MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
75, 3 (2023), 147–156
September 2023

research paper
оригинални научни рад
DOI: 10.57016/MV-nr25PM82

CONSTRUCTION OF THE TYPE 2 DEGENERATE POLY-EULER POLYNOMIALS AND NUMBERS

Waseem A. Khan

Abstract. In this paper, we introduce type 2 degenerate poly-Euler polynomials and numbers, briefly called degenerate poly-Euler polynomials and numbers, by using the modified degenerate polyexponential function and derive several properties on these polynomials and numbers. In the last section, we also consider the type 2 degenerate unipoly-Euler polynomials attached to an arithmetic function, by using the modified degenerate polyexponential function and investigate some identities of these polynomials. In particular, we give some new explicit expressions and identities of degenerate unipoly polynomials related to special numbers and polynomials.

1. Introduction

Carlitz [1, 2] initiated a study of degenerate versions of some special numbers and polynomials, namely the degenerate Bernoulli and Euler polynomials. In recent years, the idea of studying degenerate versions of many special polynomials and numbers regained interest of some mathematicians, and many interesting results were found (see [3, 6, 8]). They have been explored by employing several different tools such as combinatorial methods, generating functions, p-adic analysis, umbral calculus techniques, differential equations, and probability theory.

The aim of this paper is to introduce the type 2 degenerate poly-Euler polynomials by means of the modified degenerate polyexponential functions and to study their properties including their explicit expressions and differences. Here we note that these polynomials are slight modifications of the previously studied ones under the same name.

The outline of this paper is as follows. First, we recall the Carlitz degenerate Euler polynomials, the degenerate exponential functions, the modified degenerate polyexponential, and the degenerate Stirling numbers of the first and second kind. In Section 2,

2020 Mathematics Subject Classification: 11B73, 11B83, 05A19
Keywords and phrases: Modified degenerate polyexponential functions; degenerate poly-Euler polynomials; degenerate unipoly functions; degenerate unipoly-Euler polynomials.

we introduce the type 2 degenerate poly-Euler polynomials by means of the modified degenerate polyexponential functions. Note that they reduce to the Carlitz degenerate Euler polynomials when $k = 1$. We express the generating function of the degenerate poly-Euler polynomials as an iterated integral, from which we find an explicit expression for these polynomials when $k = 2$. We find explicit expressions for the degenerate poly-Euler polynomials in terms of the Carlitz degenerate Euler polynomials and the degenerate Stirling numbers of the first and second kind. Also, we find certain expressions for certain differences of the degenerate poly-Euler polynomials and Changhee polynomials. In Section 3, we introduce the degenerate unipoly-Euler polynomials by means of the degenerate unipoly functions. We find explicit expressions for the degenerate unipoly-Euler polynomials in terms of the Carlitz degenerate Euler polynomials, degenerate Daehee polynomials and the degenerate Stirling numbers of the first and second kind.

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by $e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}$ $e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{\frac{1}{\lambda}}$ (see [6,8]). Here we note that $e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}$, where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$ $(n \geq 1)$.

Carlitz considered the degenerate Euler polynomials which are given by (see [1,2]):

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1}$$

When $x = 0$, $E_{n,\lambda} = E_{n,\lambda}(0)$ are called degenerate Euler numbers.

The modified degenerate polyexponential function is defined by (see [8]):

$$\mathrm{Ei}_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\lambda}x^n}{(n-1)!\,n^k} \quad (k \in \mathbb{Z}). \tag{2}$$

Note that $\mathrm{Ei}_{1,\lambda}(x) = \sum\limits_{n=1}^{\infty} \frac{(1)_{n,\lambda}x^n}{n!} = e_\lambda(x) - 1$. The degenerate poly-Genocchi polynomials are defined by (see [8]):

$$\frac{2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)}{e_\lambda(t)+1}e_\lambda^n(t) = \sum_{n=0}^{\infty} G_{n,\lambda}^{(k)}(n)\frac{t^n}{n!} \quad (k \in \mathbb{Z}).$$

In the case when $x = 0$, $G_{n,\lambda}^{(k)} = G_{n,\lambda}^{(k)}(0)$ are called the degenerate poly-Genocchi numbers. Lee-Kim-Jang [10] introduced the type 2 degenerate poly-Euler polynomials defined by

$$\frac{\mathrm{Ei}_k(\log(1+2t))}{t(e_\lambda(t)+1)}e_\lambda^x(t) = \sum_{n=0}^{\infty} E_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.$$

When $x = 0$, $E_{n,\lambda}^{(k)} = E_{n,\lambda}^{(k)}(0)$ are called the type 2 degenerate poly-Euler numbers.

The Changhee polynomials $Ch_n(x)$ are defined by (see [5]):

$$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}. \tag{3}$$

In the case when $x = 0, Ch_n = Ch_n(0)$ are called the Changhee numbers.

The degenerate Daehee polynomials $D_{n,\lambda}(x)$ are defined by (see [9]):

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!}. \tag{4}$$

When $x = 0, D_{n,\lambda} = D_{n,\lambda}(0)$ are called the degenerate Daehee numbers.

The degenerate Bernoulli polynomials of the second kind are defined by (see [6]):

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!}. \tag{5}$$

For $x = 0$, $b_{n,\lambda} = b_{n,\lambda}(0)$ are called degenerate Bernoulli numbers of the second kind. In [7], the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l \quad (n \geq 0). \tag{6}$$

As an inversion formula of (6), the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda} \quad (n \geq 0). \tag{7}$$

From (6) and (7), we note that

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad \text{(see [7,8])} \tag{8}$$

and

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad \text{(see [6,8,9])},$$

where $\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1)$ is the compositional inverse of $e_\lambda(t)$ satisfying $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t$.

2. Type 2 degenerate poly-Euler polynomials and numbers

In this section, we define the type 2 degenerate Euler numbers and polynomials by using the modified degenerate polyexponential function which are called the type 2 degenerate poly-Euler numbers and polynomials as follows.

For $k \in \mathbb{Z}$, we consider the type 2 degenerate Euler polynomials which are called the type 2 degenerate poly-Euler polynomials and given by

$$\frac{2\mathrm{Ei}_{k,\lambda}(\log_\lambda(1+t))}{t(e_\lambda(t)+1)}e_\lambda^x(t) = \sum_{n=0}^{\infty} E_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}. \tag{9}$$

In the special case when $x = 0$, $E_{n,\lambda}^{(k)} = E_{n,\lambda}^{(k)}(0)$ are called the type 2 degenerate poly-Euler numbers. Note that

$$\sum_{n=0}^{\infty} E_{n,\lambda}^{(1)}(x)\frac{t^n}{n!} = \frac{2e_\lambda^x(t)}{t(e_\lambda(t)+1)}\mathrm{Ei}_{1,\lambda}\left(\log_\lambda(1+t)\right) = \frac{2e_\lambda^x(t)}{e_\lambda(t)+1} = \sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!}. \quad (10)$$

From (9), for $n \geq 0$, we have $\lim_{\lambda\to 0} E_{n,\lambda}^{(k)}(x) = E_n^{(k)}(x)$.

THEOREM 2.1. *Let $n \geq 0$. Then*

$$E_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n}\binom{n}{m}E_{m,\lambda}^{(k)}(x)_{n-m,\lambda}.$$

Proof. From (9), we note that

$$\sum_{n=0}^{\infty} E_{n,\lambda}^{(k)}(x)\frac{t^n}{n!} = \frac{2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)}{t(e_\lambda(t)+1)}e_\lambda^x(t)$$

$$= \sum_{m=0}^{\infty} E_{m,\lambda}^{(k)}\frac{t^m}{m!}\sum_{n=0}^{\infty}\frac{(x)_{n,\lambda}}{n!}t^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}E_{m,\lambda}^{(k)}(x)_{n-m,\lambda}\right)\frac{t^n}{n!}. \qquad \square$$

THEOREM 2.2. *For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$E_{n,\lambda}^{(k)} = \sum_{m=0}^{n}\binom{n}{m}\sum_{m_1,m_2,\cdots,m_{k-1}=m}^{\infty}\binom{m}{m_1,m_2,\cdots,m_k}\frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\frac{b_{n_2,\lambda}(\lambda-1)}{m_1+m_2+1}\times$$

$$\cdots\times\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}E_{n-m,\lambda}.$$

Proof. From (2), we see that

$$\frac{d}{dx}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+x)\right) = \frac{d}{dx}\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{(m-1)!m^k}\left(\log_\lambda(1+x)\right)^m$$

$$= \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)}\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{(m-1)!m^{k-1}}\left(\log_\lambda(1+x)\right)^{m-1} = \frac{(1+x)^{\lambda-1}}{\log_\lambda(1+x)}\mathrm{Ei}_{k-1,\lambda}\left(\log_\lambda(1+x)\right).$$

From here, for $k \geq 2$, we have

$$\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+x)\right) = \int_0^t \underbrace{\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t\cdots\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}}_{(k-2)-\text{times}}t\,dt\,dt\cdots dt. \quad (11)$$

By using (5), (9) and (11), we obtain

$$\sum_{n=0}^{\infty} E_{n,\lambda}^{(k)}\frac{x^n}{n!} = \frac{1}{x(e_\lambda(x)+1)}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+x)\right), \quad \text{i.e.}$$

$$\sum_{n=0}^{\infty} E_{n,\lambda}^{(k)}\frac{t^n}{n!} = \frac{2}{x(e_\lambda(x)+1)}\int_0^x \underbrace{\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t\cdots\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}\int_0^t\frac{(1+t)^{\lambda-1}}{\log_\lambda(1+t)}}_{(k-2)-\text{times}}t\,dt\,dt\cdots dt$$

$$=\frac{2x}{x(e_\lambda(x)+1)}\sum_{m_1,m_2,\cdots,m_{k-1}=m}^{\infty}\binom{m}{m_1,m_2,\cdots,m_k}\frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\frac{b_{n_2,\lambda}(\lambda-1)}{m_1+m_2+1}$$

$$\times\cdots\times\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\sum_{m_1,m_2,\cdots,m_{k-1}=m}^{\infty}\binom{m}{m_1,m_2,\cdots,m_k}\frac{b_{m_1,\lambda}(\lambda-1)}{m_1+1}\frac{b_{n_2,\lambda}(\lambda-1)}{m_1+m_2+1}\right.$$

$$\left.\times\cdots\times\frac{b_{m_{k-1},\lambda}(\lambda-1)}{m_1+\cdots+m_{k-1}+1}E_{n-m,\lambda}\right)\frac{x^n}{n!}.$$

Therefore by comparing the coefficients of t on both sides of the previous equation, we obtain the result. □

Theorem 2.3. *Let $k=2$. Then*

$$E_{n,\lambda}^{(2)}=\sum_{m=0}^{n}\binom{n}{m}\frac{b_{m,\lambda}(\lambda-1)}{m+1}E_{n-m,\lambda}=\sum_{m=0}^{n}\binom{n}{m}\frac{b_{n-m,\lambda}(\lambda-1)}{n-m+1}E_{m,\lambda}.$$

Proof. For $k=2$, by Theorem 2.2, we get

$$\sum_{n=0}^{\infty}E_{n,\lambda}^{(2)}\frac{x^n}{n!}=\frac{2}{x(e_\lambda(x)+1)}\int_0^x\frac{x}{\log_\lambda(1+x)}(1+x)^{\lambda-1}dx$$

$$=\frac{2}{x(e_\lambda(x)+1)}\int_0^x\sum_{m=0}^{\infty}b_{m,\lambda}(\lambda-1)\frac{x^m}{m!}dx=\frac{2}{e_\lambda(x)+1}\sum_{m=0}^{\infty}\frac{b_{m,\lambda}(\lambda-1)}{m+1}\frac{x^m}{m!}$$

$$=\left(\sum_{n=0}^{\infty}E_{n,\lambda}\frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{b_{m,\lambda}(\lambda-1)}{m+1}\frac{x^m}{m!}\right)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\frac{b_{m,\lambda}(\lambda-1)}{m+1}E_{n-m,\lambda}\right)\frac{x^n}{n!}.$$ □

Theorem 2.4. *For $n\geq 0$, we have*

$$E_{n,\lambda}^{(k)}(x)=\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\frac{(1)_{m+1,\lambda}}{(m+1)^{k-1}}\frac{S_{1,\lambda}(l+1,m+1)}{l+1}E_{n-l,\lambda}(x).$$

Proof. From (9), we note that

$$\sum_{n=0}^{\infty}E_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}=\frac{2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)}{t(e_\lambda(t)+1)}e_\lambda^x(t)=\frac{2}{e_\lambda(t)+1}e_\lambda^x(t)\frac{1}{t}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)$$

$$=\left(\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}\right)\frac{1}{t}\left(\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{(m-1)!m^k}(\log_\lambda(1+t))^m\right)$$

$$=\left(\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}\right)\frac{1}{t}\left(\sum_{m=0}^{\infty}\frac{(1)_{m+1,\lambda}}{(m+1)^{k-1}}\frac{1}{(m+1)!}(\log_\lambda(1+t))^{m+1}\right)$$

$$=\left(\sum_{n=0}^{\infty}E_{n,\lambda}(x)\frac{t^n}{n!}\right)\frac{1}{t}\left(\sum_{m=0}^{\infty}\frac{(1)_{m+1,\lambda}}{(m+1)^{k-1}}\sum_{l=m+1}^{\infty}S_{1,\lambda}(l,m+1)\frac{t^l}{l!}\right)$$

$$= \left(\sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!} \right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{(1)_{m+1,\lambda}}{(m+1)^{k-1}} \frac{S_{1,\lambda}(l+1, m+1)}{l+1} \frac{t^l}{l!} \right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{(1)_{m+1,\lambda}}{(m+1)^{k-1}} \frac{S_{1,\lambda}(l+1, m+1)}{l+1} E_{n-l,\lambda}(x) \right) \frac{t^n}{n!}.$$

By, comparing the coefficients of t on both sides of the previous equation, we obtain the result. □

THEOREM 2.5. *For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$E_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} (x)_m S_{2,\lambda}(l, m) E_{n-l,\lambda}^{(k)}.$$

Proof. From (8) and (9), we have

$$\sum_{n=0}^{\infty} E_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} = \frac{2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)}{t(e_\lambda(t)+1)} e_\lambda^x(t) = \left(\frac{2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)}{t(e_\lambda(t)+1)} \right) (e_\lambda(t) - 1 + 1)^x$$
$$= \left(\frac{2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)}{t(e_\lambda(t)+1)} \right) \left(\sum_{m=0}^{\infty} \binom{x}{m} (e_\lambda(t) - 1)^m \right)$$
$$= \left(\frac{2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)}{t(e_\lambda(t)+1)} \right) \left(\sum_{m=0}^{\infty} (x)_m \sum_{m=l}^{\infty} S_{2,\lambda}(l, m) \frac{t^l}{l!} \right)$$
$$= \left(\sum_{n=0}^{\infty} E_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} (x)_m S_{2,\lambda}(l, m) \frac{t^l}{l!} \right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} (x)_m S_{2,\lambda}(l, m) E_{n-l,\lambda}^{(k)} \right) \frac{t^n}{n!}.$$

By comparing the coefficients of t^n on both sides, we obtain the result. □

THEOREM 2.6. *Let $k \in \mathbb{Z}$ and $n \geq 1$. Then*

$$\sum_{m=0}^{n-1} \binom{n-1}{m} E_{n-m-1,\lambda}^{(k)} (1)_{m,\lambda} + E_{n-1,\lambda}^{(k)} = \frac{2}{n} \sum_{m=1}^{n} \frac{(1)_{m,\lambda}}{m^{k-1}} S_{1,\lambda}(n, m).$$

Proof. From (9), we observe that

$$2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right) = t(e_\lambda(t)+1) \sum_{n=0}^{\infty} E_{n,\lambda}^{(k)} \frac{t^n}{n!} = t \left(\sum_{m=0}^{\infty} \frac{(1)_{m,\lambda} t^m}{m!} + 1 \right) \sum_{n=0}^{\infty} E_{n,\lambda}^{(k)} \frac{t^n}{n!}$$
$$= t \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} E_{n-m,\lambda}^{(k)} (1)_{m,\lambda} + E_{n,\lambda}^{(k)} \right) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} E_{n-m,\lambda}^{(k)} (1)_{m,\lambda} + E_{m,\lambda}^{(k)} \right) \frac{t^{n+1}}{n!}$$

$$=\sum_{n=1}^{\infty}\left(\sum_{m=0}^{n-1}\binom{n-1}{m}E^{(k)}_{n-m-1,\lambda}(1)_{m,\lambda}+E^{(k)}_{n-1,\lambda}\right)\frac{t^n}{(n-1)!}$$

$$=\sum_{n=1}^{\infty}n\left(\sum_{m=0}^{n-1}\binom{n-1}{m}E^{(k)}_{n-m-1,\lambda}(1)_{m,\lambda}+E^{(k)}_{n-1,\lambda}\right)\frac{t^n}{n!}. \tag{12}$$

On the other hand,

$$2\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)=2\sum_{m=1}^{\infty}\frac{(1)_{m,\lambda}}{m^{k-1}}\frac{1}{m!}(\log_\lambda(1+t))^m$$

$$=2\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(1)_{m,\lambda}}{m^{k-1}}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}. \tag{13}$$

Therefore, by equations (12) and (13), we get the result. □

Theorem 2.7. *Let $n\geq 0$. Then*

$$Ch_n(x)=\sum_{m=0}^{n}E_{m,\lambda}(x)S_{1,\lambda}(n,m).$$

Proof. Replacing t by $\log_\lambda(1+t)$ in (1) and using (3), we get

$$\frac{2}{2+t}(1+t)^x=\sum_{m=0}^{\infty}E_{m,\lambda}(x)\frac{(\log_\lambda(1+t))^m}{m!}=\sum_{m=0}^{\infty}E_{m,\lambda}(x)\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}E_{m,\lambda}(x)S_{1,\lambda}(n,m)\frac{t^n}{n!}. \tag{14}$$

On the other hand,

$$\frac{2}{2+t}(1+t)^x=\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!}. \tag{15}$$

Therefore, by (14) and (15), we obtain the result. □

3. Type 2 degenerate unipoly-Euler numbers and polynomials

Let p be any arithmetic function which is a real or complex valued function defined on the set of positive integers $\mathbb{N}$. Kim-Kim [4] defined a unipoly function attached to polynomials $p(x)$ by $u_k(x|p)=\sum_{n=1}^{\infty}\frac{p(n)}{n^k}x^n$, $(k\in\mathbb{Z})$. Moreover, $u_k(x|1)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}=\mathrm{Li}_k(x)$ is the ordinary polylogarithm function.

The degenerate unipoly function attached to polynomials $p(x)$ is given as follows $u_{k,\lambda}(x|p)=\sum_{i=1}^{\infty}p(i)\frac{(1)_{i,\lambda}}{i^k}x^i$ (see [3]). It is worthy to note that $u_{k,\lambda}(x|1/\Gamma)=\mathrm{Ei}_{k,\lambda}(x)$ is the modified degenerate polyexponential function.

Now, we define the type 2 degenerate unipoly-Euler polynomials attached to polynomials $p(x)$ by

$$\frac{2u_{k,\lambda}\left(\log_\lambda(1+t)|p\right)}{t(e_\lambda(t)+1)}e_\lambda^x(t)=\sum_{n=0}^{\infty}E_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!}. \tag{16}$$

In the case when $x=0$, $E_{n,\lambda,p}^{(k)}=E_{n,\lambda,p}^{(k)}(0)$ are called the type 2 degenerate unipoly-Euler numbers attached to p.

From (16), we see

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,\lambda,1/\Gamma}^{(k)}\frac{t^n}{n!}&=\frac{2}{t(e_\lambda(t)+1)}u_{k,\lambda}\left(\log_\lambda(1+t)|1/\Gamma\right)=\frac{2}{t(e_\lambda(t)+1)}\sum_{r=1}^{\infty}\frac{(1)_{r,\lambda}(\log_\lambda(1+t))^r}{r^k(r-1)!}\\&=\frac{2}{t(e_\lambda(t)+1)}\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)=\sum_{n=0}^{\infty}E_{n,\lambda}^{(k)}\frac{t^n}{n!}.\end{aligned} \tag{17}$$

Thus, by (17), we have $E_{n,\lambda,1/\Gamma}^{(k)}=E_{n,\lambda}^{(k)}$

THEOREM 3.1. *Let $n\geq 0$ and $k\in\mathbb{Z}$. Then*

$$E_{n,\lambda,p}^{(k)}=\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{m!p(m)(1)_{m,\lambda}}{m^k}\frac{S_{1,\lambda}(l+1,m)}{l+1}E_{n-l,\lambda}.$$

Proof. From (16), we have

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,\lambda,p}^{(k)}\frac{t^n}{n!}&=\frac{2}{t(e_\lambda(t)+1)}u_{k,\lambda}\left(\log_\lambda(1+t)|p\right)\\&=\frac{2}{t(e_\lambda(t)+1)}\sum_{m=1}^{\infty}\frac{p(m)(1)_{m,\lambda}}{m^k}(\log_\lambda(1+t))^m\\&=\frac{2}{t(e_\lambda(t)+1)}\sum_{m=1}^{\infty}\frac{m!p(m)(1)_{m,\lambda}}{m^k}\frac{1}{m!}(\log_\lambda(1+t))^m\\&=\frac{2}{t(e_\lambda(t)+1)}\sum_{m=1}^{\infty}\frac{m!p(m)(1)_{m,\lambda}}{m^k}\sum_{l=m}^{\infty}S_{1,\lambda}(l,m)\frac{t^l}{l!}\\&=\left(\frac{2}{t(e_\lambda(t)+1)}\right)\sum_{l=1}^{\infty}\left(\sum_{m=1}^{l}\frac{m!p(m)(1)_{m,\lambda}}{m^k}\right)S_{1,\lambda}(l,m)\frac{t^l}{l!}\\&=\left(\sum_{n=0}^{\infty}E_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\left(\sum_{m=1}^{l+1}\frac{m!p(m)(1)_{m,\lambda}}{m^k}\frac{S_{1,\lambda}(l+1,m)}{l+1}\right)\right)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{m!p(m)(1)_{m,\lambda}}{m^k}\frac{S_{1,\lambda}(l+1,m)}{l+1}E_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned}$$

□

THEOREM 3.2. *Let n be a nonnegative integer and $k\in\mathbb{Z}$. Then*

$$E_{n,\lambda,p}^{(k)}(x)=\sum_{q=0}^{n}\sum_{i=0}^{q}\binom{n}{q}(x)_iS_{2,\lambda}(q,i)E_{n-q,\lambda,p}^{(k)}.$$

Proof. Using (16), we see that

$$\sum_{n=0}^{\infty} E_{n,\lambda,p}^{(k)}(x)\frac{t^n}{n!} = \frac{2u_{k,\lambda}\left(\log_\lambda(1+t)|p\right)}{t(e_\lambda(t)+1)}\left(e_\lambda(t)-1+1\right)^x$$

$$= \frac{2u_{k,\lambda}\left(\log_\lambda(1+t)|p\right)}{t(e_\lambda(t)+1)}\sum_{i=0}^{\infty}(x)_i\frac{(e_\lambda(t)-1)^i}{i!}$$

$$= \sum_{n=0}^{\infty} E_{n,\lambda,p}^{(k)}\frac{t^n}{n!}\sum_{i=0}^{\infty}(x)_i\sum_{q=i}^{\infty}S_{2,\lambda}(q,i)\frac{t^q}{q!}$$

$$= \sum_{n=0}^{\infty} E_{n,\lambda,p}^{(k)}\frac{t^j}{j!}\sum_{q=0}^{\infty}\sum_{i=0}^{q}(x)_iS_{2,\lambda}(q,i)\frac{t^q}{q!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{q=0}^{n}\sum_{i=0}^{q}\binom{n}{q}(x)_iS_{2,\lambda}(q,i)E_{n-q,\lambda,p}^{(k)}\right)\frac{t^n}{n!}. \qquad \square$$

THEOREM 3.3. *Let $n \geq 0$ and $k \in \mathbb{Z}$. Then*

$$E_{n,\lambda,p}^{(k)} = \sum_{s=0}^{n}\binom{n}{s}\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{l}\frac{m!p(m+1)(1)_{m+1,\lambda}}{(m+1)^k}S_{1,\lambda}(l,m)D_{s-l,\lambda}E_{n-s,\lambda}.$$

Proof. By using (1), (4) and (16), we note that

$$\sum_{n=0}^{\infty} E_{n,\lambda,p}^{(k)}\frac{t^n}{n!} = \frac{2}{t(e_\lambda(t)+1)}u_{k,\lambda}\left(\log_\lambda(1+t)|p\right)$$

$$= \frac{2}{t(e_\lambda(t)+1)}\sum_{m=1}^{\infty}\frac{p(m)(1)_{m,\lambda}}{m^k}(\log_\lambda(1+t))^m$$

$$= \frac{2}{t(e_\lambda(t)+1)}\sum_{m=0}^{\infty}\frac{p(m+1)(1)_{m+1,\lambda}}{(m+1)^k}(\log_\lambda(1+t))^{m+1}$$

$$= \frac{2}{e_\lambda(t)+1}\frac{\log_\lambda(1+t)}{t}\sum_{m=0}^{\infty}\frac{m!p(m+1)(1)_{m+1,\lambda}}{(m+1)^k}\frac{1}{m!}(\log_\lambda(1+t))^m$$

$$= \sum_{n=0}^{\infty}E_{n,\lambda}\frac{t^n}{n!}\sum_{s=0}^{\infty}D_{s,\lambda}\frac{t^s}{s!}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{m!p(m+1)(1)_{m+1,\lambda}}{(m+1)^k}S_{1,\lambda}(l,m)\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty}E_{n,\lambda}\frac{t^n}{n!}\sum_{s=0}^{\infty}\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{l}\frac{m!p(m+1)(1)_{m+1,\lambda}}{(m+1)^k}S_{1,\lambda}(l,m)D_{s-l,\lambda}\frac{t^s}{s!},$$

i.e., R.H.S. becomes

$$\sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}\sum_{l=0}^{s}\binom{s}{l}\sum_{m=0}^{l}\frac{m!p(m+1)(1)_{m+1,\lambda}}{(m+1)^k}S_{1,\lambda}(l,m)D_{s-l,\lambda}E_{n-s,\lambda}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of t on both sides, we acquire at the desired result. $\square$

ACKNOWLEDGEMENT. The author wish to express their appreciation to the reviewers for their helpful suggestions which greatly improved the presentation of this paper. Also, the author thanks to Prince Mohammad Bin Fahd University, Saudi Arabia for providing facilities and support.

REFERENCES

[1] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., **15** (1979), 51–88.

[2] L. Carlitz, *A degenerate Staudt-Clausen theorem*, Arch. Math. (Basel), **7** (1956), 28–33.

[3] D. V. Dolgy, W. A. Khan, *A note on type two degenerate poly-Changhee polynomials of the second kind*, Symmetry, **13(579)** (2021), 1–12.

[4] D. S. Kim, T. Kim, *A note on polyexponential and unipoly functions*, Russ. J. Math. Phys., **26(1)** (2019), 40–49.

[5] D. S. Kim, T. Kim, J. Seo, *A note on Changhee polynomials and numbers*, Adv. Stud. Theor. Phys., **7(1)** (2013), 993–1003.

[6] D. S. Kim, T. Kim, *A note on a new type of degenerate Bernoulli numbers*, Russ. J. Math. Phys., **27(2)** (2020), 227–235.

[7] T. Kim, *A note on degenerate Stirling polynomials of the second kind*, Proc. Jangjeon Math. Soc., **20(3)** (2017), 319–331.

[8] T. Kim, D. S. Kim, J. Kwon, H.Y. Kim, *A note on degenerate Genocchi and poly-Genocchi numbers and polynomials*, J. Ineq. Appl., **2020(110)** (2020), 13 pp.

[9] T. Kim, D. S. Kim, H. Y.-Y. Kim, J. Kwon, *Some results on degenerate Daehee and Bernoulli numbers and polynomials*, Adv. Difference Equ., **311** (2020).

[10] D. S. Lee, H.-Y. Kim, L.-C. Jang, *Type 2 degenerate poly-Euler polynomials*, Symmetry, **12** (2020), Article ID 1011.

(received 10.12.2020; in revised form 16.06.2022; available online 10.02.2023)

Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O Box 1664, Al Khobar 31952, Saudi Arabia

E-mail: wkhan1@pmu.edu.sa