MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
75, 1 (2023), 58–70
March 2023

research paper
оригинални научни рад
DOI: 10.57016/MV-XFOQ5120

SHARP ASYMPTOTIC ANALYSIS OF POSITIVE SOLUTIONS OF A COMBINED STURM-LIOUVILLE PROBLEM

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Abstract. In this work, we investigate a class of nonlinear combined Sturm-Liouville problems with zero Dirichlet boundary conditions. Using the Karamata regular variation theory and the Schauder fixed point theorem, we prove the existence of a unique positive solution satisfying a precise asymptotic behavior where a competition between singular and non singular terms in the nonlinearity appears.

1. Introduction

The study of behavioral properties of solutions of differential equations is of huge importance and it continues to attract many scholar's attention. In this paper, we will present some recent contributions to the asymptotic analysis of positive solutions of the following combined Sturm-Liouville problem:

$$\begin{cases} -\frac{1}{A}(Au')' = pu^r + qu^s, \ \text{ in } \ (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $r, s \in (-1, 1)$, p, q are nonnegative functions in $C^{\gamma}_{loc}((0, 1))$, $0 < \gamma < 1$, and A verifies the following hypothesis:
(H_0) A is a differentiable positive function in $(0, 1)$, satisfying $A(t) \approx t^{\alpha_1}(1-t)^{\alpha_2}$, $t \in (0, 1)$, with $\alpha_1, \alpha_2 < 1$.
Here and throughout the paper, for two nonnegative functions g and h defined on a set D, the notation $g(x) \approx h(x)$, $x \in D$, means that there exists $c > 0$ such that for every $x \in D$, $\frac{1}{c}\, h(x) \leqslant g(x) \leqslant c\, h(x)$. We note that the function A may be singular at $t = 0$ and/or $t = 1$ and the function $\frac{1}{A}$ is integrable on $(0, 1)$. Without loss of generality, we may suppose that $\int_0^1 \frac{d\xi}{A(\xi)} = 1$.

Our main purpose is to investigate the problem (1). Under suitable hypotheses on p and q, we prove the existence of a unique positive classical solution to (1) which

2020 Mathematics Subject Classification: 26A12, 34B16, 34B18, 34B27
Keywords and phrases: Asymptotic analysis; Sturm-Liouville equation; Dirichlet problem; Green function; Karamata class; Schäuder's fixed point theorem.

satisfies an asymptotic behavior similar to that of the nonlinearities p and q. For convenience, let us fix some notations and give some assumptions.

We denote by $B^+((0,1))$ the set of nonnegative measurable functions. We recall the definition of the potential kernel V defined on $B^+((0,1))$ by:

$$Vf(x) = \int_0^1 G(t,x)f(t)\,dt,\ x \in (0,1),$$

where G denotes the Green function of the Sturm-Liouville operator $u \longmapsto -\frac{1}{A}(Au')'$ on $(0,1)$ with Dirichlet boundary conditions $u(0) = u(1) = 0$, given by

$$G(t,x) = A(t)\left(\int_0^{\min(t,x)} \frac{dr}{A(r)}\right)\left(\int_{\max(t,x)}^1 \frac{dr}{A(r)}\right), \quad (t,x) \in (0,1)\times(0,1). \tag{2}$$

Further, if A satisfies (H_0), then we obtain that

$$G(t,x) \approx t^{\alpha_1}(1-t)^{\alpha_2}(\min(t,x))^{1-\alpha_1}(1-\max(t,x))^{1-\alpha_2},\ (t,x) \in (0,1)\times(0,1). \tag{3}$$

We refer to $C_0([0,1])$ as the collection of all maps h in $C([0,1])$ satisfying $h(0) = h(1) = 0$.

Remark 1.1. If $f \in B^+((0,1))$ satisfies $\int_0^1 t(1-t)f(t)\,dt < \infty$, then by (2) we have $Vf \in C_0([0,1])$.

Throughout this paper, $\mathcal{K}$ denotes the collection of Karamata functions L defined by

$$L(x) := c\exp\left(\int_x^\eta \frac{z(t)}{t}\,dt\right),\ x \in (0,\eta],$$

for some $\eta > 0$, $c > 0$ and $z \in C([0,\eta])$ satisfying $z(0) = 0$.

Here, we have to mention that the functions in the class $\mathcal{K}$ are slowly varying. In [13, 14], Karamata improved the initial theory in this field. These functions were first used in the generalization of the Abelian and Tauberian theorems, as well as in the theory of trigonometric series [11, 15]. We also point out that Cirstea and Radulescu were the first who exploited the Karamata regular variation theory to study the asymptotic and qualitative behavior near the boundary of positive solutions of nonlinear differential problems [7–10].

Our motivation in this work are recent advances in the study of nonlinear problems including both singular and non singular nonlinearities which have wide applications to physical models. Indeed, the study of combined problems, involving different differential operators in both bounded and unbounded domains subject to different boundary conditions, has received a lot of interest and lots of excellent results have been obtained; see for example [1–3, 5, 6, 20].

In 2012, by using Karamata regular variation theory and by means of the sub-super solution method, Chammam et al. [6] established the existence of a positive continuous solution for

$$\begin{cases} -\Delta u = pu^r + qu^s, & \text{in } D,\\ u = 0, & \text{on } \partial D,\end{cases}$$

where D is a bounded $C^{1,1}$ domain, $r, s < 1$ and p, q are nonnegative functions in $C^\gamma_{loc}(D)$, $0 < \gamma < 1$, satisfying some suitable assumptions related to Karamata theory.

Most recently, Bachar and Mâagli proved in [1] the existence of a unique positive continuous solution of the boundary value problem

$$\begin{cases} -\frac{1}{A}(Au')' = pu^r + qu^s, & \text{in } (0,\infty), \\ \lim\limits_{t\to 0^+} u(t) = 0, \lim\limits_{t\to\infty} \frac{u(t)}{\rho(t)} = 0, \end{cases}$$

where $r, s < 1$, A is a continuous function on $[0,\infty)$ which is positive differentiable on $(0,\infty)$, such that $\int_0^1 \frac{d\xi}{A(\xi)} < \infty$ and $\int_0^\infty \frac{d\xi}{A(\xi)} = \infty$ with $\rho(t) = \int_0^t \frac{d\xi}{A(\xi)}$, $t \geqslant 0$. The functions p, q are nonnegative and may be singular at $t = 0$.

For the special case $p \equiv 0$ and $A \equiv 1$, problem (1) is reduced to the following

$$\begin{cases} -u'' = qu^s, & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

Problems of the form (4) can properly describe many phenomena in non-Newtonian fluid theory, such as boundary layer theory [4] and the transport of coal slurries down conveyor belts [17].

Taliaferro showed in [19], that the singular boundary value problem (4) has a solution in $C([0,1]) \cap C^1((0,1))$ when $s < 0$, q is a nonnegative function in $C((0,1))$ satisfying $\int_0^1 t(1-t)q(t)\,dt < \infty$.

In recent paper [12], Dridi et al. considered problem (1) when $q > 0$ and $p \equiv 0$. More precisely, the authors dealt with the following semilinear problem

$$\begin{cases} -\frac{1}{A}(Au')' = qu^s, & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{5}$$

where $s < 1$, the weight A satisfies (H_0) and q satisfies the following assumption:
(H) $q \in C^{\gamma}_{loc}((0,1))$, $0 < \gamma < 1$, such that $q(t) \approx t^{-\gamma_1}(1-t)^{-\gamma_2} L_1(t) L_2(1-t)$, $t \in (0,1)$, for $i = 1,2$, $\gamma_i \leqslant 2$ and $L_i \in \mathcal{K}$ defined on $(0,\eta]$, $\eta > 1$ satisfying

$$\int_0^\eta \xi^{1-\gamma_i} L_i(\xi)\, d\xi < \infty.$$

By employing some potential theory tools and properties of functions in the class $\mathcal{K}$, the authors established the following.

Theorem 1.2. *Suppose that* (H_0)*-*(H) *hold. Then, there exists a unique classical solution u of* (5) *such that for $x \in (0,1)$,*

$$u(x) \approx x^{\min(1-\alpha_1, \frac{2-\gamma_1}{1-s})} (1-x)^{\min(1-\alpha_2, \frac{2-\gamma_2}{1-s})} \psi_1(x) \psi_2(1-x). \tag{6}$$

Here for $i = 1,2$, ψ_i is given on $(0,1)$ by:

$$\psi_i(t) := \begin{cases} 1, & \text{if } \gamma_i < \alpha_i + 1 + s(1-\alpha_i), \\ \left(\int_t^\eta \frac{L_i(\xi)}{\xi}\, d\xi\right)^{\frac{1}{1-s}}, & \text{if } \gamma_i = \alpha_i + 1 + s(1-\alpha_i), \\ (L_i(t))^{\frac{1}{1-s}}, & \text{if } \alpha_i + 1 + s(1-\alpha_i) < \gamma_i < 2, \\ \left(\int_0^t \frac{L_i(\xi)}{\xi}\, d\xi\right)^{\frac{1}{1-s}}, & \text{if } \gamma_i = 2. \end{cases}$$

In the present paper, we generalize the previous result to deal with problem (1).

Inspired by [12], we assume the following condition:
(H_1) Let $p, q \in C^{\gamma}_{loc}((0,1)), 0<\gamma<1$, satisfy for $x \in (0,1)$,

$$p(x) \approx x^{-\mu_1}(1-x)^{-\mu_2} M_1(x) M_2(1-x),$$
$$q(x) \approx x^{-\lambda_1}(1-x)^{-\lambda_2} N_1(x) N_2(1-x),$$

where for $i \in \{1,2\}$, $\mu_i, \lambda_i \leqslant 2$, and $M_i, N_i \in \mathcal{K}$ defined on $(0,\eta]$, for some $\eta > 1$, satisfy

$$\int_0^{\eta} \xi^{1-\mu_i} M_i(\xi)\, d\xi < \infty \text{ and } \int_0^{\eta} \xi^{1-\lambda_i} N_i(\xi)\, d\xi < \infty.$$

We note that the estimates (6) depend closely on $\min(1-\alpha_i, \frac{2-\gamma_i}{1-s})$, $i=1,2$. Also, as it will be seen, for $i=1,2$, the numbers $\nu_i = \min(1-\alpha_i, \frac{2-\mu_i}{1-r})$, $\xi_i = \min(1-\alpha_i, \frac{2-\lambda_i}{1-s})$ play an important role in the combined effects of singular and non singular nonlinearities in (1) and lead to a competition. Without loss of generality, we may assume that $\frac{2-\mu_1}{1-r} \leqslant \frac{2-\lambda_1}{1-s}$ and $\frac{2-\mu_2}{1-r} \leqslant \frac{2-\lambda_2}{1-s}$. We consider θ, the function defined on $(0,1)$, by

$$\theta(x) = x^{\nu_1} K_1(x)(1-x)^{\nu_2} K_2(1-x), \tag{7}$$

where, for $i \in \{1,2\}$,

$$K_i = \begin{cases} \tilde{M}_i, & \text{if } \nu_i < \xi_i, \\ \tilde{M}_i + \tilde{N}_i, & \text{if } \nu_i = \xi_i. \end{cases} \tag{8}$$

Here for $i=1,2$, $\tilde{M}_i$ and $\tilde{N}_i$ are respectively given on $(0,1)$ by:

$$\tilde{M}_i(x) := \begin{cases} 1, & \text{if } \mu_i < \alpha_i + 1 + r(1-\alpha_i), \\ \left(\int_x^{\eta} \frac{M_i(t)}{t}\, dt\right)^{\frac{1}{1-r}}, & \text{if } \mu_i = \alpha_i + 1 + r(1-\alpha_i), \\ (M_i(x))^{\frac{1}{1-r}}, & \text{if } \alpha_i + 1 + r(1-\alpha_i) < \mu_i < 2, \\ \left(\int_0^{x} \frac{M_i(t)}{t}\, dt\right)^{\frac{1}{1-r}}, & \text{if } \mu_i = 2, \end{cases}$$

and

$$\tilde{N}_i(x) := \begin{cases} 1, & \text{if } \lambda_i < \alpha_i + 1 + s(1-\alpha_i), \\ \left(\int_x^{\eta} \frac{N_i(t)}{t}\, dt\right)^{\frac{1}{1-s}}, & \text{if } \lambda_i = \alpha_i + 1 + s(1-\alpha_i), \\ (N_i(x))^{\frac{1}{1-s}}, & \text{if } \alpha_i + 1 + s(1-\alpha_i) < \lambda_i < 2, \\ \left(\int_0^{x} \frac{N_i(t)}{t}\, dt\right)^{\frac{1}{1-s}}, & \text{if } \lambda_i = 2. \end{cases}$$

Our main results are the following.

THEOREM 1.3. *Let $r, s \in (-1,1)$ and θ be the function given by (7). If (H_0)-(H_1) are satisfied. then*

$$V(p\,\theta^r + q\,\theta^s)(x) \approx \theta(x), \quad x \in\in (0,1). \tag{9}$$

THEOREM 1.4. *Let $r, s \in (-1,1)$ and θ be the function defined by (7). Assume that (H_0)-(H_1) are satisfied. Then problem (1) admits a unique positive classical solution u satisfying $u(x) \approx \theta(x)$, $x \in (0,1)$.*

The outline of the paper is as follows. In Section 2, we state some preliminaries involving some already known results on Karamata functions and some potential theory tools useful for our study. Sections 3 and 4 are respectively devoted to the proofs of Theorems 1.3 and 1.4. The last section is reserved to an example illustrating our Theorem 1.4.

2. Preliminary results

In what follows, we state some fundamental facts on the functions belonging to the Karamata class, which will be used in the proofs of our main results. For more details, please refer to [16, 18].

It is clear that a function $L \in \mathcal{K}$ if and only if there exists $\eta > 0$, such that L is a positive function in $C^1((0,\eta])$, satisfying

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0.$$

We give bellow a standard example of Karamata functions.

$$L(t) = \prod_{k=1}^{m} \left(\log_k(\frac{\omega}{t})\right)^{-\mu_k},$$

where $\log_k(x)$ denotes the k-th iteration of the logarithm, $m \in \mathbb{N}^*$, $(\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m$ and $\omega > 0$ sufficiently large such that L is defined and positive on $(0,\eta]$, for some $\eta > 1$.

According to the Karamata integration theorem, we obtain the following.

Lemma 2.1. *Let $\alpha \in \mathbb{R}$ and $L \in \mathcal{K}$ be defined on $(0,\eta]$, $\eta > 0$. Then*

1. *If $\alpha > -1$, then $\int_0^\eta t^\alpha L(t)\,dt$ converges and $\int_0^x t^\alpha L(t)\,dt \underset{x\to 0^+}{\sim} \frac{x^{1+\alpha}L(x)}{1+\alpha}$.*
2. *If $\alpha < -1$, then $\int_0^\eta t^\alpha L(t)\,dt$ diverges and $\int_x^\eta t^\alpha L(t)\,dt \underset{x\to 0^+}{\sim} -\frac{x^{1+\alpha}L(x)}{1+\alpha}$.*

Lemma 2.2. *(i) Let $L_1, L_2 \in \mathcal{K}$, and $p \in \mathbb{R}$. Then the functions L_1L_2, $L_1 + L_2$ and L_1^p belong to $\mathcal{K}$.*

(ii) Let $L \in \mathcal{K}$. Then, for any $\varepsilon > 0$, $\lim\limits_{x\to 0^+} x^\varepsilon L(x) = 0$.

(iii) Let $L \in \mathcal{K}$ be defined on $(0,\eta]$, $\eta > 0$. Then,

$$\lim_{x\to 0^+} \frac{L(x)}{\int_x^\eta \frac{L(\xi)}{\xi}\,d\xi} = 0.$$

In particular,

$$x \longmapsto \int_x^\eta \frac{L(\xi)}{\xi}\,d\xi \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(\xi)}{\xi}\,d\xi < \infty$, then we have

$$\lim_{x\to 0^+} \frac{L(x)}{\int_0^x \frac{L(\xi)}{\xi}\,d\xi} = 0.$$

In particular, $$x \longmapsto \int_0^x \frac{L(\xi)}{\xi}\,d\xi \in \mathcal{K}.$$

Lemma 2.3 ([6]). *For $x,t>0$ and $r,s<1$, we have:*
$$2^{-\max(1-r,1-s)}(t+x) \leqslant t^{1-r}(t+x)^r + x^{1-s}(t+x)^s \leqslant 2(t+x).$$

Lemma 2.4 ([6]). *Let $r,s<1$ and $M,N\in\mathcal{K}$ be defined on $(0,\eta]$, $\eta>1$.*

Put for $x\in(0,\eta)$, $$I(x) = \left(\int_x^\eta \frac{M(\xi)}{\xi}\,d\xi\right)^{\frac{1}{1-r}} + \left(\int_x^\eta \frac{N(\xi)}{\xi}\,d\xi\right)^{\frac{1}{1-s}}.$$

Then, for $x\in(0,\eta)$, $$\int_x^\eta \frac{(I^rM+I^sN)(\xi)}{\xi}\,d\xi \approx I(x).$$

Lemma 2.5. *Let $r,s<1$ and $M,N\in\mathcal{K}$ be defined on $(0,\eta]$, $\eta>1$, satisfying*
$$\int_0^\eta \frac{M(\xi)}{\xi}\,d\xi < \infty \ \text{ and } \int_0^\eta \frac{N(\xi)}{\xi}\,d\xi < \infty.$$

Put for $x\in(0,\eta]$, $$H(x) = \left(\int_0^x \frac{M(\xi)}{\xi}\,d\xi\right)^{\frac{1}{1-r}} + \left(\int_0^x \frac{N(\xi)}{\xi}\,d\xi\right)^{\frac{1}{1-s}}.$$

Then we have $$\int_0^x \frac{(H^rM+H^sN)(\xi)}{\xi}\,d\xi \approx H(x),\ x\in(0,\eta].$$

Next, we give some potential theory results which are taken from [12]. Indeed, we recall properties of some potential functions including estimates and a careful analysis of continuity. The next lemma plays a crucial role in establishing an existence result for problem (1).

Lemma 2.6. *Let $h\in C((0,1))$ satisfy $\int_0^1 \xi(1-\xi)h(\xi)\,d\xi<\infty$. Then Vh is the unique solution in $C([0,1])\cap C^2((0,1))$ of:*
$$\begin{cases} -\frac{1}{A}(Av')' = h, & \text{in } (0,1),\\ v(0)=v(1)=0. \end{cases}$$

Lemma 2.7. *Assume that condition (H$_0$) is satisfied and let q be a function satisfying (H). Then for $x\in(0,1)$,*
$$Vq(x) \approx x^{\min(1-\alpha_1,2-\gamma_1)}(1-x)^{\min(1-\alpha_2,2-\gamma_2)}\tilde{L_1}(x)\tilde{L_2}(1-x),$$
where for $i=1,2$,
$$\tilde{L_i}(x) := \begin{cases} 1, & \text{if } \gamma_i<\alpha_i+1,\\ \int_x^\eta \frac{L_i(t)}{t}\,dt, & \text{if } \gamma_i=\alpha_i+1,\\ L_i(x), & \text{if } \alpha_i+1<\gamma_i<2,\\ \int_0^x \frac{L_i(t)}{t}\,dt, & \text{if } \gamma_i=2. \end{cases}$$

Lemma 2.8. *Let $h\in B^+((0,1))$ such that $\int_0^1 t(1-t)h(t)\,dt<\infty$. Then the family of functions $\mathcal{F}_h := \{Vp;\, p\in B^+((0,1)),\ p\leqslant h\}$ is relatively compact in $C_0([0,1])$.*

3. Proof of Theorem 1.3

Let p, q be two functions satisfying (H_1) and θ be the function given by (7). Notice that

$$\theta(x) \approx x^{\nu_1} K_1(x), \text{ on } (0, \frac{1}{2}) \quad \text{and} \quad \theta(x) \approx (1-x)^{\nu_2} K_2(1-x), \text{ on } (\frac{1}{2}, 1), \tag{10}$$

where for $i = 1, 2$, K_i is the function given by (8). So to prove (9), it is enough to prove that

$$V(p\,\theta^r + q\,\theta^s)(x) \approx x^{\nu_1} K_1(x), \quad \text{on } (0, \frac{1}{2}) \tag{11}$$

and

$$V(p\,\theta^r + q\,\theta^s)(x) \approx (1-x)^{\nu_2} K_2(1-x), \quad \text{on } (\frac{1}{2}, 1). \tag{12}$$

As it can be seen, there is a complete analogy between (11) and (12). Indeed, (12) can be recovered from (11) by interchanging x by $1-x$, ν_1 by ν_2 and K_1 by K_2. So, we limited the proof to the interval $(0, \frac{1}{2})$.

Let $i = 1, 2$, since $\nu_i < \xi_i$ is equivalent to $\frac{2-\mu_i}{1-r} < \frac{2-\lambda_i}{1-s}$ and $\mu_i > 1 + \alpha_i + r(1-\alpha_i)$, then we deduce that,

$$K_i(x) \approx \begin{cases} M_i^{\frac{1}{1-r}}(x), & \text{if } 1+\alpha_i+r(1-\alpha_i)<\mu_i<2\,,\ \frac{2-\mu_i}{1-r}<\frac{2-\lambda_i}{1-s}, \\ \left(\int_0^x \frac{M_i(t)}{t}\,dt\right)^{\frac{1}{1-r}}, & \text{if } \mu_i=2\,,\ \lambda_i<2, \\ \left(\int_x^\eta \frac{M_i(t)}{t}\,dt\right)^{\frac{1}{1-r}}, & \text{if } \mu_i=1+\alpha_i+r(1-\alpha_i)\,,\ \lambda_i<1+\alpha_i+s(1-\alpha_i), \\ 1, & \text{if } \mu_i<1+\alpha_i+r(1-\alpha_i)\,,\ \lambda_i<1+\alpha_i+s(1-\alpha_i), \\ M_i^{\frac{1}{1-r}}(x)+N_i^{\frac{1}{1-s}}(x), & \text{if } 1+\alpha_i+r(1-\alpha_i)<\mu_i<2\,,\ \frac{2-\mu_i}{1-r}=\frac{2-\lambda_i}{1-s}, \\ \left(\int_x^\eta \frac{M_i(t)}{t}\,dt\right)^{\frac{1}{1-r}}+\left(\int_x^\eta \frac{N_i(t)}{t}\,dt\right)^{\frac{1}{1-s}}, & \text{if } \mu_i=1+\alpha_i+r(1-\alpha_i)\,,\ \lambda_i=1+\alpha_i+s(1-\alpha_i), \\ \left(\int_0^x \frac{M_i(t)}{t}\,dt\right)^{\frac{1}{1-r}}+\left(\int_0^x \frac{N_i(t)}{t}\,dt\right)^{\frac{1}{1-s}}, & \text{if } \mu_i=2\,,\ \lambda_i=2. \end{cases}$$

On $(0, 1)$, we put $\omega = p\theta^r + q\theta^s$. From hypothesis (H_1) and (7), we have for $x \in (0, 1)$,

$$\begin{aligned} \omega(x) \approx{}& x^{-\mu_1+\nu_1 r}\left(M_1 K_1^r\right)(x)(1-x)^{-\mu_2+\nu_2 r}\left(M_2 K_2^r\right)(1-x) \\ &+ x^{-\lambda_1+\nu_1 s}\left(N_1 K_1^s\right)(x)(1-x)^{-\lambda_2+\nu_2 s}\left(N_2 K_2^s\right)(1-x). \end{aligned}$$

In order to prove (11), we will apply Lemma 2.7. For this, we have to verify that hypothesis (H) is fulfilled. So, we remark that due to Lemmas 2.1, 2.2 and hypothesis (H_1), one can see easily that assumption (H) is verified. That is for $i \in \{1, 2\}$, the functions $M_i K_i^r$ and $N_i K_i^s$ are in $\mathcal{K}$ and satisfy:

$$\int_0^\eta t^{1-\mu_i+\nu_i r}(M_i K_i^r)(t)\,dt < \infty \qquad \text{and} \qquad \int_0^\eta t^{1-\lambda_i+\nu_i s}(N_i K_i^s)(t)\,dt < \infty.$$

In what follows, we differentiate four cases based on the comparison between ν_i and ξ_i, for $i = 1, 2$.

Case 1. $\nu_1 < \xi_1$ and $\nu_2 < \xi_2$.

Let $x \in (0, 1)$. By simple calculation and using Lemma 2.2, we obtain that $\omega(x) \approx x^{-\mu_1+\nu_1 r}(1-x)^{-\mu_2+\nu_2 r}\left(M_1 \tilde{M}_1^{\,r}\right)(x)\left(M_2 \tilde{M}_2^{\,r}\right)(1-x)$, for $x \in (0, 1)$. So applying

Lemma 2.7 for $i \in \{1,2\}$, $\gamma_i = \mu_i - \nu_i r$ and $L_i = M_i\tilde{M_i}^r$, on $(0,\frac{1}{2})$, we obtain

$$V\omega(x) \approx \begin{cases} x^{\frac{2-\mu_1}{1-r}} M_1^{\frac{1}{1-r}}(t), & \text{if } 1+\alpha_1+r(1-\alpha_1)<\mu_1<2,\ \frac{2-\mu_1}{1-r}<\frac{2-\lambda_1}{1-s}, \\ \int_0^x \frac{M_1(t)}{t}\left(\int_0^t \frac{M_1(y)}{y}dy\right)^{\frac{r}{1-r}} dt, & \text{if } \mu_1 = 2, \end{cases}$$

which gives that on $(0,\frac{1}{2})$, $V\omega(x) \approx x^{\nu_1}K_1(x)$. That is on $(0,\frac{1}{2})$, $V\omega(x) \approx \theta(x)$.

Case 2. $\nu_1 = \xi_1$ and $\nu_2 < \xi_2$. In this case, we have for $x \in (0,1)$,

$$\begin{aligned} \omega(x) \approx & x^{-\mu_1+\nu_1 r}(1-x)^{-\mu_2+\nu_2 r}M_1(x)\left(\tilde{M_1}+\tilde{N_1}\right)^r(x)(M_2\tilde{M_2}^r)(1-x) \\ & + x^{-\lambda_1+\nu_1 s}(1-x)^{-\lambda_2+\nu_2 s}N_1(x)\left(\tilde{M_1}+\tilde{N_1}\right)^s(x)(N_2\tilde{M_2}^s)(1-x) \\ :=& \varphi(x)+\psi(x). \end{aligned}$$

Now, we distinguish five subcases.

Subcase 1. $\mu_1 = 1+\alpha_1+r(1-\alpha_1)\,,\lambda_1 < 1+\alpha_1+s(1-\alpha_1)$.

By (10), we obtain that for $x \in (0,\frac{1}{2})$, $\theta(x) \approx x^{1-\alpha_1}\left(\int_x^\eta \frac{M_1(t)}{t}\,dt\right)^{\frac{1}{1-r}}$. By calculus, we have for $x \in (0,1)$,

$$\varphi(x) \approx x^{-1-\alpha_1}M_1(x)\left(\left(\int_x^\eta \frac{M_1(t)}{t}\,dt\right)^{\frac{1}{1-r}}+1\right)^r(1-x)^{-\mu_2+\nu_2 r}(M_2\tilde{M_2}^r)(1-x)$$

and $\quad \psi(x) \approx x^{-\lambda_1+\nu_1 s}N_1(x)\left(\left(\int_x^\eta \frac{M_1(t)}{t}\,dt\right)^{\frac{1}{1-r}}+1\right)^s(1-x)^{-\lambda_2+\nu_2 s}(N_2\tilde{M_2}^s)(1-x).$

Using the fact that

$$\left(\int_x^\eta \frac{M_1(t)}{t}\,dt\right)^{\frac{1}{1-r}}+1 \approx \left(\int_x^\eta \frac{M_1(t)}{t}\,dt\right)^{\frac{1}{1-r}} \tag{13}$$

and applying Lemma 2.7, we obtain that on $(0,\frac{1}{2})$

$$V\varphi(x) \approx x^{1-\alpha_1}\left(\int_x^\eta \frac{M_1(t)}{t}\,dt\right)^{\frac{1}{1-r}} \quad \text{and} \quad V\psi(x) \approx x^{1-\alpha_1}.$$

Using again (13), we deduce that on $(0,\frac{1}{2})$,

$$V\omega(x) \approx x^{1-\alpha_1}\left(\int_x^\eta \frac{M_1(t)}{t}\,dt\right)^{\frac{1}{1-r}}.$$

Subcase 2. $\mu_1 < 1+\alpha_1+r(1-\alpha_1$, $\lambda_1 < 1+\alpha_1+s(1-\alpha_1)$. Using (10), we have on $(0,\frac{1}{2})$ $\theta(x) \approx x^{1-\alpha_1}$. Then, we can clearly see that on $(0,1)$,

$$\varphi(x) \approx x^{-\mu_1+r(1-\alpha_1)}M_1(x)(1-x)^{-\mu_2+\nu_2 r}(M_2\tilde{M_2}^r)(1-x)$$

and $\quad \psi(x) \approx x^{-\lambda_1+(1-\alpha_1)s}N_1(x)(1-x)^{-\lambda_2+\nu_2 s}(N_2\tilde{M_2}^s)(1-x).$

By applying Lemma 2.7, from here follows that on $(0,\frac{1}{2})$, $V\varphi(x) \approx x^{1-\alpha_1}$ and $V\psi(x) \approx x^{1-\alpha_1}$. Hence, we have $V\omega(x) \approx x^{1-\alpha_1}$, $x \in (0,\frac{1}{2})$.

Subcase 3. $\mu_1 = 1+\alpha_1+r(1-\alpha_1)\,,\lambda_1 = 1+\alpha_1+s(1-\alpha_1)$. In this case we have for $x \in (0, \frac{1}{2})$,

$$\theta(x) \approx x^{1-\alpha_1}\left(\left(\int_x^\eta \frac{M_1(t)}{t}\,dt\right)^{\frac{1}{1-r}} + \left(\int_x^\eta \frac{N_1(t)}{t}\,dt\right)^{\frac{1}{1-s}}\right) := x^{1-\alpha_1} I(x).$$

Then we get that for $x \in (0,1)$,

$$\varphi(x) \approx x^{-1-\alpha_1}(M_1 I^r)(x)(1-x)^{-\mu_2+\nu_2 r}(M_2 \tilde{M_2}^r)(1-x)$$

and $$\psi(x) \approx x^{-1-\alpha_1}(N_1 I^s)(x)(1-x)^{-\lambda_2+\nu_2 s}(N_2 \tilde{M_2}^s)(1-x).$$

From Lemma 2.7, we obtain that on $(0, \frac{1}{2})$,

$$V\varphi(x) \approx x^{1-\alpha_1}\left(\int_x^\eta \frac{(M_1 I^r)(t)}{t}\,dt\right) \quad \text{and} \quad V\psi(x) \approx x^{1-\alpha_1}\left(\int_x^\eta \frac{(N_1 I^s)(t)}{t}\,dt\right).$$

This imply by using Lemma 2.4 that on $(0, \frac{1}{2})$,

$$V\omega(x) \approx x^{1-\alpha_1}\left(\left(\int_x^\eta \frac{M_1(t)}{t}\,dt\right)^{\frac{1}{1-r}} + \left(\int_x^\eta \frac{N_1(t)}{t}\,dt\right)^{\frac{1}{1-s}}\right).$$

Subcase 4. $1+\alpha_1+r(1-\alpha_1) < \mu_1 < 2$, $\frac{2-\mu_1}{1-r} = \frac{2-\lambda_1}{1-s}$. By (10), we have for $x \in (0, \frac{1}{2})$,

$$\theta(x) \approx x^{\frac{2-\mu_1}{1-r}}\left(M_1^{\frac{1}{1-r}} + N_1^{\frac{1}{1-s}}\right)(x).$$

This implies that on $(0,1)$

$$\varphi(x) \approx x^{-\mu_1+\frac{2-\mu_1}{1-r}r}\left(M_1\left(M_1^{\frac{1}{1-r}} + N_1^{\frac{1}{1-s}}\right)^r\right)(x)(1-x)^{-\mu_2+\nu_2 r}(M_2 \tilde{M_2}^r)(1-x)$$

and $$\psi(x) \approx x^{-\lambda_1+\frac{2-\lambda_1}{1-s}s}(N_1\left(M_1^{\frac{1}{1-r}} + N_1^{\frac{1}{1-s}}\right)^s)(x)(1-x)^{-\lambda_2+\nu_2 s}(N_2 \tilde{M_2}^s)(1-x).$$

Applying Lemma 2.7, on $(0, \frac{1}{2})$ we get

$$V\varphi(x) \approx x^{\frac{2-\mu_1}{1-r}}\left(M_1\left(M_1^{\frac{1}{1-r}} + N_1^{\frac{1}{1-s}}\right)^r\right)(x)$$

and $$V\psi(x) \approx x^{\frac{2-\mu_1}{1-r}}\left(N_1\left(M_1^{\frac{1}{1-r}} + N_1^{\frac{1}{1-s}}\right)^s\right)(x).$$

From Lemma 2.3, it follows

$$V\omega(x) \approx x^{\frac{2-\mu_1}{1-r}}\left(M_1^{\frac{1}{1-r}} + N_1^{\frac{1}{1-s}}\right)(x), \ x \in (0, \frac{1}{2}).$$

Subcase 5. $\mu_1 = \lambda_1 = 2$. In this case, we can clearly see, that for $x \in (0, \frac{1}{2})$

$$\theta(x) \approx \left(\int_0^x \frac{M_1(t)}{t}\,dt\right)^{\frac{1}{1-r}} + \left(\int_0^x \frac{N_1(t)}{t}\,dt\right)^{\frac{1}{1-s}} := H(x).$$

This gives that for $x \in (0,1)$

$$\varphi(x) \approx x^{-2}(M_1 H^r)(x)(1-x)^{-\mu_2+\nu_2 r}(M_2 \tilde{M_2}^r)(1-x)$$

and $$\psi(x) \approx x^{-2}(N_1 H^s)(x)(1-x)^{-\lambda_2+\nu_2 s}(N_2 \tilde{M_2}^s)(1-x).$$

According to Lemma 2.7, we get that for $x \in (0, \frac{1}{2})$

$$V\varphi(x) \approx \left(\int_0^x \frac{(M_1 H^r)(t)}{t}\, dt\right) \quad \text{and} \quad V\psi(x) \approx \left(\int_0^x \frac{(N_1 H^s)(t)}{t}\, dt\right).$$

Due to Lemma 2.5, this implies that on $(0, \frac{1}{2})$

$$V\omega(x) \approx \left(\int_0^x \frac{M_1(t)}{t}\, dt\right)^{\frac{1}{1-r}} + \left(\int_0^x \frac{N_1(t)}{t}\, dt\right)^{\frac{1}{1-s}}.$$

Case 3. $\nu_1 < \xi_1$ and $\nu_2 = \xi_2$. Let $x \in (0,1)$, we have

$$\begin{aligned}\omega(x) \approx\, & x^{-\mu_1+\nu_1 r}(1-x)^{-\mu_2+\nu_2 r}\left(M_1 \tilde{M_1}^r\right)(x) M_2(1-x)\left(\tilde{M_2}+\tilde{N_2}\right)^r(1-x)\\ & + x^{-\lambda_1+\nu_1 s}(1-x)^{-\lambda_2+\nu_2 s}\left(N_1 \tilde{M_1}^s\right)(x) N_2(1-x)\left(\tilde{M_2}+\tilde{N_2}\right)^s(1-x).\end{aligned}$$

Similarly to the **Case 1.**, we can get that on $(0, \frac{1}{2})$

$$V\omega(x) \approx \begin{cases} x^{\frac{2-\mu_1}{1-r}} M_1^{\frac{1}{1-r}}(t), & \text{if } 1+\alpha_1+r(1-\alpha_1)<\mu_1<2,\ \frac{2-\mu_1}{1-r}<\frac{2-\lambda_1}{1-s},\\ \int_0^x \frac{M_1(t)}{t}\left(\int_0^t \frac{M_1(y)}{y} dy\right)^{\frac{r}{1-r}} dt, & \text{if } \mu_1 = 2.\end{cases}$$

That is, $V\omega(x) \approx \theta(x)$, $x \in (0, \frac{1}{2})$.

Case 4. $\nu_1 = \xi_1$ and $\nu_2 = \xi_2$. For $x \in (0,1)$, we get that:

$$\begin{aligned}\omega(x) \approx\, & x^{-\mu_1+\nu_1 r}(1-x)^{-\mu_2+\nu_2 r} M_1(x)\left(\tilde{M_1}+\tilde{N_1}\right)^r(x) M_2(1-x)\left(\tilde{M_2}+\tilde{N_2}\right)^r(1-x)\\ & + x^{-\lambda_1+\nu_1 s}(1-x)^{-\lambda_2+\nu_2 s} N_1(x)\left(\tilde{M_1}+\tilde{N_1}\right)^s(x) N_2(1-x)\left(\tilde{M_2}+\tilde{N_2}\right)^s(1-x).\end{aligned}$$

By the same arguments as in **Case 2.**, on $(0, \frac{1}{2})$ the following is true:

$$V\omega(x) \approx \begin{cases} x^{1-\alpha_1}\left(\int_x^\eta \frac{M_1(t)}{t}\, dt\right)^{\frac{1}{1-r}}, & \text{if } \mu_1=1+\alpha_1+r(1-\alpha_1),\ \lambda_1<1+\alpha_1+s(1-\alpha_1),\\ x^{1-\alpha_1}, & \text{if } \mu_1<1+\alpha_1+r(1-\alpha_1),\ \lambda_1<1+\alpha_1+s(1-\alpha_1),\\ x^{1-\alpha_1}\left(\left(\int_x^\eta \frac{M_1(t)}{t}\, dt\right)^{\frac{1}{1-r}}+\left(\int_x^\eta \frac{N_1(t)}{t}\, dt\right)^{\frac{1}{1-s}}\right), & \text{if } \mu_1=1+\alpha_1+r(1-\alpha_1),\ \lambda_1=1+\alpha_1+s(1-\alpha_1),\\ x^{\frac{2-\mu_1}{1-r}}\left(M_1^{\frac{1}{1-r}}+N_1^{\frac{1}{1-s}}\right)(x), & \text{if } 1+\alpha_1+r(1-\alpha_1)<\mu_1<2,\ \frac{2-\mu_1}{1-r}=\frac{2-\lambda_1}{1-s},\\ \left(\int_0^x \frac{M_1(t)}{t}\, dt\right)^{\frac{1}{1-r}}+\left(\int_0^x \frac{N_1(t)}{t}\, dt\right)^{\frac{1}{1-s}}, & \text{if } \mu_1=\lambda_1=2.\end{cases}$$

So, $V\omega$ satisfies (11) in $(0, \frac{1}{2})$.

4. Proof of Theorem 1.4

The next lemma is useful in the proof of our Theorem 1.4.

LEMMA 4.1. *Suppose that (*H_0*) is satisfied. Let θ be the function given by* (7)*, $r, s \in (-1,1)$ and p, q are nonnegative functions satisfying (*H_1*). Then we have*

$$\int_0^1 t(1-t)(p\theta^r + q\theta^s)(t)\, dt < \infty.$$

Proof. Let $x, t \in (0,1)$. Using the fact that $xt \leqslant \min(t,x)$ and $(1-x)(1-t) \leqslant 1-\max(t,x)$, we obtain from (3), that there is $c>0$ such that for $t,x \in (0,1)$

$$c\,t(1-t)x^{1-\alpha_1}(1-x)^{1-\alpha_2}(p\theta^r+q\theta^s)(t) \leqslant G(t,x)(p\theta^r+q\theta^s)(t).$$

Then, we have for $x \in (0,1)$,

$$c\,x^{1-\alpha_1}(1-x)^{1-\alpha_2}\int_0^1 t(1-t)(p\theta^r+q\theta^s)(t)\,dt \leqslant \int_0^1 G(t,x)(p\theta^r+q\theta^s)(t)\,dt<\infty. \tag{14}$$

Taking $x=\frac{1}{2}$ in (14), we deduce the desired result due to Theorem 1.3. □

We are now able to prove our Theorem 1.4.

Proof (of Theorem 1.4). Let $r,s \in (-1,1)$ and assume that (H_0)-(H_1) are satisfied. Let θ be the function defined by (7). By Theorem 1.3, we have, $V\omega(t) \approx \theta(t)$, $t \in (0,1)$, where $\omega := p\theta^r + q\theta^s$. We deduce the existence of $c>0$ such that for each $t \in (0,1)$,

$$\frac{1}{c}\,\theta(t) \leqslant V\omega(t) \leqslant c\,\theta(t). \tag{15}$$

In order to construct a solution of problem (1), we shall use a fixed point argument.

Define the convex closed set Y as follows, $Y := \{u \in C_0([0,1]);\ \frac{1}{m}\theta \leqslant u \leqslant m\theta\}$, where $m = c^{\frac{1}{1-\max(|r|,|s|)}}$. Clearly $\theta \in Y$. We consider the operator T defined on Y as follows, $Tu := V(pu^r+qu^s)$. Let $u \in Y$. Then we have on $(0,1)$,

$$\frac{1}{m^{|r|}}\,p\theta^r + \frac{1}{m^{|s|}}\,q\theta^s \leqslant pu^r+qu^s \leqslant m^{|r|}\,p\theta^r + m^{|s|}\,q\theta^s. \tag{16}$$

Hence, on $(0,1)$,

$$\frac{1}{m^{\max(|r|,|s|)}}\omega \leqslant pu^r+qu^s \leqslant m^{\max(|r|,|s|)}\,\omega. \tag{17}$$

It is obvious that $V(pu^r+qu^s) \in \mathcal{F}_{m^{\max(|r|,|s|)}\,\omega}$, then applying Lemmas 2.8 and 4.1, we conclude that TY is relatively compact in $C_0([0,1])$. Besides, from (17), we get $\frac{1}{m^{\max(|r|,|s|)}}\,V\omega(t) \leqslant Tu(t) \leqslant m^{\max(|r|,|s|)}\,V\omega(t)$, $t \in (0,1)$. Using (15) we obtain that on $(0,1)$, $\frac{1}{m}\theta \leqslant Tu \leqslant m\theta$. Then T leaves invariant the convex Y.

Next, we prove the continuity of T in Y. Consider a sequence $(u_k)_{k\in\mathbb{N}}$ of functions in Y which converges uniformly to a function u in Y. Let $k \in \mathbb{N}$ and $x \in (0,1)$; we have $|Tu_k(x)-Tu(x)| \leqslant \int_0^1 G(t,x)|(pu_k^r+qu_k^s)(t)-(pu^r+qu^r)(t)|\,dt$. Moreover, we have for $(t,x) \in (0,1)\times(0,1)$, $G(t,x)|(pu_k^r+qu_k^s)(t)-(pu^r+qu^s)(t)| \leqslant 2m^{\max(|r|,|s|)}G(t,x)\,\omega(t)$. Since $V\omega<\infty$, we conclude due to the dominated convergence theorem, that for $x \in (0,1)$, $Tu_k(x) \longrightarrow Tu(x)$, as $k\to\infty$. In view of the relative compactness of TY in $C_0([0,1])$, we obtain the uniform convergence, that is, $\|Tu_k - Tu\|_\infty \longrightarrow 0$, as $k\to\infty$. Hence, we have showed that T is a compact operator from Y into itself. As a consequence of the Schauder fixed point theorem, we conclude that the operator T has a fixed point, i.e., there is $u \in Y$ satisfying $u = V(pu^r+qu^s)$. Next we show that u is a classical solution of (1). As $u \in Y$, we have $u \in C_0([0,1])$. On the other hand, from (16) and Lemma 4.1, we obtain that

$$\int_0^1 t(1-t)(pu^r+qu^s)(t)\,dt<\infty.$$

From Lemma 2.6, it follows that u is a positive classical solution of problem (1).

To prove the uniqueness result, we suppose that (1) admits two positive classical solutions u and v satisfying $u(x) \approx \theta(x)$ and $v(x) \approx \theta(x)$, $x \in (0,1)$. Consequently, there is $t_0 \geqslant 1$, such that $\frac{1}{t_0} \leqslant \frac{u}{v} \leqslant t_0$, on $(0,1)$. Therefore, the set $\Gamma := \{t \geqslant 1, \frac{1}{t}v \leqslant u \leqslant tv\}$ is not empty. Consider $\tau_0 = \inf \Gamma$. Let $\sigma := \max(|r|,|s|)$. It follows that $u^r \leqslant \tau_0^\sigma v^r$ and $u^s \leqslant \tau_0^\sigma v^s$. So we obtain that, $u - \tau_0^\sigma v = V(p(u^r - \tau_0^\sigma v^r)) + V(q(u^s - \tau_0^\sigma v^s)) \leqslant 0$. Similarly, we have $v - \tau_0^\sigma u$ is nonnegative. Hence, $\tau_0^\sigma \in \Gamma$ and $\tau_0 \leqslant \tau_0^\sigma$. Since $\sigma \in [0,1)$, then $\tau_0 = 1$. Finally we conclude that $u = v$. □

5. Example

Let $r \in (-1,0)$, $s \in (0,1)$, α, β and $\delta < 2$, such that $\frac{2-\alpha}{1-r} \leqslant \frac{2-\delta}{1-s}$. Consider two functions p and q in $C^\gamma_{loc}((0,1))$, $0 < \gamma < 1$, such that for $x \in (0,1)$, $p(x) \approx x^{-2}(1-x)^{-\alpha}\log^{-2}(\frac{3}{x})$, $q(x) \approx x^{-\beta}(1-x)^{-\delta}\log(\frac{3}{1-x})$. We can obviously see that (H_1) is satisfied. Thanks to Theorem 1.4, problem (1) admits a unique solution $u \in C_0([0,1]) \cap C^2((0,1))$ such that for $x \in (0,1)$, $u(x) \approx \Phi(x)\Psi(1-x)$, where $\Phi(x) \approx \left(\log(\frac{3}{x})\right)^{\frac{-1}{1-r}}$, and

$$\Psi(x) \approx \begin{cases} x^{1-\alpha_2}\left(\log(\frac{3}{x})\right)^{\frac{1}{1-r}}, & \text{if } \alpha = 1+\alpha_2+r(1-\alpha_2),\ \delta < 1+\alpha_2+s(1-\alpha_2), \\ x^{1-\alpha_2}, & \text{if } \alpha < 1+\alpha_2+r(1-\alpha_2),\ \delta < 1+\alpha_2+s(1-\alpha_2), \\ x^{\frac{2-\alpha}{1-r}}, & \text{if } 1+\alpha_2+r(1-\alpha_2) < \alpha,\ \frac{2-\alpha}{1-r} < \frac{2-\delta}{1-s}, \\ x^{\frac{2-\alpha}{1-r}}\left(\log(\frac{3}{x})\right)^{\frac{1}{1-s}}, & \text{if } 1+\alpha_2+r(1-\alpha_2) < \alpha,\ \frac{2-\alpha}{1-r} = \frac{2-\delta}{1-s}, \\ x^{1-\alpha_2}\left(\log(\frac{3}{x})\right)^{\frac{2}{1-s}}, & \text{if } \alpha = 1+\alpha_2+r(1-\alpha_2),\ \delta = 1+\alpha_2+s(1-\alpha_2). \end{cases}$$

Acknowledgement. The authors thank the referees for their careful reading of the paper and helpful comments.

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(received 09.10.2021; in revised form 26.01.2022; available online 08.09.2022)

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