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SOME REMARKS ON MONOTONICALLY STAR COUNTABLE SPACES

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Abstract. A topological space X is *monotonically star countable* if for every open cover $\mathcal{U}$ of X we can assign a subspace $s(\mathcal{U}) \subseteq X$, called the kernel, such that $s(\mathcal{U})$ is a countable subset of X, and $st(s(\mathcal{U}),\mathcal{U}) = X$, and if $\mathcal{V}$ refines $\mathcal{U}$, then $s(\mathcal{U}) \subseteq s(\mathcal{V})$, where $st(s(\mathcal{U}),\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap s(\mathcal{U}) \neq \emptyset\}$. In this paper we study the relation between monotonically star countable spaces and related spaces, and we also study topological properties of monotonically star countable spaces.

1. Introduction

By a space we mean a topological space. In this section we give definitions of terms used in this paper. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$ let $st(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usual, we write $st(x,\mathcal{U})$ instead of $st(\{x\},\mathcal{U})$.

Definition 1.1 ([1–3, 9]). Let P be a topological property. A space X is said to be *star* P if whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = st(A,\mathcal{U})$. The set A is called the *star kernel* of the cover $\mathcal{U}$.

The term star P was coined in [1–3, 9], but certain star properties, in particular those that are "P=finite" and "P=countable" were first studied by van Douwen et al. in [5] and later by many other authors. A review of star covering properties with a comprehensive bibliography can be found in [5, 7]. In [7] and earlier [5] a star finite space is called starcompact and strongly 1-starcompact, a star countable space is called star Lindelöf and strongly 1-star Lindelöf.

As a monotone version of star covering properties, Popvassilev and Porter [11] introduced the following definition.

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Definition 1.2 ([11]). Let P be a topological property. A space X is *monotonically star $\mathcal{P}$* if for every open cover $\mathcal{U}$ we can assign a subspace $s(\mathcal{U}) \subseteq X$, called a kernel, such that $s(\mathcal{U})$ has property $\mathcal{P}$, and $st(s(\mathcal{U}), \mathcal{U}) = X$, and if $\mathcal{V}$ refines $\mathcal{U}$, $s(\mathcal{U}) \subseteq s(\mathcal{V})$.

From the above definitions, it is clear that every monotonically star finite space is monotonically star countable, every monotonically star countable space is star countable, and every monotonically star countable space is monotonically star Lindelöf, but the converses need not be true (see, Examples 2.2, 2.4 and 2.6). The purpose of this paper is to study the relationship between monotonically star countable spaces and related spaces, and also to study topological properties of monotonically star countable spaces.

Throughout the paper, the extension $e(X)$ of a space X is the smallest infinite cardinal κ such that any discrete closed subset of X has cardinality at most κ. The cardinality of a set A is denoted by $|A|$. Let $\mathfrak{c}$ denote the cardinality of the continuum, ω_1 denote the first uncountable cardinal, and ω denote the first infinite cardinal. For a pair of ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Each cardinal is often considered as a space with the usual order topology. Other terms and symbols that we do not define are used as in [6].

2. Monotonically star countable spaces

In this section we first give an example showing the relation between monotonically starcountable spaces and related spaces. For the next example we need the following lemma.

Lemma 2.1. *A space X with a dense Lindelöf subspace is monotonically star Lindelöf.*

Proof. Let D be a dense Lindelöf subspace of X. We show that X is monotonically star Lindelöf. Let $\mathcal{U}$ be an open cover of X. Then $s(\mathcal{U}) = D$ defines a monotonically star Lindelöf operator for X and $st(s(\mathcal{U}), \mathcal{U}) = X$ since D is a dense Lindelöf subset of X. □

Example 2.2. There exists a Tychonoff monotonically star Lindelöf-space X which is not monotonically star countable.

Let D be a discrete space of cardinality $\mathfrak{c}$ and let $D^* = D \cup \{d^*\}$ be one-point Lindelöfication of D, where $d^* \notin D$. Let $X = (D^* \times (\omega + 1)) \setminus \{\langle d^*, \omega\rangle\}$ be the subspace of the product of D^* and $\omega + 1$. Since $D^* \times \omega$ is a dense σ-compact subset of X, and it is Lindelöf, thus X is is monotonically star Lindelöf.

Next, we show that X is not monotonically star countable. We only show that X is not star countable, because every monotonically star countable is star countable. Since $|D| = \mathfrak{c}$, we can enumerate D as $\{d_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{d_\alpha\} \times (\omega + 1)$. Then $U_\alpha \cap U_{\alpha'} = \emptyset$ if $\alpha \neq \alpha'$.

Consider the open cover $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{D^* \times \omega\}$ of X. It suffices to show that for every countable subset F of X there is a point $x \in X$ such that $x \notin st(F,\mathcal{U})$. Let F be any countable subset of X. There is an $\alpha_0 < \mathfrak{c}$ such that $U_\alpha \cap F = \emptyset$ for each $\alpha > \alpha_0$, since F is countable. If we choose $\alpha' > \alpha_0$, then $\langle d_{\alpha'}, \omega\rangle \notin st(F,\mathcal{U})$, since $U_{\alpha'}$ is the only element of $\mathcal{U}$ that contains $\langle d_{\alpha'}, \omega\rangle$, which completes the proof.

For the next example we need the following lemma from [11]. It is known that star finiteness is equivalent to countably compactness for Hausdorff spaces (see [5, 7] under another name).

Lemma 2.3. *The space ω_2 is not monotonically star countable.*

Example 2.4. There exists a star finite (hence star countable) space that is not monotonically star countable.

Let $D = \omega_2$ be with the usual order topology. Then X is not monotonically star countable by Lemma 2.3. Since X is countably compact, X is star finite, thus X is star countable.

Similar to Lemma 2.1, we have the following result.

Lemma 2.5. *A space X having a dense separable subspace is monotonically star countable.*

Example 2.6. There exists a Tychonoff monotonically star countable space X which is not monotonically finite.

Let $X = \omega \cup \mathcal{A}$ be the Isbell-Mrówka space [10], where $\mathcal{A}$ is the almost disjoint family of infinite subsets of ω with $|\mathcal{A}| = \mathfrak{c}$. Then X is monotonically star countable by Lemma 2.5, since ω is a countable dense subset of X. It is well-known that X is not countably compact, hence it is not star finite, thus X is not monotonically star finite, since every monotonically star finite space is star finite.

Remark 2.7. Example 2.6 also shows that there exists a Tychonoff monotonically star countable space X such that $e(X) = \mathfrak{c}$. Matveev [8] showed that the extent of a Tychonoff star countable space can be arbitrarily big. However the authors do not know whether the extent of a Tychonoff monotonically star countable space can be arbitrarily big.

From Example 2.6 it is not hard to see that a closed subset of a monotonically star countable space need not be monotonically star countable, since $\mathcal{A}$ is a discrete closed subset of cardinality $\mathfrak{c}$. We now give a stronger example. For the next example, we give two lemmas.

Lemma 2.8. *If $X = \bigcup\{X_n : n \in \omega\}$ and every X_n is monotonically star countable, then X is monotonically star countable.*

Proof. Let $\mathcal{U}$ be an open cover of X. For every $n \in \omega$, since X_n is monotonically star countable, there exists a monotonically star countable operator $s_n(\mathcal{U})$ for X_n such that $X_n \subseteq st(s_n(\mathcal{U}),\mathcal{U})$. Let $s(\mathcal{U}) = \bigcup_{n\in\omega} S_n(\mathcal{U})$. Then $s(\mathcal{U})$ is a monotonically star countable operator for X. □

Lemma 2.9. *Let D be a discrete space of cardinality $\mathfrak{c}$ and let $D^* = D \cup \{d^*\}$ be one-point Lindelöfication of D, where $d^* \notin D$. Then D^* is monotonically star countable.*

Proof. Let $\mathcal{U}$ be an open cover of D^*, and let $s(\mathcal{U}) = \{d^*\} \cup (D^* \setminus st(d^*, \mathcal{U}))$. Then $s(\mathcal{U})$ is countable by the construction of the topology of D^*. If $\mathcal{V} \prec \mathcal{U}$, clearly $s(\mathcal{U}) \subseteq s(\mathcal{V})$. Hence $s(\mathcal{U})$ is a monotonically star countable operator for D^*. □

Example 2.10. There exists a monotonically star countable space having a G_δ regular closed subspace which is not star countable (hence not monotonically star countable).

Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$, and let $S_1 = X$ be the same space X of the above Example 2.2. Then S_1 is not monotonically star countable.

Let $S_2 = X$ be the same space X of the above Example 2.6. Then S_2 is monotonically star countable.

We assume $S_1 \cap S_2 = \emptyset$. Let $\pi : D \times \{\omega\} \to \mathcal{A}$ be a bijection and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \omega\rangle$ of S_1 with $\pi(\langle d_\alpha, \omega\rangle)$ of S_2 for every $\alpha < \mathfrak{c}$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. Then $\varphi(S_1)$ is a regular-closed subspace of X. For each $n \in \omega$, let $F_n = \{m \in \omega : m \leq n\}$. For each $n \in \omega$, let $U_n = \varphi(\{\{A\} \cup (A \setminus F_n) : A \in \mathcal{A}\}) \cup \varphi(D^* \times \omega)$. Then U_n is open in X and $\varphi(S_1) = \bigcap_{n\in\omega} U_n$. Thus $\varphi(S_1)$ is a G_δ regular close subspace of X. However $\varphi(S_1)$ is not monotonically star countable, since it is homeomorphic to S_1.

Finally we show that X is monotonically star countable. To this end, let $\mathcal{U}$ be an open cover of X. Since $\varphi(S_2)$ is homeomorphic to S_2, consequently $\varphi(S_2)$ is monotonically star countable. Let $s'(\mathcal{U})$ be a monotonically star countable operator for $\varphi(S_2)$ such that $\varphi(S_2) \subseteq st(s'(\mathcal{U}), \mathcal{U})$.

On the other hand, for each $n \in \omega$, since $\varphi(D^* \times \{n\})$ is homeomorphic to $D^* \times \{n\}$, it is monotonically star countable by Lemma 2.9. Thus $\bigcup_{n\in\omega} \varphi(D^* \times \{n\})$ is monotonically star countable by Lemma 2.8. Let $s''(\mathcal{U})$ be a monotonically star countable operator for $\varphi(D^* \times \omega)$ such that $\bigcup_{n\in\omega} \varphi(D^* \times \{n\}) \subseteq st(s''(\mathcal{U}), \mathcal{U})$.

Let $s(\mathcal{U}) = s'(\mathcal{U}) \cup s''(\mathcal{U})$. Then $s(\mathcal{U})$ is monotonically star countable operator for X such that $X = st(s(\mathcal{U}), \mathcal{U})$.

In the following, we give a positive result.

Theorem 2.11. *An open F_δ-subset of a monotonically star countable space is monotonically star countable.*

Proof. Let X be a monotonically star countable space and let $Y = \bigcup\{H_n : n \in \omega\}$ be an open F_δ-subset of X, where each H_n is a closed subset of X. To show that Y is monotonically star countable, let $\mathcal{U}$ be an open cover of Y. We have to find a monotonically star countable operator $s(\mathcal{U})$ of Y such that $st(s(\mathcal{U}), \mathcal{U}) = Y$. For each $n \in \omega$, consider the open cover $\mathcal{U}_n = \mathcal{U} \cup \{X \setminus H_n\}$ of X. Since X is monotonically star countable, there is a monotonically star countable operator s' for X such that $X = st(s'(\mathcal{U}_n), \mathcal{U}_n)$. Let $s''(\mathcal{U}_n) = s'(\mathcal{U}_n) \cap Y$. Then $s''(\mathcal{U}_n)$ is a countable subset of Y and $H_n \subseteq st(s''(\mathcal{U}_n), \mathcal{U})$. Thus $s(\mathcal{U}) = \bigcup_{n\in\omega} s''(\mathcal{U}_n)$ is a countable subset of Y such that $Y = st(s(\mathcal{U}), \mathcal{U})$.

If $\mathcal{V} \prec \mathcal{U}$, then for each $n \in \omega$, $\mathcal{V}_n \prec \mathcal{U}_n$ and $s'(\mathcal{U}_n) \subseteq s'(\mathcal{V}_n)$, hence $s''(\mathcal{U}_n) \subseteq s''(\mathcal{V}_n)$, thus $s(\mathcal{U}) = \bigcup_{n\in\omega} s''(\mathcal{U}_n) \subseteq \bigcup_{n\in\omega} s''(\mathcal{V}_n) = s(\mathcal{V})$. Hence $s(\mathcal{U})$ is monotonically star countable operator for Y. □

A *cozero-set* in a space X is a set of the form $f^{-1}(R \setminus \{0\})$ for some real-valued continuous function f on X. Since a cozero-set is an open F_δ-set, we have the following corollary of Theorem 2.11.

COROLLARY 2.12. *A cozero-set of a monotonically star countable space is monotonically star countable.*

Since every clopen subset is an open F_δ-set, we have the following corollary of Theorem 2.11.

COROLLARY 2.13. *A clopen subset of a monotonically star countable space is monotonically star countable.*

The first author [13] showed that a continuous image of a monotonically star compact space is monotonically star compact; similarly we have the following result.

THEOREM 2.14. *A continuous image of a monotonically star countable space is monotonically star countable.*

Next we turn to consider preimages. To show that the preimage of a monotonically star countable space under a closed 2-to-1 continuous map need not be monotonically star countable, we use the Alexandorff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0,1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X. It is well-known that if X is compact (Lindelöf) if and only if so is $A(X)$.

EXAMPLE 2.15. There exists a closed 2-to-1 continuous map $f : A(X) \to X$ such that X is a monotonically star countable space, but $A(X)$ is not monotonically star countable.

Let X be the space S_1 of Example 2.6. Then X is monotonically star countable and has an uncountable discrete closed subset $\mathcal{A}$. Hence the Alexandorff duplicate $A(X)$ of X is not monotonically star countable by Corollary 2.13, since $\mathcal{A} \times \{1\}$ is an uncountable infinite discrete, open and closed set in $A(X)$. Let $f : A(X) \to X$ be the projection. Then f is a closed 2-to-1 continuous map.

REMARK 2.16. The Example 2.6 also shows that there exists a Tychonoff monotonically star countable space X such that $A(X)$ is not monotonically star countable.

THEOREM 2.17. *If X is a T_1-space and $A(X)$ is a monotonically star counatble space. Then $e(X) < \omega_1$,*

Proof. Suppose that $e(X) \geq \omega_1$. Then there exists a discrete closed subset B of X such that $|B| \geq \omega_1$. Hence $B \times \{1\}$ is an open and closed subset of $A(X)$ and every point of $B \times \{1\}$ is an isolated point. Thus $A(X)$ is not monotonically star countable by Corollary 2.13, since $B \times \{1\}$ is not monotonically star countable. □

Singh [12] proved that, if X is monotonically star compact space with $e(X) < \omega$, then $A(X)$ is monotonically star compact. Similarly, we will prove the following result.

Lemma 2.18 ([4]). *For T_1-space X, $e(X) = e(A(X))$.*

Theorem 2.19. *If X is a monotonically star countable space with $e(X) < \omega_1$, then $A(X)$ is monotonically star countable.*

Proof. Let $\mathcal{U}$ be an open cover of $A(X)$. Let $\mathcal{U}' = \{U \cap (X \times \{0\}) : U \in \mathcal{U}\}$. Then $\mathcal{U}'$ is an open cover of $X \times \{0\}$. Since $X \times \{0\}$ is homeomorphic to X and X is monotonically star countable, thus $X \times \{0\}$ is a monotonically star countable, hence there exists a countable subset $s(\mathcal{U}')$ of X such that $st(s(\mathcal{U}'), \mathcal{U}') = X \times \{0\}$. Let $s(\mathcal{U})' = A(s(\mathcal{U}'))$. Then $s(\mathcal{U})'$ is a countable subset. Let $A_{\mathcal{U}} = A(X) \setminus st(s(\mathcal{U})', \mathcal{U}')$. Then $A_{\mathcal{U}}$ is a discrete closed subset of $A(X)$. By Lemma 2.6, the set $A_{\mathcal{U}}$ is countable. Let $s(\mathcal{U}) = s(\mathcal{U})' \cup A_{\mathcal{U}}$. Then $s(\mathcal{U})$ is a countable subset of $A(X)$ and $A(X) = st(s(\mathcal{U}), \mathcal{U})$.

If $\mathcal{V} \prec \mathcal{U}$, then $\mathcal{V}' \prec \mathcal{U}'$, hence $s(\mathcal{U}') \subseteq s(\mathcal{V}')$ and $A_{\mathcal{U}} \subseteq A_{\mathcal{V}}$, thus $s(\mathcal{U}) \subseteq s(\mathcal{V})$. Therefore $s(\mathcal{U})$ is monotonically star countable operator for X. □

We have the following corollary from Theorems 2.17 and 2.19.

Corollary 2.20. *If X is a T_1 monotonically star countable space, then $A(X)$ is monotonically star countable if and only if $e(X) < \omega_1$.*

In [5, Example 3.34] van Douwen-Reed-Roscoe-Tree gave an example showing that there is a star countable space X and a compact space Y such that $X \times Y$ is not star-countable. We now show that the product $X \times Y$ is not monotonically star countable. We give the construction for the sake of completeness.

Example 2.21. There exists a monotonically star countable space X and a compact space Y such that $X \times Y$ is not monotonically star countable.

Let $X = \omega \cup \mathcal{A}$ be the Isbell-Mrówka space, where $\mathcal{A}$ is the almost disjoint family of infinite subsets of ω with $|\mathcal{A}| = \mathfrak{c}$. Then X is monotonically star countable. Let D be a discrete space of cardinality $\mathfrak{c}$ and let $Y = D \cup \{d^*\}$ be one-point compactification of D, where $d^* \notin D$. However $X \times Y$ is not star countable (see [5, Example 3.34]), hence $X \times Y$ is not monotonically star countable, since every monotonically star countable is star countable.

In the following, we give another example showing the product of a monotonically star countable space X and a compact space Y such that $X \times Y$ is not monotonically star countable. For the next example, we need the following lemma from [11].

Lemma 2.22 ([11]). *The space ω_1 is monotonically star countable.*

Example 2.23. There exists a monotonically star countable space X and a compact space Y such that $X \times Y$ is not monotonically star countable.

Let $X = \omega_1$ with the usual order topology. Then X is monotonically star countable by Lemma 2.22. Let D be a discrete space of cardinality ω_1 and let $Y = D \cup \{d^*\}$

be one-point compactification of D, where $d^* \notin D$. To show that $X \times Y$ is not monotonically star countable, we have only to show that $X \times Y$ is not star countable, since every monotonically star countable space is star countable. For each $\alpha < \omega_1$, let $U_\alpha = [0, \alpha] \times [\alpha, \omega_1]$ and $V_\alpha = (\alpha, \omega_1) \times \{\alpha\}$. Consider the open cover $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}$ of $X \times Y$ and let F be a countable subset of $X \times Y$. Then $\pi(F)$ is a countable subset of X, where $\pi : X \times Y \to X$ is the projection. Thus, there exists $\beta < \omega_1$ such that $F \cap ((\beta, \omega_1) \times Y) = \emptyset$. Pick α with $\alpha > \beta$. Then $\langle \alpha, \beta \rangle \notin st(F, \mathcal{U})$, since V_β is the only element of $\mathcal{U}$ containing $\langle \alpha, \beta \rangle$. Hence $X \times Y$ is not star countable.

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