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NEW CHARACTERIZATIONS OF FUZZY TOPOLOGY

D. N. Georgiou, A. C. Megaritis and G. A. Prinos

Abstract. Following the generalization of Moore-Smith convergence of nets to fuzzy topological spaces which was given in Pu Pao-Ming, Liu Ying-Ming, *Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl., **76** (1980), 571–599, a characterization theorem between fuzzy topologies and fuzzy convergence classes was introduced in Ying-Ming Liu, *On fuzzy convergence classes*, Fuzzy Sets and Systems, **30** (1989), 47–51. Our goal in this paper is to provide modified versions of this characterization. Specifically, we will introduce the concept of fuzzy semi-convergence class to give an alternative characterization of fuzzy topology, in relation to the ordinary convergence of fuzzy nets, and then we will introduce the concept of fuzzy ideal convergence class to obtain analogous results, in relation to the ideal convergence of fuzzy nets.

1. Introduction

The fundamental notion of a fuzzy set, introduced by Zadeh [27] in 1965, provided the natural framework for generalizing many of the concepts of general topology to the fuzzy setting. It was in 1968, that Chang [5] made the first "grafting" of the notion of a fuzzy set onto general topology, by defining the notion of fuzzy topological space. Since then, much research has been carried out in the area of fuzzy topology (see, for instance, [18–20, 26]). Especially, all the theorems concerning the neighborhood structure of a point and the theory of convergence in Chapters I and II of [12] are generalized to fuzzy topological spaces (see [19]). In [18], using the notions of the fuzzy point and its quasi-coincidence neighborhood (see [19]), a designation of fuzzy topology via fuzzy convergence classes has been given.

On the other hand, the concept of statistical convergence, based on the notion of the asymptotic density, was first introduced by Fast [9] and Steinhaus [24], independently. Statistical convergence has been investigated and generalized from various points of view (see for example, in summability theory [6, 10, 23, 24], number theory

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and mathematical analysis [1, 2, 21], topological and function spaces [4, 8], topological groups [3] and fuzzy mathematical analysis [22]). Using the notion of the ideal $\mathcal{I}$, Kostyrko et al. [14] introduced the concept of the $\mathcal{I}$-convergence which is a natural generalization of the ordinary convergence and statistical convergence. In the last years, researchers investigate convergence in metric and topological spaces via an ideal $\mathcal{I}$ (for an extensive view of this subject we refer for example to [7, 11, 13, 14, 16, 17]). The concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence and $\mathcal{I}$-Cauchyness for sequences of fuzzy numbers where defined and studied in [15]. The fuzzified notion of $\mathcal{I}$-convergence of fuzzy nets in a fuzzy topological spaces was given in [25]. In this paper we introduce and study the notions of fuzzy semi-convergence class and fuzzy ideal convergence class in order to obtain new characterizations of fuzzy topology.

The rest of this paper is organized as follows. Section 2 contains preliminaries. In Section 3 we give a modification of [18, Theorem 2] by considering the notion of a fuzzy semi-convergence class. In Section 4 we introduce the notion of fuzzy ideal convergence class and develop its correlation with fuzzy topology. Particularly, we characterize fuzzy topology via fuzzy ideal convergence classes.

2. Preliminaries

In this section, we recall basic notions in fuzzy set theory and topology, which will be needed in the sequel and we refer the reader to [5, 18, 19, 27] for more details. Throughout this paper, we use the symbols I and I^X to represent the unit closed interval $[0, 1]$ and the set of all functions with domain the non-empty (ordinary) set X and codomain I, respectively. A function $A : X \to I$ is called a *fuzzy set* in X (due to Zadeh [27]) i.e. a fuzzy set in X is an element of I^X. For every $x \in X$, $A(x)$ is called the *grade of membership* of x in A. X is called the *carrier* of the fuzzy set A. The set $\{x \in X : A(x) > 0\}$ is called the *support* of A. If A takes only the values $0, 1$, then A is called a *crisp set* in X. Particularly, the crisp set which always takes the value 1 on X is denoted by X_{fuzzy}, and the crisp set which always takes the value 0 on X is denoted by $\varnothing_{fuzzy}$. Also if A, B are fuzzy sets in X we say that A is *contained* in B, which we will denote by $A \leq_{fuzzy} B$, if $A(x) \leq B(x)$, for every $x \in X$.

Let Λ be an indexed set, and $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ be a family of fuzzy sets in X. Then, the *union* $\vee\mathcal{A}$ and the *intersection* $\wedge\mathcal{A}$ of the members of the family are fuzzy sets, defined respectively by the following rules:

$$(\vee\mathcal{A})(x) = \sup\{A_\lambda(x) : \lambda \in \Lambda\},\ x \in X$$
$$(\wedge\mathcal{A})(x) = \inf\{A_\lambda(x) : \lambda \in \Lambda\},\ x \in X.$$

For a fuzzy set A, the *complement* A' of A is a fuzzy set, defined by the formula: $A'(x) = 1 - A(x),\ x \in X$. The following De Morgan's laws also hold:

$$(\vee\{A_\lambda : \lambda \in \Lambda\})' = \wedge\,\{A'_\lambda : \lambda \in \Lambda\}$$
$$(\wedge\{A_\lambda : \lambda \in \Lambda\})' = \vee\,\{A'_\lambda : \lambda \in \Lambda\}.$$

A family δ of fuzzy sets in X is called a *fuzzy topology* for X (due to Chang [5]) if
(i) $\varnothing_{fuzzy}, X_{fuzzy} \in \delta$

(ii) $A \wedge B \in \delta$, whenever $A, B \in \delta$, and

(iii) $\vee\{A_\lambda : \lambda \in \Lambda\} \in \delta$, whenever $A_\lambda \in \delta$, for each $\lambda \in \Lambda$.
Moreover, the pair (X, δ) is called a *fuzzy topological space* or *fts*, for short. Every member of δ is called a δ*-open* (or simply *open*) fuzzy set. The complement of a δ-open fuzzy set is called δ*-closed* (or simply *closed*) fuzzy set. Let δ_1 and δ_2 be two fuzzy topologies for X. We say that δ_2 is *finer* than δ_1 and δ_1 is *coarser* than δ_2 if the inclusion relation $\delta_1 \subseteq \delta_2$ holds.

In this paper we adopted the definition of a fuzzy point from [19]. A fuzzy set in X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is $\lambda \in (0, 1]$ we denote the fuzzy point by x_λ, where the point x is called its *support*. The set of all the fuzzy points in X is denoted by $\mathrm{FP}(X)$. The fuzzy point x_λ is said to be *contained* in a fuzzy set A or to *belong* to A, denoted by $x_\lambda \in_{fuzzy} A$, if $\lambda \leq A(x)$. Evidently, every fuzzy set A can be expressed as the union of all the fuzzy points which belong to A.

Two fuzzy sets A, B in X are said to be *intersecting* if there exists a point $x \in X$ such that $(A \wedge B)(x) \neq 0$. For such a case, we say that A and B *intersect at* x. Given an fts (X, δ), a fuzzy set A in X is called a *neighborhood* of fuzzy point x_λ if there exists a $B \in \delta$ such that $x_\lambda \in_{fuzzy} B \leq_{fuzzy} A$; a neighborhood A is said to be *open* if A is open. The family consisting of all the neighborhoods of x_λ is called the *system of neighborhoods of* x_λ. A fuzzy point x_λ is said to be *quasi-coincident* with the fuzzy set A, denoted by $x_\lambda \,\mathrm{q}\, A$, if $\lambda > A'(x)$, or $\lambda + A(x) > 1$. A is said to be *quasi-coincident* with B, denoted by $A \,\mathrm{q}\, B$, if there exists $x \in X$ such that $A(x) > B'(x)$, or $A(x) + B(x) > 1$. In this case, we also say that A and B are *quasi-coincident (with each other) at* x. It is clear that if A and B are quasi-coincident at x, both $A(x)$ and $B(x)$ are not zero and hence A and B intersect at x. If A is not quasi-coincident with B, then we write $A \,\overline{\mathrm{q}}\, B$. Given an fts (X, δ), a fuzzy set A in X is called a Q*-neighborhood of* x_λ if there exists a $B \in \delta$ such that $x_\lambda \,\mathrm{q}\, B \leq_{fuzzy} A$. The family consisting of all the Q-neighborhoods of x_λ, is called the *system of* Q*-neighborhoods of* x_λ. A Q-neighborhood of a fuzzy point generally does not contain the fuzzy point itself.

Let (X, δ) be an fts and A be a fuzzy set in X. The intersection of all δ-closed fuzzy sets containing A is called the *(fuzzy) closure* of A, denoted by $\overline{A}$, or by $\mathrm{cl}_\delta A$. Obviously, $\overline{A}$ is the smallest δ-closed fuzzy set containing A and $\overline{(\overline{A})} = \overline{A}$.

A map $f : I^X \to I^X$ is called a *fuzzy closure operator* on X if f satisfies the following Kuratowski closure axioms:
(FCO1) $f(\varnothing_{fuzzy}) = \varnothing_{fuzzy}$, (FCO2) $A \leq_{fuzzy} f(A)$,

(FCO3) $f(f(A)) = f(A)$ and (FCO4) $f(A \vee B) = f(A) \vee f(B)$.
In a fuzzy topological space, the map $g : I^X \to I^X$ defined by $g(A) = \overline{A}$ is a fuzzy closure operator on X. Conversely, any fuzzy closure operator on X can determine some fuzzy topology for X. For this, we have:

Proposition 2.1 ([19]). *Let f be a fuzzy closure operator on X and let $\delta = \{A' : A \in \delta'\}$, where $\delta' = \{A \in I^X : f(A) = A\}$. The family δ is a fuzzy topology for X and for every $B \in I^X$, $\mathrm{cl}_\delta B = f(B)$. The topology δ will be called the* fuzzy topology associated with the fuzzy closure operator f.

A partially preordered set $(D, \geqslant)$ (simply denoted D) is called *directed* if every two elements of D have an upper bound in D. If $\{(E_d, \geqslant_d)\}_{d\in D}$ is a family of directed sets, the cartesian product $\prod_{d\in D} E_d$ of the family is directed by $\geqslant$, where $f \geqslant g$ iff $f(d) \geqslant_d g(d)$, for all $d \in D$.

A *fuzzy net* in X is an arbitrary function $s : D \to \mathrm{FP}(X)$ where D is directed. If we set $s(d) = s_d$, for all $d \in D$, then the fuzzy net s will be denoted by $(s_d)_{d\in D}$.

A fuzzy net $t = (t_\lambda)_{\lambda\in\Lambda}$ in X is said to be a *fuzzy semisubnet* of the fuzzy net $s = (s_d)_{d\in D}$ in X if there exists a function $\varphi : \Lambda \to D$ such that $t = s \circ \varphi$, i.e. $t_\lambda = s_{\varphi(\lambda)}$ for every $\lambda \in \Lambda$. We write $(t_\lambda)^\varphi_{\lambda\in\Lambda}$ to indicate the fact that φ is the function mentioned above.

A fuzzy net $t = (t_\lambda)_{\lambda\in\Lambda}$ in X is said to be a *fuzzy subnet* of the fuzzy net $s = (s_d)_{d\in D}$ in X if t is a fuzzy semisubnet of the fuzzy net s and for every $d \in D$ there exists $\lambda_0 \in \Lambda$ such that $\varphi(\lambda) \geqslant d$ whenever $\lambda \in \Lambda$ with $\lambda \geqslant \lambda_0$.

Let A be a fuzzy set in X. A fuzzy net $s = (s_d)_{d\in D}$ in X is said to be

(i) *quasi-coincident* with A if for each $d \in D$, s_d is quasi-coincident with A,

(ii) *eventually quasi-coincident* with A if there is an element $d_0 \in D$, such that, if $d \in D$ and $d \geqslant d_0$, then s_d is quasi-coincident with A,

(iii) *frequently quasi-coincident* with A if for each $d \in D$ there is a $d' \in D$ such that $d' \geqslant d$ and $s_{d'}$ is quasi-coincident with A and

(iv) *in* A if for each $d \in D$, $s_d \in_{fuzzy} A$.

We say that a fuzzy net $s = (s_d)_{d\in D}$ in an fts (X, δ) *converges* to a fuzzy point e in X, relative to δ, if s is eventually quasi-coincident with each Q-neighborhood of e. In this case we write $\lim_{d\in D} s_d = e$.

Proposition 2.2 ([19, Theorem 11.1]). *In an fts (X, δ), a fuzzy point $e \in_{fuzzy} \overline{A}$ iff there is a fuzzy net $s = (s_d)_{d\in D}$ in A such that s converges to e.*

Suppose that D is a directed set, and for each $d \in D$ there are a directed set E^d and a fuzzy net $s^d = (s^d(n))_{n\in E^d}$. Then, under product ordering we have a directed set $F = D \times \prod_{d\in D} E_d$ and a fuzzy net s defined by $s(d, f) = s^d(f(d))$, $d \in D$, $f \in \prod_{d\in D} E_d$. The fuzzy net s is called *induced net (associated with D and each s^d).*

In what follows, let X be a non-empty set and let $\mathcal{G}$ be a class consisting of pairs (s, e), where $s = (s_d)_{d\in D}$ is a fuzzy net in X and e is a fuzzy point in X.

Definition 2.3 ([18]). We say that $\mathcal{G}$ is a *fuzzy convergence class* for X if it satisfies the conditions listed below. For convenience, we say that s *converges ($\mathcal{G}$)* to e or that $\lim_{d\in D} s_d \equiv e(\mathcal{G})$ if $(s, e) \in \mathcal{G}$.

(G1) If s is such that $s_d = e$, for each $d \in D$, then s converges ($\mathcal{G}$) to e.

(G2) If s converges ($\mathcal{G}$) to e, then so does each fuzzy subnet of s.

(G3) If s does not converge $(\mathcal{G})$ to e, then there exists a fuzzy subnet t of s, no fuzzy subnet of which converges $(\mathcal{G})$ to e.

(G4) We consider the following:

(i) D is a directed set.

(ii) E_d is a directed set, for each $d \in D$.

(iii) $s^d = (s^d(n))_{n\in E_d}$ is a fuzzy net in X, converging $(\mathcal{G})$ to s_d, for each $d \in D$ and the fuzzy net $(s_d)_{d\in D}$, thus obtained, converges $(\mathcal{G})$ to e.

Then, the induced net (associated with D and each s^d), converges $(\mathcal{G})$ to e.

(G5) For each point $x \in X$ and real directed set $D \subseteq (0,1]$, if $r \leqslant \sup D$, then the fuzzy net $(x_d)_{d\in D}$ converges $(\mathcal{G})$ to x_r.
The class of all fuzzy convergence classes for X is denoted by $Con(X)$.

THEOREM 2.4 ([18]). *Let (X,δ) be a fuzzy topological space. Then, the class of pairs $\{(s,e) :$ the fuzzy net s converges to $e\}$ is a fuzzy convergence class, denoted by $\phi(\delta)$.*

PROPOSITION 2.5 ([18]). *Let Ω be a family of fuzzy points in X and $A = \vee\Omega$. Let the class of pairs $\mathcal{G}$ satisfy the conditions (G4) and (G5). If a fuzzy net s in A converges $(\mathcal{G})$ to e, then there exists a fuzzy net $\overline{s}$ that consists of fuzzy points in Ω and converges $(\mathcal{G})$ to e.*

THEOREM 2.6 ([18, Theorem 2], fuzzy convergence classes theorem). *We consider a map $\mathrm{c} : I^X \to I^X$ induced as follows: for each $A \in I^X$, we define*

$$\mathcal{G}(A) = \{e : \text{ for some fuzzy net } s \text{ in } A,\ (s,e) \in \mathcal{G}\} \qquad \mathrm{c}(A) = \vee\mathcal{G}(A).$$

Now if $\mathcal{G}$ is a fuzzy convergence class for X, then the following hold:
(i) The correspondence $A \mapsto \mathrm{c}(A)$ is a fuzzy closure operator and the fuzzy topology thus obtained, will be denoted by $\psi(\mathcal{G})$,

(ii) $\phi(\psi(\mathcal{G})) = \mathcal{G}$ and

(iii) $\psi(\phi(\delta)) = \delta$, for a fuzzy topology δ on X.
Therefore, there exists a bijective map between the set of all fuzzy topologies δ for X and the set of all fuzzy convergence classes $\mathcal{G}$ for X. Moreover, this map is order-reversing, i.e. if $\delta_1 \supseteq \delta_2$, then $\phi(\delta_1) \subseteq \phi(\delta_2)$.

If δ_1 and δ_2 are two fuzzy topologies on X, then the following two statements are equivalent:
(i) $\delta_1 = \delta_2$.

(ii) If $(s_d)_{d\in D}$ is a fuzzy net in X and e a fuzzy point in X, then $\lim_{d\in D} s_d = e$ with respect to δ_1 iff $\lim_{d\in D} s_d = e$ with respect to δ_2.

3. A new characterization of fuzzy topology, relative to the ordinary convergence

In this section we aim to give a modification of Theorem 2.6 by considering the notion of a fuzzy semi-convergence class.

PROPOSITION 3.1. *Let (X,δ) be an fts and $s=(s_d)_{d\in D}$ be a fuzzy net that fails to converge to a fuzzy point e in X. Then there exists an open Q-neighborhood U of e and a fuzzy subnet $t=(t_m)_{m\in E}$ of s such that t_m is not quasi-coincident with U for each $m\in E$ and hence any fuzzy semisubnet of t does not converge to e.*

Proof. The proof of [19, Proposition 14.1] establishes this stronger result. □

In what follows let X be a non-empty set and let $\mathcal{C}$ be a class consisting of pairs (s,e), where $s=(s_d)_{d\in D}$ is a fuzzy net in X and e is a fuzzy point in X.

DEFINITION 3.2. We say that $\mathcal{C}$ is a fuzzy semi-convergence class for X if it satisfies the conditions listed below. For convenience, we say that s *semi-converges $(\mathcal{C})$* to e or that $\lim_{d\in D} s_d \equiv e(\mathcal{C})$ if $(s,e)\in\mathcal{C}$.

(G′1) If s is such that $s_d=e$, for each $d\in D$, then s semi-converges $(\mathcal{C})$ to e.

(G′2) If s semi-converges $(\mathcal{C})$ to e, then so does each fuzzy subnet of s.

(G′3) If s does not semi-converge $(\mathcal{C})$ to e, then there exists a fuzzy subnet t of s, no fuzzy semisubnet of which semi-converges $(\mathcal{C})$ to e.

(G′4) We consider the following:

(i) D is a directed set.

(ii) E_d is a directed set, for each $d\in D$.

(iii) $s^d=(s^d(n))_{n\in E_d}$ is a fuzzy net in X, semi-converging $(\mathcal{C})$ to s_d, for each $d\in D$ and the fuzzy net $(s_d)_{d\in D}$, thus obtained, semi-converges $(\mathcal{C})$ to e.

Then, the induced net (associated with D and each s^d), semi-converges $(\mathcal{C})$ to e.

(G′5) For each point $x\in X$ and real directed set $D\subseteq(0,1]$, if $r\leqslant\sup D$, then the fuzzy net $(x_d)_{d\in D}$ semi-converges $(\mathcal{C})$ to x_r.

The class of all semi-convergence classes for X is denoted by $Con^s(X)$.

Note that Definitions 3.2 and 2.3 differ only in the third axiom.

THEOREM 3.3. *Let (X,δ) be a fuzzy topological space. Then, the class of pairs $\{(s,e):$ the fuzzy net s converges to $e\}$ is a fuzzy semi-convergence class, denoted by $\phi'(\delta)$.*

Proof. It follows from Proposition 3.1 and Theorem 2.4. □

An immediate observation is that $Con(X)\subseteq Con^s(X)$. Indeed, let $\mathcal{G}\in Con(X)$. By Theorem 2.6 we have that $\mathcal{G}=\{(s,e):$ the fuzzy net s converges to e with respect to $\psi(\mathcal{G})\}$, which, by Theorem 3.3, is a member of $Con^s(X)$. Moreover $\phi(\delta)=\phi'(\delta)$. (However, we will continue to use different symbolism for the map ϕ' till the end of this section.)

The proof of the following proposition follows from Proposition 2.5.

Proposition 3.4. *Let Ω be a family of fuzzy points in X and $A = \vee\Omega$. Let the class of pairs $\mathcal{C}$ satisfy the conditions (G′4) and (G′5). If a fuzzy net s in A semi-converges $(\mathcal{C})$ to e, then there exists a fuzzy net $\overline{s}$ that consists of fuzzy points in Ω and semi-converges $(\mathcal{C})$ to e.*

The following theorem sets up a one-to-one correspondence between the fuzzy topologies for a non-empty set X and the fuzzy semi-convergence classes on it.

Theorem 3.5 (fuzzy semi-convergence classes theorem). *Let $\mathcal{C}$ be a fuzzy semi-convergence class for a non-empty set X. We consider a map $\mathrm{cl} : I^X \to I^X$ induced as follows: for each $A \in I^X$, we define $\mathrm{cl}(A) \in I^X$ to be such that a fuzzy point $e \in_{fuzzy} \mathrm{cl}(A)$ iff there exists a fuzzy net s in A such that s semi-converges $(\mathcal{C})$ to e, i.e. $(s, e) \in \mathcal{C}$.*
(i) The correspondence $A \mapsto \mathrm{cl}(A)$ is a fuzzy closure operator and the topology thus obtained, will be denoted by $\psi'(\mathcal{C})$,
(ii) $\phi'(\psi'(\mathcal{C})) = \mathcal{C}$ and
(iii) $\psi'(\phi'(\delta)) = \delta$, for a fuzzy topology δ on X.
Therefore, there exists a bijective map between the set of all fuzzy topologies δ for X and the set of all fuzzy semi-convergence classes $\mathcal{C}$ for X. Moreover, this map is order-reversing, i.e. if $\delta_1 \supseteq \delta_2$, then $\phi'(\delta_1) \subseteq \phi'(\delta_2)$.

Proof. (i) We first prove that the map cl is well-defined. Let $A \in I^X$, the set

$$\mathcal{C}(A) = \{e : \text{ there exists a fuzzy net } s \text{ in } A, \text{ where } (s, e) \in \mathcal{C}\},$$

and put $\mathrm{cl}(A) = \vee\mathcal{C}(A)$. It is enough to prove that for each fuzzy point $e \in_{fuzzy} \mathrm{cl}(A)$ there exists a fuzzy net s in A such that s semi-converges $(\mathcal{C})$ to e. Let $e \in_{fuzzy} \mathrm{cl}(A)$ and denote the support point and the membership grade of e, by x and $\lambda \in (0, 1]$ respectively i.e. $e = x_\lambda$. Set

$$R = \{r \in (0, 1] : \text{ there exists a fuzzy net } s^r \text{ in } A, \text{ such that } (s^r, x_r) \in \mathcal{C}\}.$$

Clearly $R \neq \varnothing$ and $\sup R \geqslant \lambda$. Therefore from (G′5) the fuzzy net $(x_r)_{r\in R}$ semi-converges $(\mathcal{C})$ to e. Now from the definition of R there exists a fuzzy net s^r in A, such that s^r semi-converges $(\mathcal{C})$ to x_r, for each $r \in R$. It follows from (G′4) that there exists an induced fuzzy net in A, that semi-converges $(\mathcal{C})$ to e.

We next verify the conditions of a fuzzy closure operator.
(FCO1) is clear.
(FCO2) Let $A \in I^X$ and $e \in_{fuzzy} A$. By (G′1) the constant fuzzy net $(s_d)_{d\in D}$ with value e semi-converges $(\mathcal{C})$ to e. Therefore, $e \in_{fuzzy} \mathrm{cl}(A)$.
(FCO3) Let $A, B \in I^X$. Then, clearly $\mathrm{cl}(A) \leq_{fuzzy} \mathrm{cl}(A \vee B)$ and $\mathrm{cl}(B) \leq_{fuzzy} \mathrm{cl}(A \vee B)$. Therefore, $\mathrm{cl}(A) \vee \mathrm{cl}(B) \leq_{fuzzy} \mathrm{cl}(A \vee B)$. We prove that $\mathrm{cl}(A \vee B) \leq_{fuzzy} \mathrm{cl}(A) \vee \mathrm{cl}(B)$. Let $e \in_{fuzzy} \mathrm{cl}(A \vee B)$. Then there exists a fuzzy net $(s_d)_{d\in D}$ in $A \vee B$ that semi-converges $(\mathcal{C})$ to e. Denote

$$D_A = \{d \in D : s_d \in_{fuzzy} A\} \text{ and } D_B = \{d \in D : s_d \in_{fuzzy} B\}.$$

Then at least one of D_A, D_B is cofinal in D, since $D_A \cup D_B = D$. Without loss of generality assume that this is D_A. Then, we get a fuzzy subnet $(s_d)_{d\in D_A}$ of $(s_d)_{d\in D}$, in A, which by (G′2) semi-converges ($\mathcal{C}$) to e. Thus, $e \in_{fuzzy} \mathrm{cl}(A)$. It follows that $e \in_{fuzzy} \mathrm{cl}(A) \vee \mathrm{cl}(B)$.

(FCO4) Let $A \in I^X$. We prove that $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$. We have $A \leq_{fuzzy} \mathrm{cl}(A)$ and so $\mathrm{cl}(A) \leq_{fuzzy} \mathrm{cl}(\mathrm{cl}(A))$. We prove that $\mathrm{cl}(\mathrm{cl}(A)) \leq_{fuzzy} \mathrm{cl}(A)$. Let $e \in_{fuzzy} \mathrm{cl}(\mathrm{cl}(A))$. Then, there exists a fuzzy net $(t_d)_{d\in D}$ in $\mathrm{cl}(A)$ that semi-converges ($\mathcal{C}$) to e. Since $t_d \in_{fuzzy} \mathrm{cl}(A)$ for every $d \in D$, there exists a directed set E_d and a fuzzy net $(s^d(n))_{n\in E_d}$ in A, that semi-converges ($\mathcal{C}$) to t_d. By condition (G′4) there exists an induced fuzzy net in A, that semi-converges ($\mathcal{C}$) to e. Hence, $e \in_{fuzzy} \mathrm{cl}(A)$.

(ii) We prove that if the fuzzy net $s = (s_d)_{d\in D}$ semi-converges ($\mathcal{C}$) to e, then $(s_d)_{d\in D}$ converges to e with respect to the fuzzy topology $\psi'(\mathcal{C})$. Suppose that $(s_d)_{d\in D}$ semi-converges ($\mathcal{C}$) to e and does not converge to e with respect to $\psi'(\mathcal{C})$. By Proposition 3.1, there exists an open Q-neighborhood U of e such that there exists a fuzzy subnet t of s in U', which in view of (G′2), semi-converges ($\mathcal{C}$) to e. But U' is closed, and therefore $e \in_{fuzzy} \mathrm{cl}(U') = U'$. This contradicts the fact that e is quasi-coincident with U.

We prove that if the fuzzy net $s = (s_d)_{d\in D}$ converges to e with respect to the fuzzy topology $\psi'(\mathcal{C})$, then s semi-converges ($\mathcal{C}$) to e. Suppose that s converges to e with respect to $\psi'(\mathcal{C})$, and does not semi-converge ($\mathcal{C}$) to e. By condition (G′3) of the Definition 3.2, there exists a fuzzy subnet $t = (t_\lambda)_{\lambda\in\Lambda}$ of s such that no fuzzy semisubnet of t semi-converges ($\mathcal{C}$) to e. By Theorem 2.5, t converges to e, with respect to $\psi'(\mathcal{C})$. Set $A = \vee\{t_\lambda : \lambda \in \Lambda\}$. The fuzzy net t is in A so by Proposition 2.2, $e \in_{fuzzy} \mathrm{cl}(A)$. By the definition of $\mathrm{cl}(A)$, there exists a fuzzy net w in A such that w semi-converges ($\mathcal{C}$) to e. By Proposition 3.4, there exists a fuzzy net $\overline{w}$ that consists of fuzzy points in $\{t_\lambda : \lambda \in \Lambda\}$ (so $\overline{w}$ is a fuzzy semisubnet of t), that semi-converges ($\mathcal{C}$) to e, which is a contradiction.

Thus, we have proved that $\phi'(\psi'(\mathcal{C})) = \mathcal{C}$.

(iii) Let δ be a fuzzy topology on X. By Proposition 2.2, the fuzzy closure operator induced by $\phi'(\delta)$ coincides with the one associated with δ. Therefore, $\psi'(\phi'(\delta)) = \delta$.

Finally, if $\delta_1 \supseteq \delta_2$ then it is clear that $\phi'(\delta_1) \subseteq \phi'(\delta_2)$. □

Proposition 3.6. *Let X be a non-empty set. Then, $Con(X) = Con^s(X)$.*

Proof. It remains to show that $Con^s(X) \subseteq Con(X)$. Let $\mathcal{C} \in Con^s(X)$. By Theorem 3.5 we have that $\mathcal{C} = \{(s, e)$: the fuzzy net s converges to e with respect to $\psi'(\mathcal{C})\}$, which, by Theorem 2.4, is a member of $Con(X)$. □

Corollary 3.7. *Let X be a non-empty set. Then, $\phi' = \phi$ and $\psi' = \psi$.*

4. A new characterization of fuzzy topology, relative to the $\mathcal{I}$-convergence

In this section our purpose is to give a generalization of Theorem 2.6 for the case of fuzzy ideal convergence of fuzzy nets.

Let D be a non-empty set. A family $\mathcal{I}$ of subsets of D is called *ideal* if $\mathcal{I}$ has the following properties:

(i) $\varnothing \in \mathcal{I}$.

(ii) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

(iii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
The ideal $\mathcal{I}$ is called *proper* if $D \notin \mathcal{I}$.

Suppose that $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ is a fuzzy semisubnet of the fuzzy $(s_d)_{d\in D}$ in X. For every ideal $\mathcal{I}$ of the directed set D, the family $\{A \subseteq \Lambda : \varphi(A) \in \mathcal{I}\}$ is an ideal of the directed set Λ, which will be denoted by $\mathcal{I}_\Lambda(\varphi)$.

Let D be a directed set. For all $d \in D$ we set $M_d = \{d' \in D : d' \geqslant d\}$ (see [17]).

Definition 4.1 ([25]). A proper ideal $\mathcal{I}$ of a directed set D is called *admissible*, if $D \setminus M_d \in \mathcal{I}$, for all $d \in D$.

In what follows we recall the concept of convergence of fuzzy nets in an fts via ideal $\mathcal{I}$. We emphasize that although in [25, Section 3] the ideals are supposed to be admissible and therefore proper, the proofs are formulated for arbitrary proper ideals.

Definition 4.2 ([25]). Let (X, δ) be an fts and $\mathcal{I}$ an ideal of a directed set D. We say that a fuzzy net $(s_d)_{d\in D}$ *$\mathcal{I}$-converges* to a fuzzy point e in X, relative to δ, if for every open Q-neighborhood U of e, we have $\{d \in D : s_d \,\overline{\mathrm{q}}\, U\} \in \mathcal{I}$. In this case we write $\mathcal{I} - \lim_{d\in D} s_d = e$ and we say that e is the *$\mathcal{I}$-limit* of the fuzzy net $(s_d)_{d\in D}$.

Let (X, δ) be a fuzzy T_2 (Hausdorff) topological space. If a fuzzy net $\mathcal{I}$-converges, where $\mathcal{I}$ is a proper ideal of D, to two distinct fuzzy points, then their supports are the same (see [25]).

Proposition 4.3 ([17, 25]). *Let (X, δ) be an fts, e a fuzzy point in X, and D a directed set. Then, $\mathcal{I}_0(D) = \{A \subseteq D : A \subseteq D \setminus M_d$ for some $d \in D\}$ is a proper ideal of D. Moreover, a fuzzy net $(s_d)_{d\in D}$ converges to e iff $(s_d)_{d\in D}$ $\mathcal{I}_0(D)$-converges to e.*

Proposition 4.4 ([25, Theorem 3.5]). *Let (X, δ) be an fts and A a fuzzy set in X. If there is a fuzzy net $(s_d)_{d\in D}$ in A that $\mathcal{I}$-converges to the fuzzy point e in X, where $\mathcal{I}$ is a proper ideal of D, then $e \in_{fuzzy} \mathrm{cl}_\delta(A)$.*

In addition, the converse part of Proposition 4.4 also holds, if we take into account Propositions 2.2 and 4.3.

In what follows (X, δ) is an fts, e a fuzzy point in X, $(s_d)_{d\in D}$ is a fuzzy net in X, and $\mathcal{I}$ is an ideal of the directed set D.

Proposition 4.5. *If $(s_d)_{d\in D}$ is a fuzzy net such that $s_d = e$ for every $d \in D$, then $\mathcal{I} - \lim_{d\in D} s_d = e$.*

Proof. For every open Q-neighborhood U of e we have $\{d \in D : s_d \,\overline{\mathrm{q}}\, U\} = \varnothing \in \mathcal{I}$. Thus, $\mathcal{I} - \lim_{d\in D} s_d = e$. □

Proposition 4.6. *If $\mathcal{I} - \lim_{d\in D} s_d = e$, then for every fuzzy semisubnet $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the fuzzy net $(s_d)_{d\in D}$ we have $\mathcal{I}_\Lambda(\varphi) - \lim_{\lambda\in\Lambda} t_\lambda = e$.*

Proof. Let $(t_\lambda)^\varphi_{\lambda\in\Lambda}$ be a fuzzy semisubnet of the fuzzy net $(s_d)_{d\in D}$ and U be an open Q-neighborhood of e. We shall prove that $\{\lambda\in\Lambda : t_\lambda\,\overline{\mathrm{q}}\,U\}\in\mathcal{I}_\Lambda(\varphi)$.

Let $A=\{\lambda\in\Lambda : t_\lambda\,\overline{\mathrm{q}}\,U\}$. It suffices to prove that $\varphi(A)\in\mathcal{I}$. The case $A=\varnothing$ is clear. Let $A\neq\varnothing$. Since $(t_\lambda)^\varphi_{\lambda\in\Lambda}$ is a fuzzy semisubnet of the fuzzy net $(s_d)_{d\in D}$, for the function $\varphi:\Lambda\to D$ we have that $t_\lambda=s_{\varphi(\lambda)}$ for every $\lambda\in\Lambda$. Let $\varphi(\lambda)\in\varphi(A)$, where $\lambda\in A$. Since $t_\lambda\,\overline{\mathrm{q}}\,U$ and, therefore, $s_{\varphi(\lambda)}\,\overline{\mathrm{q}}\,U$, we have $\varphi(\lambda)\in\{d\in D : s_d\,\overline{\mathrm{q}}\,U\}$ which means that $\varphi(A)\subseteq\{d\in D : s_d\,\overline{\mathrm{q}}\,U\}$. However, $\mathcal{I}-\lim_{d\in D}s_d=e$. So, we have $\{d\in D : s_d\,\overline{\mathrm{q}}\,U\}\in\mathcal{I}$ and, therefore, $\varphi(A)\in\mathcal{I}$. □

Proposition 4.7. *If $\mathcal{I}-\lim_{d\in D}s_d=e$, where $\mathcal{I}$ is a proper ideal of D, then there exists a fuzzy semisubnet $(t_\lambda)_{\lambda\in\Lambda}$ of the fuzzy net $(s_d)_{d\in D}$ such that $\mathcal{I}_0(\Lambda)-\lim_{\lambda\in\Lambda}t_\lambda=e$.*

Proof. Let $A=\vee\{s_d : d\in D\}$. Since $(s_d)_{d\in D}$ is a fuzzy net in A $\mathcal{I}$-converging to e and the ideal $\mathcal{I}$ is proper, by Proposition 4.4 we have that $e\in_{fuzzy}\mathrm{cl}_\delta(A)$. Therefore, there exists a fuzzy net w in A converging to e (see Proposition 2.2). By Proposition 2.5, there exists a fuzzy net $\overline{w}=(t_\lambda)_{\lambda\in\Lambda}$ that consists of fuzzy points in $\{s_d : d\in D\}$ (so $\overline{w}$ is a fuzzy semisubnet of the fuzzy net $(s_d)_{d\in D}$) and converges to e. By Proposition 4.3, we have $\mathcal{I}_0(\Lambda)-\lim_{\lambda\in\Lambda}t_\lambda=e$. □

Proposition 4.8. *If the fuzzy net $(s_d)_{d\in D}$ does not $\mathcal{I}_D$-converge to e, where $\mathcal{I}_D$ is a proper ideal of D, then there exists a fuzzy semisubnet $(t_\lambda)^\varphi_{\lambda\in\Lambda}$ of the fuzzy net $(s_d)_{d\in D}$ such that:*

(i) $\Lambda\subseteq D$.

(ii) $\varphi(\lambda)=\lambda$, *for every* $\lambda\in\Lambda$.

(iii) No fuzzy semisubnet $(r_k)^f_{k\in K}$ of the fuzzy net $(t_\lambda)^\varphi_{\lambda\in\Lambda}$ $\mathcal{I}_K$-converges to e, for every proper ideal $\mathcal{I}_K$ of K.

(iv) $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal of Λ.

Proof. Since the fuzzy net $(s_d)_{d\in D}$ does not $\mathcal{I}_D$-converge to e, there exists an open Q-neighborhood U of the fuzzy point e such that $\{d\in D : s_d\,\overline{\mathrm{q}}\,U\}\notin\mathcal{I}_D$. Let $\Lambda=\{d\in D : s_d\,\overline{\mathrm{q}}\,U\}\subseteq D$ and $\varphi:\Lambda\to D$ be the inclusion map. We can consider Λ as directed by $\geqslant_\Lambda$ (regardless $\geqslant_D$), therefore the map $t=s\circ\varphi$ as a fuzzy semisubnet of s, where $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal of Λ, because $\varphi(\Lambda)=\Lambda\notin\mathcal{I}_D$.

We prove that no fuzzy semisubnet $(r_k)^f_{k\in K}$ of $(t_\lambda)_{\lambda\in\Lambda}$ $\mathcal{I}_K$-converges to e, where $\mathcal{I}_K$ is a proper ideal of K. Let $(r_k)^f_{k\in K}$ be a fuzzy semisubnet of $(t_\lambda)_{\lambda\in\Lambda}$ and $\mathcal{I}_K$ a proper ideal of K. Then, the function $f:K\to\Lambda$ is such that $r_k=t_{f(k)}$ for every $k\in K$. It suffices to prove that $\{k\in K : r_k\,\overline{\mathrm{q}}\,U\}\notin\mathcal{I}_K$. Indeed, let $k\in K$. Then, $r_k=t_{f(k)}=s_{\varphi(f(k))}=s_{f(k)}$. Since $f(k)\in\Lambda$, from the definition of Λ we have $s_{f(k)}\,\overline{\mathrm{q}}\,U$. Hence, $\{k\in K : r_k\,\overline{\mathrm{q}}\,U\}=K$. Since $\mathcal{I}_K$ is a proper ideal of K, $\{k\in K : r_k\,\overline{\mathrm{q}}\,U\}=K\notin\mathcal{I}_K$. □

Proposition 4.9. *We consider the following:*

(i) D is a directed set.

(ii) $\mathcal{I}_0(D)$ is a proper ideal of D (see Proposition 4.3).

(iii) E_d *is a directed set, for each* $d \in D$.

(iv) $\mathcal{I}_0(E_d)$ *is a proper ideal of* E_d, *for each* $d \in D$.

(v) $(s(d,e))_{e \in E_d}$ *is a fuzzy net in* X, *for each* $d \in D$.

(vi) $\mathcal{I}_0(D) - \lim_{d \in D} t_d = e$ *(e is a fuzzy point in* X*), where* $\mathcal{I}_0(E_d) - \lim_{e \in E_d} s(d,e) = t_d$ *(*t_d *is a fuzzy point in* X*), for every* $d \in D$.
Then, the fuzzy net $r : D \times \prod_{d \in D} E_d \to X$, *where* $r(d,f) = s(d, f(d))$, *for every* $(d,f) \in D \times \prod_{d \in D} E_d$, $\mathcal{I}_0(D \times \prod_{d \in D} E_d)$*-converges to* e.

Proof. It is actually the ideal version of [19, Theorem 12.2] taking into account Proposition 4.3. □

Proposition 4.10. *For each point* $x \in X$ *and real directed set* $D \subseteq (0,1]$, *if* $r \leqslant \sup D$, *then the fuzzy net* $(x_d)_{d \in D}$ $\mathcal{I}_0(D)$*-converges to* x_r.

Proof. It is actually the ideal version of [18, Definition 2, condition (G5)] taking into account Proposition 4.3. □

The following result is evident.

Proposition 4.11. *If* $(s_d)_{d \in D}$ *is a fuzzy net in* X, *then* $\mathcal{P}(D) - \lim_{d \in D} s_d = e$, *for every fuzzy point* e *in* X, *where* $\mathcal{P}(D)$ *denotes the powerset of* D.

Definition 4.12. Let X be a non-empty set and let $\mathcal{H}$ be a class consisting of triads $(s, e, \mathcal{I})$, where $s = (s_d)_{d \in D}$ is a fuzzy net in X, e is a fuzzy point in X and $\mathcal{I}$ is an ideal of D. We say that $\mathcal{H}$ is a *fuzzy ideal convergence class* for X if it satisfies the conditions listed below. For convenience, we say that s *$\mathcal{I}$-converges ($\mathcal{H}$)* to e or that $\mathcal{I} - \lim_{d \in D} s_d \equiv e(\mathcal{H})$ if $(s, e, \mathcal{I}) \in \mathcal{H}$.
(C′1) If $(s_d)_{d \in D}$ is a fuzzy net such that $s_d = e$ for every $d \in D$ and $\mathcal{I}$ is an ideal of D, then $\mathcal{I} - \lim_{d \in D} s_d \equiv e(\mathcal{H})$.

(C′2) If $\mathcal{I} - \lim_{d \in D} s_d \equiv e(\mathcal{H})$, where $\mathcal{I}$ is an ideal of D, then for every fuzzy semisubnet $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ of the fuzzy net $(s_d)_{d \in D}$ we have $\mathcal{I}_\Lambda(\varphi) - \lim_{\lambda \in \Lambda} t_\lambda \equiv e(\mathcal{H})$.

(C′3) If $\mathcal{I} - \lim_{d \in D} s_d \equiv e(\mathcal{H})$, where $\mathcal{I}$ is a proper ideal of D, then there exists a fuzzy semisubnet $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ of the fuzzy net $(s_d)_{d \in D}$ such that $\mathcal{I}_0(\Lambda) - \lim_{\lambda \in \Lambda} t_\lambda \equiv e(\mathcal{H})$.

(C′4) Let D be a directed set and $\mathcal{I}_D$ a proper ideal of D. If the fuzzy net $(s_d)_{d \in D}$ does not $\mathcal{I}_D$-converge ($\mathcal{H}$) to e, then there exists a fuzzy semisubnet $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ of the fuzzy net $(s_d)_{d \in D}$ such that:

(i) No fuzzy semisubnet $(r_k)^{f}_{k \in K}$ of $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ $\mathcal{I}_K$-converges ($\mathcal{H}$) to e, for every proper ideal $\mathcal{I}_K$ of K.

(ii) $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal of Λ.

(C′5) We consider the following:

(i) D is a directed set.

(ii) $\mathcal{I}_0(D)$ is a proper ideal of D.

(iii) E_d is a directed set, for each $d \in D$.

(iv) $\mathcal{I}_0(E_d)$ is a proper ideal of E_d, for each $d \in D$.

(v) $(s(d,e))_{e\in E_d}$ is a fuzzy net in X for each $d \in D$.

(vi) $\mathcal{I}_0(D)-\lim_{d\in D} t_d \equiv e(\mathcal{H})$ (e is a fuzzy point in X), where $\mathcal{I}_0(E_d)-\lim_{e\in E_d} s(d,e) \equiv t_d(\mathcal{H})$ (t_d is a fuzzy point in X), for every $d \in D$.

Then, the fuzzy net $r : D \times \prod_{d\in D} E_d \to X$, where $r(d,f) = s(d, f(d))$, for every $(d,f) \in D \times \prod_{d\in D} E_d$, $\mathcal{I}_0(D \times \prod_{d\in D} E_d)$-converges $(\mathcal{H})$ to e.

(C$'$6) For each point $x \in X$ and real directed set $D \subseteq (0,1]$, if $r \leqslant \sup D$, then the fuzzy net $(x_d)_{d\in D}$ $\mathcal{I}_0(D)$-converges $(\mathcal{H})$ to x_r.

(C$'$7) If $(s_d)_{d\in D}$ is a fuzzy net in X, then $\mathcal{P}(D) - \lim_{d\in D} s_d \equiv e(\mathcal{H})$, for every fuzzy point $e \in X$.
The class of all fuzzy ideal convergence classes for X is denoted by $Con_I(X)$.

Remark 4.13. Let (X,δ) be a fuzzy topological space. Then, the class consisting of triads $((s_d)_{d\in D}, e, \mathcal{I})$, where $(s_d)_{d\in D}$ is a fuzzy net in X, e is a fuzzy point in X, $\mathcal{I}$ is an ideal of D and $(s_d)_{d\in D}$ $\mathcal{I}$-converges to X, relative to δ, is a fuzzy ideal convergence class, denoted by $\Phi(\delta)$, since it satisfies all the conditions of Definition 4.12. We say that the fuzzy topology δ *generates* the fuzzy ideal convergence class $\Phi(\delta)$.

Proposition 4.14. *Let Ω be a family of fuzzy points in X and $A = \vee\Omega$. Let the class of triads $\mathcal{H}$ satisfy the conditions (C$'$3), (C$'$5) and (C$'$6). If a fuzzy net $s = (s_d)_{d\in D}$ in A $\mathcal{I}$-converges $(\mathcal{H})$ to e, where $\mathcal{I}$ is a proper ideal of D, then there exists a fuzzy net $\overline{s} = (\overline{s}_k)_{k\in K}$ that consists of fuzzy points in Ω and $\mathcal{I}_0(K)$-converges $(\mathcal{H})$ to e.*

Proof. Suppose that a fuzzy net $s = (s_d)_{d\in D}$ in A $\mathcal{I}$-converges $(\mathcal{H})$ to e, where $\mathcal{I}$ is a proper ideal of D. Then by condition (C$'$3) there exists a fuzzy semisubnet $t = (t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the fuzzy net $(s_d)_{d\in D}$ such that $\mathcal{I}_0(\Lambda)$-converges $(\mathcal{H})$ to e. From this point we continue as in the proof of Proposition 2.5. For each $\lambda \in \Lambda$, let y and r be the support point and the membership grade, respectively, of t_λ i.e. $t_\lambda = y_r$. Since $y_r \in_{fuzzy} A$ we can consider the family of fuzzy points $\{y_{r_n}\} \subseteq \Omega$, such that $y_r \leq_{fuzzy} \vee\{y_{r_n}\}$. If we denote by E^λ the set of reals r_n, we get a fuzzy net $t^\lambda = (y_{r_n})_{r_n\in E^\lambda}$. Since $r \leq \sup E^\lambda$, by condition (C$'$6) we have that t_λ $\mathcal{I}_0(E^\lambda)$-converges $(\mathcal{H})$ to $y_r = t_\lambda$. Now condition (C$'$5) applies and we get the desired fuzzy net. □

The following theorem sets up a one-to-one correspondence between the fuzzy topologies for a non-empty set X and the fuzzy ideal convergence classes on it.

Theorem 4.15 (fuzzy ideal convergence classes theorem). *Let $\mathcal{H}$ be a fuzzy ideal convergence class for a non-empty set X. We consider a map $\mathrm{cl} : I^X \to I^X$ induced as follows: for each $A \in I^X$, we define $\mathrm{cl}(A) \in I^X$ to be such that a fuzzy point $e \in_{fuzzy} \mathrm{cl}(A)$ iff for some fuzzy net $(s_d)_{d\in D}$ in A and a proper ideal $\mathcal{I}$ of the directed set D, $(s_d)_{d\in D}$ $\mathcal{I}$-converges $(\mathcal{H})$ to e i.e. $(s,e,\mathcal{I}) \in \mathcal{H}$. Then, cl is a fuzzy closure operator for a fuzzy topology denoted by $\Psi(\mathcal{H})$ on X and $((s_d)_{d\in D}, e, \mathcal{I}) \in \mathcal{H}$ iff $(s_d)_{d\in D}$ $\mathcal{I}$-converges to e with respect to $\Psi(\mathcal{H})$.*

Proof. We first prove that the map cl is well defined. Let $\mathcal{C}(A)$ be the set of all fuzzy points e in X for which there exists a fuzzy net $s = (s_d)_{d\in D}$ in A and $\mathcal{I}$ be a proper ideal of D such that $(s, e, \mathcal{I}) \in \mathcal{H}$ and put $\mathrm{cl}(A) = \vee\mathcal{C}(A)$. It is enough to prove that for each fuzzy point $e \in_{fuzzy} \mathrm{cl}(A)$ there exists a fuzzy net $s = (s_d)_{d\in D}$ in A such that s $\mathcal{I}$-converges $(\mathcal{H})$ to e, where $\mathcal{I}$ is a proper ideal of D. Let $e \in_{fuzzy} \mathrm{cl}(A)$ and denote the support point and the membership grade of e, by x and $\lambda \in (0, 1]$, respectively i.e. $e = x_\lambda$. Let R be the set of all $r \in (0, 1]$ for which there exists a fuzzy net $s^r = (s^r(m))_{m\in M^r}$ in A, and $\mathcal{I}$ be a proper ideal of M^r, such that $(s^r, x_r, \mathcal{I}) \in \mathcal{H}$. Clearly $R \neq \varnothing$ and $\sup R \geqslant \lambda$. Therefore, from (C$'$6) the fuzzy net $(x_r)_{r\in R}$ $\mathcal{I}_0(R)$-converges $(\mathcal{H})$ to e. Now from the definition of R there exists a fuzzy net $s^r = (s^r(m))_{m\in M^r}$ in A, and a proper ideal $\mathcal{I}$ of M^r, such that s^r $\mathcal{I}$-converges $(\mathcal{H})$ to x_r, for each $r \in R$. By condition (C$'$3) of the Definition 4.12, there exists a fuzzy semisubnet $t^r = (t^r(n))_{n\in N^r}$ of s^r, in A, such that t^r $\mathcal{I}_0(N^r)$-converges $(\mathcal{H})$ to x_r, for each $r \in R$. It follows from (C$'$5) that there exists a fuzzy net in A such that $\mathcal{I}_0(R \times \prod_{r\in R} N^r)$-converges $(\mathcal{H})$ to e.

Next we prove that cl is a fuzzy closure operator on X.

(FCO1) is clear.

(FCO2) Let $A \in I^X$ and $e \in_{fuzzy} A$. We consider the fuzzy net $(s_d)_{d\in D}$ in A, where $s_d = e$ for every $d \in D$. By condition (C$'$1) of the Definition 4.12, we have $\mathcal{I} - \lim_{d\in D} s_d \equiv e(\mathcal{H})$ for every proper ideal $\mathcal{I}$ of the directed set D. Therefore, $e \in_{fuzzy} \mathrm{cl}(A)$.

(FCO3) Let $A, B \in I^X$. Then, clearly $\mathrm{cl}(A) \leq_{fuzzy} \mathrm{cl}(A \vee B)$ and $\mathrm{cl}(B) \leq_{fuzzy} \mathrm{cl}(A \vee B)$. Therefore, $\mathrm{cl}(A) \vee \mathrm{cl}(B) \leq_{fuzzy} \mathrm{cl}(A \vee B)$. We prove that $\mathrm{cl}(A \vee B) \leq_{fuzzy} \mathrm{cl}(A) \vee \mathrm{cl}(B)$. Let $e \in_{fuzzy} \mathrm{cl}(A \vee B)$. Then there exists a fuzzy net $(s_d)_{d\in D}$ in $A \vee B$ and a proper ideal $\mathcal{I}$ of the directed set D such that $(s_d)_{d\in D}$ $\mathcal{I}$-converges $(\mathcal{H})$ to e. Denote

$$D_A = \{d \in D : s_d \in_{fuzzy} A\} \text{ and } D_B = \{d \in D : s_d \in_{fuzzy} B\}.$$

Then, we have $D_A \notin \mathcal{I}$ or $D_B \notin \mathcal{I}$, otherwise $D_A \cup D_B = D \in \mathcal{I}$, which is a contradiction. Without loss of generality assume that $D_A \notin \mathcal{I}$. We can consider the set D_A as directed by $\geqslant_A$ (regardless $\geqslant_D$). Let the following function and fuzzy net

(i) $\varphi_A : D_A \to D$, where $\varphi_A(d) = d$, for every $d \in D_A$.

(ii) $(t^A_d)_{d\in D_A}$, where $t^A_d = s_{\varphi_A(d)}$.

Obviously, $(t^A_d)_{d\in D_A}$ is a fuzzy semisubnet of $(s_d)_{d\in D}$ in A. Since $(s_d)_{d\in D}$ $\mathcal{I}$-converges $(\mathcal{H})$ to e, by condition (C$'$2) of the Definition 4.12 we have that $(t^A_d)_{d\in D_A}$ $\mathcal{I}_{D_A}(\varphi_A)$-converges $(\mathcal{H})$ to e. Moreover, the ideal $\mathcal{I}_{D_A}(\varphi_A)$ of D_A is proper, since $\varphi_A(D_A) = D_A \notin \mathcal{I}$. Thus, $e \in_{fuzzy} \mathrm{cl}(A)$ and therefore $e \in_{fuzzy} \mathrm{cl}(A) \vee \mathrm{cl}(B)$.

(FCO4) We prove that $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$. We have $A \leq_{fuzzy} \mathrm{cl}(A)$ and so $\mathrm{cl}(A) \leq_{fuzzy} \mathrm{cl}(\mathrm{cl}(A))$. We prove that $\mathrm{cl}(\mathrm{cl}(A)) \leq_{fuzzy} \mathrm{cl}(A)$. Let $e \in_{fuzzy} \mathrm{cl}(\mathrm{cl}(A))$. Then, there exists a fuzzy net $t = (t_d)_{d\in D}$ in $\mathrm{cl}(A)$ and a proper ideal $\mathcal{I}_D$ of the directed set D such that $(t_d)_{d\in D}$ $\mathcal{I}_D$-converges $(\mathcal{H})$ to e. Then, from (C$'$3) there is a fuzzy semisubnet $(w_n)_{n\in N}$ of t (in $\mathrm{cl}(A)$), such that $\mathcal{I}_0(N)$-converges $(\mathcal{H})$ to e. Therefore, for every $n \in N$ there exist a directed set E_n, a fuzzy net $(s(n, \epsilon))_{\epsilon\in E_n}$ in A and a proper ideal $\mathcal{I}_{E_n}$ of the directed set E_n such that $(s(n, \epsilon))_{\epsilon\in E_n}$ $\mathcal{I}_{E_n}$-converges $(\mathcal{H})$ to w_n.

Then, from (C$'$3) and for every $n \in N$ there is a fuzzy semisubnet $(\sigma(n,m))_{m \in M_n}$ of $(s(n,\epsilon))_{\epsilon \in E_n}$, in A, such that $\mathcal{I}_0(M_n)$-converges $(\mathcal{H})$ to w_n. By condition (C$'$5) of the Definition 4.12 there exists a fuzzy net in A, such that $\mathcal{I}_0(N \times \prod_{n \in N} M_n)$-converges $(\mathcal{H})$ to e. Hence, $e \in_{fuzzy} \mathrm{cl}(A)$.

By Proposition 4.11 and condition (C$'$7) of the Definition 4.12, it is enough to show that ideal convergence $(\mathcal{H})$, over proper ideals, coincides with ideal convergence with respect to the fuzzy topology $\Psi(\mathcal{H})$.

We prove that if the fuzzy net $(s_d)_{d \in D}$ $\mathcal{I}_D$-converges $(\mathcal{H})$ to the fuzzy point e in X, where $\mathcal{I}_D$ is a proper ideal of D, then $(s_d)_{d \in D}$ $\mathcal{I}_D$-converges to e with respect to $\Psi(\mathcal{H})$. Suppose that $(s_d)_{d \in D}$ $\mathcal{I}_D$-converges $(\mathcal{H})$ to e and does not $\mathcal{I}_D$-converge to e with respect to $\Psi(\mathcal{H})$. By Proposition 4.8 and its proof there exist an open Q-neighborhood U of e and a fuzzy semisubnet $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ of the fuzzy net $(s_d)_{d \in D}$ such that:

(i) $\Lambda \subseteq D$.

(ii) $\varphi(\lambda) = \lambda$, for every $\lambda \in \Lambda$.

(iii) No fuzzy semisubnet $(r_k)^{f}_{k \in K}$ of $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ $\mathcal{I}_K$-converges to e with respect to $\Psi(\mathcal{H})$, for every proper ideal $\mathcal{I}_K$ of K.

(iv) $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal of Λ.

(v) $t_\lambda \in_{fuzzy} U'$, for every $\lambda \in \Lambda$.

Since $(s_d)_{d \in D}$ $\mathcal{I}_D$-converges $(\mathcal{H})$ to e, by condition (C$'$2) of the Definition 4.12, $(t_\lambda)_{\lambda \in \Lambda}$ $\mathcal{I}_\Lambda(\varphi)$-converges $(\mathcal{H})$ to e. Therefore, $e \in_{fuzzy} \mathrm{cl}(U') = U'$. This contradicts the fact that e is quasi-coincident with U.

We prove that if the fuzzy net $(s_d)_{d \in D}$ $\mathcal{I}_D$-converges to the fuzzy point e in X, with respect to $\Psi(\mathcal{H})$, where $\mathcal{I}_D$ is a proper ideal of D, then $(s_d)_{d \in D}$ $\mathcal{I}_D$-converges $(\mathcal{H})$ to e. Suppose that $(s_d)_{d \in D}$ $\mathcal{I}_D$-converges to e with respect to $\Psi(\mathcal{H})$, where $\mathcal{I}_D$ is a proper ideal of D, and does not $\mathcal{I}_D$-converge $(\mathcal{H})$ to e. By condition (C$'$4) of the Definition 4.12, there exists a fuzzy semisubnet $t = (t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ of the fuzzy net $(s_d)_{d \in D}$ such that no fuzzy semisubnet $(r_k)^{f}_{k \in K}$ of t $\mathcal{I}_K$-converges $(\mathcal{H})$ to e, for every proper ideal $\mathcal{I}_K$ of K, where $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal of Λ. From Proposition 4.6, $(t_\lambda)_{\lambda \in \Lambda}$ $\mathcal{I}_\Lambda(\varphi)$-converges to e, with respect to $\Psi(\mathcal{H})$. Set $A = \vee\{t_\lambda : \lambda \in \Lambda\}$. The fuzzy net t is in A so by Proposition 4.4, $e \in_{fuzzy} \mathrm{cl}(A)$. By the definition of $\mathrm{cl}(A)$, there exists a fuzzy net $w = (w_n)_{n \in N}$ in A such that w $\mathcal{I}_N$-converges $(\mathcal{H})$ to e, where $\mathcal{I}_N$ is a proper ideal of N. By Proposition 4.14, there exists a fuzzy net $\overline{w} = (\overline{w}_k)_{k \in K}$ that consists of fuzzy points in $\{t_\lambda : \lambda \in \Lambda\}$ (so $\overline{w}$ is a fuzzy semisubnet of t), that $\mathcal{I}_0(K)$-converges $(\mathcal{H})$ to e, which is a contradiction. □

Corollary 4.16. *Let $\mathcal{H}$ be a fuzzy ideal convergence class and δ be a fuzzy topology for a non-empty set X. We have the following:*
(i) $\Phi(\Psi(\mathcal{H})) = \mathcal{H}$ and (ii) $\Psi(\Phi(\delta)) = \delta$.
Therefore, there exists a bijective map between the set of all fuzzy topologies δ for X and the set of all fuzzy ideal convergence classes $\mathcal{H}$ for X. Moreover, this map is order-reversing, i.e. if $\delta_1 \supseteq \delta_2$, then $\Phi(\delta_1) \subseteq \Phi(\delta_2)$.

Proof. (i) It follows directly from the fact that a fuzzy net $(s_d)_{d\in D}$ $\mathcal{I}_D$-converges $(\mathcal{H})$ to the fuzzy point e in X, where $\mathcal{I}_D$ is a proper ideal of D, iff $(s_d)_{d\in D}$ $\mathcal{I}_D$-converges to e with respect to $\Psi(\mathcal{H})$, which was proved in the Theorem 4.15.

(ii) Let δ be a fuzzy topology on X. By Proposition 4.4, the fuzzy closure operator induced by $\Phi(\delta)$ coincides with the one associated with δ (note that the converse part of Proposition 4.4 also holds). Therefore $\Psi(\Phi(\delta)) = \delta$.

Finally, if $\delta_1 \supseteq \delta_2$ then it is clear that $\Phi(\delta_1) \subseteq \Phi(\delta_2)$. □

Proposition 4.17. *Let X be a non-empty set. There exists a one-to-one map m of $Con_I(X)$ onto $Con(X)$ such that for every $\mathcal{H} \in Con_I(X)$ the following properties hold:*

(i) $\Psi(\mathcal{H}) = \psi(m(\mathcal{H}))$,

(ii) $m(\mathcal{H})$ can be considered as a subclass of the class $\mathcal{H}$ in the sense that there exists a one-to-one map $\varepsilon : m(\mathcal{H}) \to \mathcal{H}$ and each $((s_d)_{d\in D}, e) \in m(\mathcal{H})$ is identified with $\varepsilon((s_d)_{d\in D}, e) \in \mathcal{H}$.

Proof. We define a map $m : Con_I(X) \to Con(X)$ as follows:

$$m(\mathcal{H}) = \{((s_d)_{d\in D}, e) : ((s_d)_{d\in D}, e, \mathcal{I}_0(D)) \in \mathcal{H}\}, \text{ for every } \mathcal{H} \in Con_I(X).$$

The map m is well-defined, that is, for each $\mathcal{H} \in Con_I(X)$ the class $m(\mathcal{H})$ is a fuzzy convergence class. Indeed, we have the following equivalences: $((s_d)_{d\in D}, e) \in m(\mathcal{H})$ iff $((s_d)_{d\in D}, e, \mathcal{I}_0(D)) \in \mathcal{H}$ iff $\mathcal{I}_0(D) - \lim_{d\in D} s_d = e$ with respect to $\Psi(\mathcal{H})$ iff $\lim_{d\in D} s_d = e$ with respect to $\Psi(\mathcal{H})$. Hence, $m(\mathcal{H})$ is the fuzzy convergence class generated from the fuzzy topology $\Psi(\mathcal{H})$, that is $m(\mathcal{H}) = \phi(\Psi(\mathcal{H}))$ (see Theorem 2.4). We shall prove that the map m is onto. Let $\mathcal{G} \in Con(X)$ and $\Phi(\psi(\mathcal{G}))$ be the fuzzy ideal convergence class generated from the topology $\psi(\mathcal{G})$ (see Remark 4.13). Then, $m(\Phi(\psi(\mathcal{G}))) = \mathcal{G}$.

(i) Let $\mathcal{H} \in Con_I(X)$. We have $\psi(m(\mathcal{H})) = \psi(\phi(\Psi(\mathcal{H}))) = \Psi(\mathcal{H})$.

(ii) Let $\mathcal{H} \in Con_I(X)$. Then, $((s_d)_{d\in D}, e) \in m(\mathcal{H})$ iff $((s_d)_{d\in D}, e, \mathcal{I}_0(D)) \in \mathcal{H}$. Therefore, we can define a one-to-one map $\varepsilon : m(\mathcal{H}) \to \mathcal{H}$ as follows: $\varepsilon((s_d)_{d\in D}, e) = ((s_d)_{d\in D}, e, \mathcal{I}_0(D))$. So, we can consider the class $m(\mathcal{H})$ as a subclass of the class $\mathcal{H}$ by identifying $m(\mathcal{H})$ with its image $\varepsilon(m(\mathcal{H}))$.

Finally, we prove that m is one-to-one. Let $\mathcal{H}_1, \mathcal{H}_2 \in Con_I(X)$ and assume that $m(\mathcal{H}_1) = m(\mathcal{H}_2)$. Then, $\psi(m(\mathcal{H}_1)) = \psi(m(\mathcal{H}_2))$ and so, by property (i), $\Psi(\mathcal{H}_1) = \Psi(\mathcal{H}_2)$. By Theorem 4.15 we have $\mathcal{H}_1 = \mathcal{H}_2$. □

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University of Patras, Department of Mathematics, 265 00 Patras, Greece

E-mail: georgiou@math.upatras.gr

University of Thessaly, Department of Physics, 35 100 Lamia, Greece

E-mail: thanasismeg13@gmail.com

University of Patras, Department of Mathematics, 265 00 Patras, Greece

E-mail: gprinos161168@yahoo.gr