\left\|\left(\operatorname{Re}\left(e^{i\theta}A\right)\right)^2\right\| + 4\left\||A|^{2(1-t)} + |A^*|^{2t}\right\|\left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\| \\
&= \left\|\left(\frac{2|A|^{2(1-t)} + 2|A^*|^{2t}}{2}\right)^2\right\| + 4\left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\|^2 + 4\left\||A|^{2(1-t)} + |A^*|^{2t}\right\|\left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\| \\
&\le 2\left\||A|^{4(1-t)} + |A^*|^{4t}\right\| + 4\left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\|^2 + 4\left\||A|^{2(1-t)} + |A^*|^{2t}\right\|\left\|\operatorname{Re}\left(e^{i\theta}A\right)\right\| \\
&\le 2\left\||A|^{4(1-t)} + |A^*|^{4t}\right\| + 4\omega^2(A) + 4\left\||A|^{2(1-t)} + |A^*|^{2t}\right\|\omega(A).
\end{aligned}
$$

Thus,

$$
\omega(A) \le \frac{1}{4}\sqrt{2\left\||A|^{4(1-t)} + |A^*|^{4t}\right\| + 4\omega^2(A) + 4\left\||A|^{2(1-t)} + |A^*|^{2t}\right\|\omega(A)}. \qquad \square
$$

Remark 2.7. Letting $t = \frac{1}{2}$, we get

$$
\omega(A) \le \frac{1}{4}\sqrt{2\left\||A|^2 + |A^*|^2\right\| + 4\omega^2(A) + 4\left\||A| + |A^*|\right\|\omega(A)}.
$$

Theorem 2.8. *Let A be a non-zero operator on $\mathscr{B}(\mathscr{H})$. Then*

$$
\omega(A) \le \frac{1}{2}\left\|\widetilde{A}_t\right\|\left\||A|^{2(1-t)} + |A|^{2(t-1)}\right\|.
$$

Proof. Let $x \in \mathscr{H}$ be a unit vector. Utilizing the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we can write

$$
|\langle Ax, x\rangle| = |\langle U\,|A|\,x, x\rangle| = \left|\left\langle |A|^{t-1}|A|^{1-t}U|A|^t|A|^{1-t}x, x\right\rangle\right|
$$

$$\begin{aligned}&= \left|\left\langle |A|^{t-1}\widetilde{A}_t|A|^{1-t}x, x\right\rangle\right| = \left|\left\langle \widetilde{A}_t|A|^{1-t}x, |A|^{t-1}x\right\rangle\right| \\ &\leq \left\|\widetilde{A}_t\right\| \sqrt{\left\langle |A|^{2(1-t)}x, x\right\rangle \left\langle |A|^{2(t-1)}x, x\right\rangle} \\ &\leq \frac{1}{2}\left\|\widetilde{A}_t\right\| \left(\left\langle |A|^{2(1-t)}x, x\right\rangle + \left\langle |A|^{2(t-1)}x, x\right\rangle\right) \\ &= \frac{1}{2}\left\|\widetilde{A}_t\right\| \left\langle \left(|A|^{2(1-t)} + |A|^{2(t-1)}\right)x, x\right\rangle \leq \frac{1}{2}\left\|\widetilde{A}_t\right\| \left\||A|^{2(1-t)} + |A|^{2(t-1)}\right\|.\end{aligned}$$

Thus,

$$\omega(A) \leq \frac{1}{2}\left\|\widetilde{A}_t\right\| \left\||A|^{2(1-t)} + |A|^{2(t-1)}\right\|. \qquad \square$$

In the next result, we give a lower bound for $\|A + A^*\|$. Let $m(A)$ be the nonnegative number defined by $m(A) = \inf_{\|x\|=1,\, x\in\mathscr{H}} |\langle Ax, x\rangle|$.

THEOREM 2.9. *Let $A \in \mathscr{B}(\mathscr{H})$ and $0 \leq t \leq 1$. Then*

$$\sqrt{\left\||A|^2 + |A^*|^2\right\| + m\left(A^2 + (A^*)^2\right)} \leq \|A + A^*\|.$$

Proof. We have

$$\begin{aligned}\|A + A^*\| &= \sqrt{\left\|(A + A^*)^2\right\|} = \sqrt{\left\|A^2 + |A^*|^2 + |A|^2 + (A^*)^2\right\|} \\ &\geq \sqrt{\left|\left\langle \left(A^2 + |A^*|^2 + |A|^2 + (A^*)^2\right)x, x\right\rangle\right|} \\ &= \sqrt{\left|\left\langle \left(|A|^2 + |A^*|^2\right)x, x\right\rangle + \left\langle \left(A^2 + (A^*)^2\right)x, x\right\rangle\right|} \\ &= \sqrt{\left|\left\langle \left(|A|^2 + |A^*|^2\right)x, x\right\rangle\right| + \left|\left\langle \left(A^2 + (A^*)^2\right)x, x\right\rangle\right|} \\ &= \sqrt{\left|\left\langle \left(|A|^2 + |A^*|^2\right)x, x\right\rangle\right| + m\left(A^2 + (A^*)^2\right)}\end{aligned}$$

Thus,

$$\sqrt{\left|\left\langle \left(|A|^2 + |A^*|^2\right)x, x\right\rangle\right| + m\left(A^2 + (A^*)^2\right)} \leq \|A + A^*\|.$$

By taking supremum over all unit vectors $x \in \mathscr{H}$,

$$\sqrt{\left\||A|^2 + |A^*|^2\right\| + m\left(A^2 + (A^*)^2\right)} \leq \|A + A^*\|. \qquad \square$$

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