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SOME RESULTS ON SCALABLE K-FRAMES

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Abstract. We investigate the scalability of K-frames and derive a characterization for scalable K-frames. We investigate whether or not a particular K-frame is scalable, as well as the existence and uniqueness of scalings. Using the concept of trace of an operator, we analyse the possible scalings, if a given K-frame is scalable. In $\mathbb{C}^n$, we look at the scalability of K-frames independently.

1. Introduction

Frames in Hilbert spaces were introduced by R. J. Duffin and A. C. Schaffer while working on nonharmonic Fourier series. Later Daubechies, Grossmann and Meyer gave a strong place to frames in harmonic analysis. Frame theory have wide range of applications in signal processing, sampling theory, coding and communications etc. Both orthonormal bases and frames in separable Hilbert spaces can be used to express any vector in the Hilbert space. However, the advantage of frames over orthonormal bases is their redundancy. Some particular types of frames have been suggested in theory for various applications. One such frame is K-frame. Notion of K-frames were introduced by L. Gavruta, to study atomic systems with respect to bounded linear operators. K-frames are more general than frames. The span limit of K-frames is restricted to $R(K)$. Scalability of frames was introduced in [6].

In this paper we study about the scalability of K-frames. Recent studies on K-frames show that Parseval K-frames can be used to manage data loss in signal communication. So the construction of Parseval K-frames is desirable and scaling is the easiest way for this construction. In this paper we deal with K-frames which can be scaled to Parseval K-frames and tight K-frames and we term such K-frames as scalable K-frames and A-scalable K-frames, respectively. We prove some of the results related to scalable K-frames. Also we give characterization result for scalable K-frames.

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Throughout this paper H is a complex separable Hilbert space, $B(H)$ the space of all linear bounded operators on H. For $K \in B(H)$, we denote by $R(K)$ the range of K and $D(K)$ domain of K and $\mathrm{Tr}\,(K)$ denotes the trace of K. $\mathscr{H}^{n\times n}$ denotes the space of all $n \times n$ Hermitian matrices. Let M and N denote the index sets with m and n elements, respectively.

2. Preliminaries

This section contains some basic definitions and results about frames and K-frames which are required in the next sections. For a detailed study of frames and K-frames we refer to [4, 5].

Definition 2.1 ([4]). For a separable Hilbert space H, a sequence $\{f_n\}_{n\in\mathbb{N}} \subset H$ is said to be a **frame** for H if there exist $A,\ B > 0$ such that $A\|x\|^2 \le \sum_{n\in\mathbb{N}} |\langle x, f_n\rangle|^2 \le B\|x\|^2$, for all $x \in H$. The scalars A and B are called lower and upper frame bounds. If $A = B$ then we call it an A-tight frame and if $A = B = 1$ we call it a Parseval frame.

Definition 2.2 ([5]). Let $K \in B(H)$. We say that $\{f_n\}_{n\in N} \subset H$ is a ***K*-frame** for H if there exist constants $A,\ B > 0$ such that $A\|K^*x\|^2 \le \sum_{n\in\mathbb{N}} |\langle x, f_n\rangle|^2 \le B\|x\|^2$, for all $x \in H$. Again A and B are called lower and upper K-frame bounds.

If the equality $\sum_{n\in\mathbb{N}} |\langle x, f_n\rangle|^2 = A\|K^*x\|^2$ holds then we call it an A-tight K-frame. If the equality $\sum_{n\in\mathbb{N}} |\langle x, f_n\rangle|^2 = \|K^*x\|^2$ holds then we call it a Parseval K-frame.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a frame or K-frame. Then we can define two operators as follows. The mapping $T_F\colon H \longrightarrow l^2(\mathbb{N})$ defined by $T_F(f) = \{\langle f, f_n\rangle\}_{n\in\mathbb{N}}$ is called the associated analysis operator. The adjoint operator $T_F^*\colon l^2(\mathbb{N}) \longrightarrow H$ defined by $T_F^*(\{c_n\}_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} c_n f_n$ is called the synthesis operator. From the properties of T_F, it follows that the frame operator $S_F\colon H \longrightarrow H$ defined by $S_F f = TT^* = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle f_n$, for all $f \in H$ is a bounded and positive self-adjoint operator on H. If $\{f_n\}_{n\in\mathbb{N}}$ is a frame, then S_F is invertible. But in the case of K-frames, S_F is not invertible on H, in general. However, S_F is invertible on $R(K)$.

Theorem 2.3. *$\{f_n\}_{n\in\mathbb{N}}$ is a Parseval K-frame for H if and only if $S = KK^*$, $KK^*f = \sum_{i\in I} \langle f, f_i\rangle f_i$, for all $f \in H$.*

Theorem 2.4 ([3]). *Let $K \in B(H)$ and $\{e_i\}_{i\in I}$ be an orthonormal basis for H, then* $\mathrm{Tr}\,(K) = \sum_{i\in I} \langle Ke_i, e_i\rangle$.

Definition 2.5 ([6]). A frame $\{f_n\}_{n\in\mathbb{N}}$ for H is said to be a scalable frame for H if there exist non-negative scalars $\{a_n\}_{n\in\mathbb{N}}$ such that $\{a_n f_n\}_{n\in\mathbb{N}}$ is a Parseval frame for H.

Theorem 2.6 ([2]). *Given a set of points $\{f_j\}_{j\in M} \subseteq \mathbb{R}^n$, and $y \in \mathrm{con}\{f_j\}$, then there exists a subset $J \subseteq M$ such that $y \in \mathrm{con}\{f_j\}_{j\in J}$ and $\{f_j\}_{j\in J}$ is affinely independent.*

Theorem 2.7 ([2]). *Given a collection of unit norm vectors $\{f_j\}_{j\in M}$ in $\mathbb{C}^{\mathbb{K}}$, $\{f_j {f_j}^*\}_{j\in M}$ is linearly independent if and only if it is affinely independent.*

Definition 2.8 ([2]). Given a collection of vectors $\{f_j\}_{j\in M} \subseteq \mathbb{R}^d$ we define their affine span as $\text{aff}\{f_j\}_{j\in M} = \{\sum_{j\in M} c_j f_j : \sum_{j\in M} c_j = 1\}$ and we say that $\{f_j\}_{j\in M}$ is affinely independent if $f_i \notin \text{aff}\{f_j\}_{j\neq i}$ for every j. We define their convex hull as $\text{con}\{f_j\}_{j\in M} = \{\sum_{j\in M} c_j f_j : c_j \geq 0; \sum_{j\in M} c_j = 1\}$.

Definition 2.9 ([6]). Diagonal operator D_a in $l^2(\mathbb{N})$ corresponding to a sequence $a = \{a_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$ is defined by $D_a\{v_n\}_{n\in\mathbb{N}} = \{a_n v_n\}_{n\in\mathbb{N}}.D_a$ (possibly unbounded) is a self-adjoint operator.

Definition 2.10 ([1]). A sequence $\{a_n\}_{n\in N}$ is said to be semi-normalized if there exist $a, b > 0$ such that $a \leq a_n \leq b$ for all n.

Definition 2.11 ([7]). A sequence $\{a_n\}_{n\in N}$ is said to be positively confined if $0 < \inf_n c_n \leq \sup_n c_n < +\infty$.

3. Scalable K-frames

We commence this section with the following definitions.

Definition 3.1. A K-frame $\{f_n\}_{n\in\mathbb{N}}$ for H is said to be a scalable K-frame for H if there exist non-negative scalars $\{a_n\}_{n\in\mathbb{N}}$ such that $\{a_n f_n\}_{n\in\mathbb{N}}$ is a Parseval K-frame for H.

Definition 3.2. A K-frame $\{f_n\}_{n\in\mathbb{N}}$ for H is said to be a A-scalable K-frame for H if there exist non-negative scalars $\{a_n\}_{n\in\mathbb{N}}$ such that $\{a_n f_n\}_{n\in\mathbb{N}}$ is an A-tight K-frame for H.

If there exist positive scalars $\{a_n\}_{n\in\mathbb{N}}$ such that $\{a_n f_n\}_{n\in\mathbb{N}}$ is a Parseval K-frame for H (or A-tight K-frame), then we say $\{f_n\}_{n\in\mathbb{N}}$ is a strictly scalable K-frame (or strictly A-scalable K-frame) for H. The following two results help us to identify two types of scalings.

Theorem 3.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a K-frame for H and $\{a_n\}_{n\in\mathbb{N}}$ be a semi-normalized sequence. Then $\{a_n f_n\}_{n\in\mathbb{N}}$ is also a K-frame.*

Proof. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a $K-$frame for H. Therefore there exist $A,\ B > 0$ such that $A\|K^*f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$, for all $x \in H$. Since $\{a_n\}_{n\in\mathbb{N}}$ is a semi-normalized sequence, there exist $a,\ b > 0$ such that $\sum_{n\in\mathbb{N}} |\langle f, a_n f_n\rangle|^2 = \sum_{n\in\mathbb{N}} a_n^2 |\langle f, f_n\rangle|^2 \leq b^2 B\|f\|^2$ and $\sum_{n\in\mathbb{N}} |\langle f, a_n f_n\rangle|^2 \geq a^2 A\|K^*f\|^2$, for all $f \in H$, where A and B are optimal K-frame bounds for$\{f_n\}_{n\in\mathbb{N}}$. Hence $\{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame with bounds $A' = Aa^2$ and $B' = Bb^2$ □

Theorem 3.4. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a K-frame for H and $\{a_n\}_{n\in\mathbb{N}}$ be a positively confined sequence. Then $\{a_n f_n\}_{n\in\mathbb{N}}$ is also a K-frame for H.*

Proof. For all $f \in H$ we have

$$\sum_{n\in\mathbb{N}} |\langle f, a_n f_n\rangle|^2 = \sum_{n\in\mathbb{N}} a_n^2 |\langle f, f_n\rangle|^2,$$

$$(\inf_{n\in\mathbb{N}} a_n)^2 \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \le \sum_{n\in\mathbb{N}} a_n^2 |\langle f, f_n\rangle|^2 \le (\sup_{n\in\mathbb{N}} a_n)^2 \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2,$$

$$A(\inf_{n\in\mathbb{N}} a_n)^2 \|K^* f\|^2 \le \sum_{n\in\mathbb{N}} |\langle f, a_n f_n\rangle|^2 \le B(\sup_{n\in\mathbb{N}} a_n)^2 \|f\|^2.$$

Hence $\{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame for H . □

THEOREM 3.5. *Let $T \in B(H)$ be an isomorphism. Then a K-frame $\{f_n\}_{n\in N}$ for H is scalable if and only if TK-frame $\{Tf_n\}_{n\in N}$ is scalable.*

Proof. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a scalable K-frame. This implies that $\sum_{n\in\mathbb{N}} |\langle f, a_n f_n\rangle|^2 = \|K^* f\|^2$, for all $f \in H$. Consider, $\sum_{n\in\mathbb{N}} |\langle f, a_n T f_n\rangle|^2 = \sum_{n\in\mathbb{N}} |\langle T^* f, a_n f_n\rangle|^2 = \|K^* T^* f\|^2 = \|(TK)^* f\|^2$, for all $f \in H$. Hence $\{Tf_n\}_{n\in\mathbb{N}}$ is a scalable TK-frame.

Conversely, suppose that $\{Tf_n\}_{n\in\mathbb{N}}$ is a scalable TK-frame. This implies that $\sum_{n\in\mathbb{N}} |\langle f, a_n T f_n\rangle|^2 = \|(TK)^* f\|^2$, for all $f \in H$. It follows that $\sum_{n\in\mathbb{N}} |\langle T^* f, a_n f_n\rangle|^2 = \|K^* T^* f\|^2$, for all $f \in H$. Thus $\sum_{n\in\mathbb{N}} |\langle g, a_n f_n\rangle|^2 = \|K^* g\|^2$, for all $g \in R(T^*) = H$. Hence $\{f_n\}_{n\in\mathbb{N}}$ is a scalable K-frame. □

If $\{f_n\}_{n\in\mathbb{N}}$ is a scalable K-frame for H, then

$$\sum_{n\in\mathbb{N}} |\langle f, a_n K f_n\rangle|^2 = \sum_{n\in\mathbb{N}} |\langle K^* f, a_n f_n\rangle|^2 = \|K^*(K^* f)\|^2 = \|(K^2)^* f\|^2,$$

for all $f \in H$. This implies that $\{a_n K f_n\}_{n\in\mathbb{N}}$ is a Parseval K^2-frame for H. In general, $\{a_n K^s f_n\}_{n\in\mathbb{N}}$ is a Parseval K^{s+1}-frame for H and hence $\{K^s f_n\}_{n\in\mathbb{N}}$ is a scalable K^{s+1}-frame for H.

If $T \in B(H)$ and $\{f_n\}_{n\in\mathbb{N}}$ is a scalable frame for H, then $\{Tf_n\}_{n\in\mathbb{N}}$ is a scalable T-frame for H. But the converse holds only when T is an isomorphism. That is, if $T \in B(H)$ is an isomorphism and $\{Tf_n\}_{n\in\mathbb{N}}$ is a scalable T-frame for H, then $\{f_n\}_{n\in\mathbb{N}}$ is a scalable frame for H.

THEOREM 3.6. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a scalable K-frame. Then $\{f_n\}_{n\in\mathbb{N}}$ is a scalable $(KK^*)^{\frac{1}{2}}$-frame.*

Proof. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a scalable K-frame for H. This implies, $\sum_{n\in\mathbb{N}} |\langle f, a_n f_n\rangle|^2 = \|K^* f\|^2$, for all $f \in H$. Let S be the frame operator of $\{c_n f_n\}_{n\in\mathbb{N}}$. So we get $\langle S^{\frac{1}{2}} S^{\frac{1}{2}^*} f, f\rangle = \langle KK^* f, f\rangle$ and $\|S^{\frac{1}{2}^*} f\|^2 = \|K^* f\|^2$ for all $f \in H$. Thus we obtain $\sum_{n\in\mathbb{N}} |\langle f, a_n f_n\rangle|^2 = \|S^{\frac{1}{2}^*} f\|^2$, for all $f \in H$. Thus $\{a_n f_n\}_{n\in\mathbb{N}}$ is a Parseval $S^{\frac{1}{2}}$-frame and hence is a Parseval $(KK^*)^{\frac{1}{2}}$-frame. That is $\{f_n\}_{n\in\mathbb{N}}$ is a scalable $(KK^*)^{\frac{1}{2}}$-frame. □

THEOREM 3.7. *Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be scalable K-frames for H with scalings $a = \{a_n\}_{n\in\mathbb{N}}$ and $b = \{b_n\}_{n\in\mathbb{N}}$, respectively. Suppose the frame operators T_{aF} and T_{bF} of the scaled frames satisfy $T_{aF}^* T_{bG} = 0$. Then $\{a_n f_n + b_n g_n\}_{n\in\mathbb{N}}$ is a $2c^2$-scalable K-frame. In particular, $\{a_n f_n + b_n g_n\}_{n\in\mathbb{N}}$ is a scalable K-frame.*

Proof. For all $f \in H$ we have, $\sum_{n\in\mathbb{N}} |\langle f, a_n f_n\rangle|^2 = \|K^* f\|^2$ and $\sum_{n\in\mathbb{N}} |\langle f, b_n g_n\rangle|^2 = \|K^* f\|^2$. Also, since $T^*_{aF} T_{bG} = 0$, we get, $\sum_{n\in\mathbb{N}} a_n b_n \langle f, g_n\rangle f_n = 0$.

Take $c_n = c$ for all n where $c > 0$. Then $\{c_n\}_{n\in\mathbb{N}}$ is a non-negative sequence and

$$\sum_{n\in\mathbb{N}} |\langle f, c_n(a_n f_n + b_n f_n)\rangle|^2 =$$

$$\sum_{n\in\mathbb{N}} |\langle f, c_n a_n f_n\rangle|^2 + \sum_{n\in\mathbb{N}} |\langle f, c_n b_n f_n|^2 + \sum_{n\in\mathbb{N}} 2Re\langle f, c_n a_n f_n\rangle\langle f, c_n b_n g_n\rangle = 2c^2\|K^* f\|^2.$$

Therefore, $\{a_n f_n + b_n g_n\}_{n\in\mathbb{N}}$ is a $2c^2$-scalable K-frame.

Taking $c = \frac{1}{\sqrt{2}}$, $\{a_n f_n + b_n g_n\}_{n\in\mathbb{N}}$ is a scalable K-frame. □

We prove some results which lead us to a characterization theorem for scalable K-frames.

THEOREM 3.8. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a K-frame for H with analysis operator T_F and let $a = \{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative scalars. If $G = \{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame for H then $R(T_F) \subset D(D_a)$ and $D_a|_{R(T_F)}$ is bounded.*

Proof. Suppose $\{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame for H and T_G is the coresponding analysis operator. Then, for any $f \in H$, $T_G f = \{\langle f, a_n f_n\rangle\}_{n\in\mathbb{N}} = \{a_n\langle f, f_n\rangle\}_{n\in\mathbb{N}} = D_a T_F f$. Thus $T_G = D_a T_F$ and $R(T_F) \subset D(D_a)$.

Now let $v \in R(T_F)$ so that $v = T_F f$ for some $f \in H$.

Consider, $\|D_a v\| = \|D_a T_F f\| = \|T_G f\| \leq A_1\|f\|^2 \leq A_1\|T_F^{-1} v\| \leq A_2\|T_F\|^{-1}\|\|v\|$. Thus, $D_a|R(T_F)$ is bounded. □

THEOREM 3.9. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a K-frame for H with analysis operator T_F and let $a = \{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative scalars. Then the following conditions are equivalent.*

(i) $G = \{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame for H.

(ii) There exists a diagonal operator D_a in $l^2(\mathbb{N})$ such that $R(T_F) \subset D(D_a)$ and $D_a|_{R(T_F)}$ is bounded and $R(K) \subseteq R(D_a T_F^*)$.

In particular, the frame operator of $G = \{a_n f_n\}_{n\in\mathbb{N}}$ is given by $S_G = T_F^* D^2 T_F$.

Proof. Suppose $G = \{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame for H. Then $R(T_F) \subset D(D_a)$ and $D_a|_{R(T_F)}$ is bounded by Theorem 3.8. Since $\{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame, we have

$$A\|K^* f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, a_n f_n\rangle|^2 = \sum_{n\in\mathbb{N}} |a_n\langle f, f_n\rangle|^2 = \|a_n\{\langle f, f_n\rangle\}_{n\in N}\|^2 = \|D_a T_F f\|^2.$$

Using Douglas Majorization Theorem, we get, $R(K) \subseteq R((D_a T_F)^*)$.

To prove the converse, let $v \in R(T_F)$. Then $v = T_F f$ for some $f \in H$. Since $D_a|R(T_F)$ is bounded, we have, $\|D_a v\| \leq \alpha\|v\|$, for some $\alpha > 0$ and for all $v \in R(T_F)$. Thus $\|D_a T_F f\|^2 \leq \alpha\|T_F f\|^2 \leq \alpha\|T_F\|^2\|f\|^2$ and we get $\sum_{n\in\mathbb{N}} |\langle f, a_n f_n\rangle|^2 \leq B\|f\|^2$, where $B = \alpha\|T_F\|^2$. Also since $R(K) \subseteq R(D_a T_F^*)$, we get $A\|K^* f\|^2 \leq \sum_{n\in\mathbb{N}}\langle f, a_n f_n\rangle|^2$. Hence, $\{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame for H. Moreover, $S_G = T_G^* T_G = (D_a T_F)^*(D_a T_F) = T_F^* D_c^2 T_F$. □

THEOREM 3.10. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a K-frame for H with analysis operator T_F and let $a = \{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative scalars. Also assume that* $\inf_{n\in\mathbb{N}} \|f_n\| > 0$. *Then the following conditions are equivalent.*
(i) $G = \{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame for H.

(ii) D_a is bounded and $R(K) \subseteq R(D_a T_F^*)$.

Proof. Suppose that $\{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame. Then there exist $A,\ B > 0$ such that $A\|K^* f\|^2 \le \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \le B\|f\|^2$, for all $f \in H$. From the right inequality we get D_a is bounded and from the left inequality we get $R(K) \subseteq R(D_a T_F^*)$.

Conversely, suppose D_a is bounded and $R(K) \subseteq R(D_a T_F^*)$. Then using Theorem 3.9 we get $\{a_n f_n\}_{n\in\mathbb{N}}$ is a K-frame for H. $\square$

THEOREM 3.11. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a K-frame for H. If $\{f_n\}_{n\in\mathbb{N}}$ is a scalable K-frame for H, then there exists a non-negative diagonal operator D in $l^2(\mathbb{N})$ such that $KK^* = T_F^* D^2 T_F$.*

Proof. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a scalable K-frame for H. This implies that there exists $a = \{a_n\}_{n\in\mathbb{N}}$ where $a_n \ge 0$ such that $\{a_n f_n\}_{n\in\mathbb{N}}$ is a Parseval K-frame. Then by Theorem 3.9 frame operator of $\{a_n f_n\}_{n\in\mathbb{N}}$ is $S_G = T_F^* D_a^2 T_F$. But frame operator of Parseval K-frame is KK^*. Thus we obtain $T_F^* D^2 T_F = KK^*$ where $D = D_a$. $\square$

REMARK 3.12. Using Theorem 3.9 it is clear that, if there exists a semi-normalized diagonal operator D_a in $l^2(\mathbb{N})$ such that $KK^* = T_F^* D^2 T_F$, then $\{f_n\}_{n\in\mathbb{N}}$ is a scalable K-frame for H.

THEOREM 3.13. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a K-frame for H such that* $\inf_{n\in\mathbb{N}} \|f_n\| > 0$. *Then the following conditions are equivalent.*
(i) $\{f_n\}_{n\in\mathbb{N}}$ is a scalable K-frame.

(ii) There exists a non-negative bounded diagonal operator D in $l^2(\mathbb{N})$ such that $KK^* = T_F^* D^2 T_F$.

Proof. (i) $\Longrightarrow$ (ii) holds from Theorem 3.10 and Theorem 3.11.

Conversely, suppose that there exists a non-negative bounded diagonal operator D in $l^2(J)$ such that $KK^* = T_F^* D^2 T_F$. Then for all $f \in H$, $\langle T_F^* D^2 T_F f, f\rangle = \langle KK^* f, f\rangle$. This implies $\|DT_F f\|^2 = \|K^* f\|^2$ and we get $\sum_{n\in\mathbb{N}} |a_n \langle f, f_n\rangle|^2 = \|K^* f\|^2$. Thus $\{f_n\}_{n\in\mathbb{N}}$ is a scalable K-frame for H. $\square$

4. Scaling sequences-finite K-frames

In Section 3, we have discussed the scalability of K-frames. That is, the existence of scaling sequence or scalings, so that the scaled frame is a Parseval K-frame. We now examine the various scaling sequences for a finite K-frame.

The following example from [9] gives a scalable K-frame with more than one scaling sequence.

Example 4.1. Let $H = \mathbb{C}^3$ and $\{e_1,\ e_2,\ e_3\}$ be the standard orthonormal basis for H. Let $K \in B(\mathbb{C}^3)$, defined by $Ke_1 = e_1,\ Ke_2 = e_1,\ Ke_3 = e_2$. Then $F = \{f_j\}_{j\in J} = \{Ke_1,\ Ke_2,\ Ke_3\} = \{e_1,\ e_1,\ e_2\}$ is a K-frame for H. Here $\inf_j \|f_j\| > 0$. Let $D : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ be the diagonal operator defined by $D(\{v_j\}) = \{a_j v_j\}_{j\in J}$ where $a_j \geq 0$. We have,

$$T_F{}^* D^2 T_F(f) = \sum_{j\in J} a_j{}^2 \langle f, Ke_j\rangle Ke_j = (a_1{}^2 + a_2{}^2)\langle f, e_1\rangle e_1 + a_3{}^2 \langle f, e_2\rangle e_2, \text{ and}$$

$$KK^*(f) = K(K^*f) = K(\sum_{j\in J}\langle f, Ke_i\rangle e_i) = \sum_{j\in J}\langle f, Ke_i\rangle Ke_i = 2\langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_2.$$

Then using Theorem 3.13, $F = \{f_j\}_{j\in J}$ is a scalable K-frame if and only if $a_1{}^2 + a_2{}^2 = 2$ and $a_3{}^2 = 1$ if and only if $(a_1,\ a_2,\ a_3) = (1,\ 1,\ 1)$ or $(a_1,\ a_2,\ a_3) = (\sqrt{2},\ 0,\ 1)$ or $(a_1,\ a_2,\ a_3) = (0,\ \sqrt{2},\ 1)$.

The K-frame given above is strictly scalable if and only if $(a_1,\ a_2,\ a_3) = (1,\ 1,\ 1)$.

Lemma 4.2. *Let $\{f_i\}_{i\in M}$ be a Parseval K-frame for H. Then,* $\mathrm{Tr}\,(KK^*) = \sum_{i\in M} \|f_i\|^2$.

Proof. Let $\{e_j\}_{j\in N}$ be an orthonormal basis for H. Then,

$$\begin{aligned}\mathrm{Tr}\,(KK^*) &= \sum_{j\in N}\langle KK^*e_j, e_j\rangle = \sum_{j\in N}\left\langle \sum_{i\in M}\langle e_j, f_i\rangle f_i, e_j\right\rangle \\ &= \sum_{i\in M}\sum_{j\in N}\langle e_j, f_i\rangle\langle f_i, e_j\rangle = \sum_{i\in M}\left\langle f_i, \sum_{j\in N}\langle f_i, e_j\rangle e_j\right\rangle = \sum_{i\in M}\|f_i\|^2. \qquad \square\end{aligned}$$

Lemma 4.3. *Let $\{f_i\}_{i\in M}$ be a K-frame for H. Then,* $\mathrm{Tr}\,(KK^*) = \sum_{i\in M}\langle Kg_i, f_i\rangle$, *where $\{g_i\}$ is a K-dual of $\{f_i\}$.*

Proof. Let $\{g_i\}$ be a K-dual of $\{f_i\}$. We have

$$\mathrm{Tr}\,(KK^*) = \sum_{j\in N}\langle KK^*e_j, e_j\rangle = \sum_{j\in N}\left\langle K(\sum_{i\in M}\langle e_j f_i\rangle g_i), e_j\right\rangle$$

$$= \sum_{j\in N}\sum_{i\in M}\langle e_j, f_i\rangle\langle Kg_i, e_j\rangle = \sum_{i\in M}\left\langle Kg_i, \sum_{j\in N}\langle f_i, e_j\rangle e_j\right\rangle = \sum_{i\in M}\langle Kg_i, f_i\rangle. \qquad \square$$

Theorem 4.4. *Let $\{f_i\}_{i\in M}$ be a scalable K-frame with scaling sequence $\{c_i\}_{i\in M}$. Let $s = \sum_{i\in M} c_i{}^2$. Then the following statements hold.*
(i) If $\|f_i\| = p$ for all i, then the other possible scalings are non-negative sequences $\{d_i\}_{i\in M}$ such that $\sum_{i\in M} d_i{}^2 = s$.

(ii) If vectors of $\{f_i\}_{i\in M}$ are of non-uniform norm then $\{c_i\}_{i\in M}$ is the unique scaling of $\{f_i\}_{i\in M}$.

Proof. (i) Since $\{f_i\}_{i\in M}$ is a scalable K-frame with scaling $\{c_i\}_{i\in M}$, we have $\{c_i f_i\}_{i\in M}$ is a Parseval K-frame. Then by Lemma 4.2,

$$\mathrm{Tr}\,(KK^*) = \sum_{i\in M}\|c_i f_i\|^2 = p^2\sum|c_i|^2 = p^2\sum_{i\in M} c_i^2.$$

This implies that any other scaling $\{d_i\}_{i\in M}$ such that $\{c_i d_i\}_{i\in M}$ is a Parseval K-frame, should satisfy the condition $\sum_{i\in M} {d_i}^2 = s$.

(ii) We have, $\mathrm{Tr}\,(KK^*) = \sum_{i\in M} {c_i}^2\|f_i\|^2$. If $\{d_i\}_{i\in M}$ is any other scaling so that $\{d_i f_i\}_{i\in M}$ is a Parseval K-frame, then we get $\mathrm{Tr}\,(KK^*) = \sum_{i\in M} {c_i}^2\|f_i\|^2 = \sum_{i\in M} {d_i}^2\|f_i\|^2$. This implies that ${c_i}^2 = {d_i}^2$ and hence $c_i = d_i$ for all i. Therefore, $\{c_i\}_{i\in M}$ is the unique scaling of $\{f_i\}_{i\in M}$. □

Remark 4.5. From part (ii) of Theorem 4.4, it is clear that, if $\{f_i\}_{i\in M}$ is a non-uniform norm K-frame, then it can have at most one scaling.

Theorem 4.6. *Let $\{f_j\}_{j\in J}$ be a scalable K-frame for H. Let F denote the collection of all scaling sequences of $\{f_j\}_{j\in J}$. Let $F_* = \{d = \{d_j\}_{j\in J} \in F : T_d^*T_c(f) = T_c^*T_d(f) = KK^*(f)$ for each $c = \{c_j\}_{j\in J} \in F\}$. Thus F_* is a convex set.*

Proof. Let $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$ and $\{a_j\}_{j\in J}, \{b_j\}_{j\in J} \in F_a$. Consider,

$$\sum |\langle f, (\lambda a_j + \mu b_j) f_j)\rangle|^2 = \sum |\langle f, \lambda a_j f_j\rangle + \langle f, \mu b_j f_j\rangle|^2$$

$$= \sum |\langle f, \lambda a_j f_j\rangle|^2 + \sum |\langle f, \mu b_j f_j\rangle|^2 + 2\sum Re\langle f, \lambda a_j f_j\rangle \overline{\langle f, \mu b_j f_j\rangle}$$

$$= \lambda^2\|K^*f\|^2 + \mu^2\|K^*f\|^2 + 2\lambda\mu Re\langle f, \sum \langle f, b_j f_j\rangle a_j f_j\rangle$$

$$= \lambda^2\|K^*f\|^2 + \mu^2\|K^*f\|^2 + 2\lambda\mu Re\langle f, KK^*f\rangle = \lambda^2\|K^*f\|^2 + \mu^2\|K^*f\|^2 + 2\lambda\mu\|K^*f\|^2$$

$$= (a+b)^2\|K^*f\|^2\| = \|K^*f\|^2.$$

Thus F_c is a convex set. □

5. Scalability and K-frames in $\mathbb{C}^n$

In this section we deal with K-frames in $\mathbb{C}^n$. We rely on $\mathscr{H}^{n\times n}$ to analyse the scalability of K-frames in $\mathbb{C}^n$. We will follow the same setting as in [2]. $\mathscr{H}^{n\times n}$ is an n^2-dimensional inner product space over $\mathbb{R}$ with inner product $\langle P, Q\rangle_{\mathscr{H}} = \mathrm{Tr}\,(PQ)$ and the induced norm $\|P\|^2{}_{\mathscr{F}} = \langle P, P\rangle_{\mathscr{H}}$; $\|P\|^2{}_{\mathscr{F}}$ is the Frobenius norm. We start with a mapping $A : \mathbb{C}^n \to \mathscr{H}^{n\times n}$ given by $Af = \mathrm{ff}^*$ where ff^* is the outer product of f with its conjugate. Let $\{f_j\}_{j\in M}$ be a K-frame for $\mathbb{C}^n$. Under this setting the K-frame operator for $\{f_j\}_{j\in M}$ is given by $S = \sum_{j\in M} f_j {f_j}^*$.

Theorem 5.1. *A K-frame $\{f_j\}_{j\in M}$ in $\mathbb{C}^n$ is a scalable K-frame if and only if there exist non-negative scalars $\{a_j\}_{j\in M}$ such that $\sum_{j\in M} a_j f_j {f_j}^* = KK^*$.*

Proof. Since $\{f_j\}_{j\in M}$ in $\mathbb{C}^n$ is a scalable K-frame, $\{b_j f_j\}_{j\in M}$ is a Parseval K-frame for some collection of non-negative scalars $\{b_j\}_{j\in M}$, so that

$$KK^* = \sum_{j\in M} b_j f_j (b_j f_j)^* = \sum_{j\in M} {b_j}^2 f_j {f_j}^* = \sum_{j\in M} a_j f_j {f_j}^*,$$

where $a_j = {b_j}^2$.

To prove the converse, suppose $\{f_j\}_{j\in M}$ in $\mathbb{C}^n$ is a K-frame such that $\sum_{j\in M} a_j f_j {f_j}^* = KK^*$. From Theorem 3.4, it follows that $\{\sqrt{a_j} f_j\}_{j\in M}$ is also a K-frame for $\mathbb{C}^n$.

Therefore, K-frame operator is given by $S = \sum_{j\in M} \sqrt{a_j} f_j (\sqrt{a_j} f_j)^* = \sum_{j\in M} a_j f_j {f_j}^* = KK^*$. Thus, $\{\sqrt{a_j} f_j\}_{j\in M}$ is a Parseval K-frame and hence $\{f_j\}_{j\in M}$ in $\mathbb{C}^n$ is a scalable K-frame. $\square$

THEOREM 5.2. *Let $\{f_j\}_{j\in M}$ be a K-frame in $\mathbb{C}^n$ such that $\|f_j\| = 1$ for all j. The following statements hold.*

(i) $\{f_j\}_{j\in M}$ is a scalable K-frame if and only if $\dfrac{KK^*}{\|K^*\|^2} \in \operatorname{con}\{f_j {f_j}^*\}_{j\in M}$.

(ii) If $\alpha KK^* \in \operatorname{con}\{f_j {f_j}^*\}_{j\in M}$, then $\alpha = \dfrac{1}{\|K^*\|^2}$.

(iii) If $\sum_{j\in M} a_j f_j {f_j}^* = \dfrac{KK^*}{\|K^*\|^2}$ then $\sum_{j\in M} a_j = 1$.

Proof. (i) Suppose that $\{f_j\}_{j\in M}$ is scalable K-frame in $\mathbb{C}^n$. This implies that there exist $\{a_j\}_{j\in M}$ where $a_j \geq 0$ for all j such that $\{a_j f_j\}_{j\in M}$ is a Parseval K-frame and its K-frame operator is $KK^* = \sum_{j\in M} {a_j}^2 f_j {f_j}^*$. Consider,

$$\|K^*\|^2 = \langle K^*, K^*\rangle = \langle KK^*, I\rangle = \langle \sum_{j\in M} {a_j}^2 f_j {f_j}^*, I_n\rangle = \sum_{j\in M} {a_j}^2 \langle f_j, f_j\rangle = \sum_{j\in M} {a_j}^2.$$

Thus, $\dfrac{\sum_{j\in M} {a_j}^2}{\|K^*\|^2} = 1.$

Since $a_j \geq 0$, for every $j \in M$ we get, $\sum_{j\in M} \dfrac{{a_j}^2}{\|K^*\|^2} f_j {f_j}^* \in \operatorname{con}\{f_j {f_j}^*\}_{j\in M}$ and thus $\dfrac{KK^*}{\|K^*\|^2} \in \operatorname{con}\{f_j {f_j}^*\}_{j\in M}$.

Conversely, suppose that $\dfrac{KK^*}{\|K^*\|^2} \in \operatorname{con}\{f_j {f_j}^*\}_{j\in M}$. This implies, $\dfrac{KK^*}{\|K^*\|^2} = \sum_{j\in M} a_j f_j {f_j}^*$, where $\sum_{j\in M} a_j = 1, a_j \geq 0$ and we get

$$KK^* = \sum_{j\in M} (\sqrt{a_j}\|K^*\|)^2 f_j {f_j}^* = \sum_{j\in M} (\sqrt{a_j}\|K^*\| f_j)(\sqrt{a_j}\|K^*\| f_j)^*.$$

Therefore, $\{\sqrt{a_j}\|K^*\| f_j\}_{j\in M}$ is a K-frame with KK^* as frame operator and hence $\{\sqrt{a_j}\|K^*\| f_j\}_{j\in M}$ is Parseval. This implies $\{f_j\}_{j\in M}$ is a scalable K-frame.

(ii) Now suppose $\alpha KK^* \in \operatorname{con}\{f_j {f_j}^*\}_{j\in M}$, then $\alpha KK^* = \sum_{j\in M} a_j f_j {f_j}^*$ and $\sum_{j\in M} a_j = 1; a_j \geq 0$. Then we have

$$\alpha\|K^*\|^2 = \langle \alpha K^*, K^*\rangle = \langle \alpha KK^*, I_n\rangle = \langle \sum_{j\in M} a_j f_j {f_j}^*, I_n\rangle = \sum_{j\in M} a_j = 1, \text{ and } \alpha = \frac{1}{\|K^*\|^2}.$$

(iii) Suppose $\sum_{j\in M} a_j f_j {f_j}^* = \dfrac{KK^*}{\|K^*\|^2}$. Then,

$$\begin{aligned}\|K^*\|^2 = \langle K^*, K^*\rangle = \langle KK^*, I_n\rangle &= \langle \|K^*\|^2 \sum_{j\in M} a_j f_j {f_j}^*, I_d\rangle \\ &= \|K^*\|^2 \sum_{j\in M} a_j \langle f_j {f_j}^*, I_d\rangle = \|K^*\|^2 \sum_{j\in M} a_j.\end{aligned}$$

This implies $\sum_{j\in M} a_j = 1$. □

THEOREM 5.3. *Let $\{f_j\}_{j\in M}$ be a scalable K-frame in $\mathbb{C}^n$ with $\|f_j\| = 1$ for all j. Then M has a subset N such that $\{f_j\}_{j\in N}$ is scalable and $\{f_j {f_j}^*\}_{j\in N}$ is linearly independent.*

Proof. Using Theorem 5.2, we get, $\dfrac{KK^*}{\|K^*\|^2} \in \operatorname{con}\{f_j {f_j}^*\}_{j\in M}$. From Theorem 2.6 and Theorem 2.7, it follows that there exists a subset $J \subseteq M$ such that $\dfrac{KK^*}{\|K^*\|^2} \in \operatorname{con}\{f_j {f_j}^*\}_{j\in J}$. and $\{f_j {f_j}^*\}_{j\in J}$ is linearly independent. □

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