МАТЕМАТИЧКИ ВЕСНИК
46 (1994), 41–50

UDC 512.64
оригинални научни рад
research paper

DETERMINANTAL REPRESENTATION OF WEIGHTED MOORE-PENROSE INVERSE

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Abstract. In this paper we introduce determinantal representation of *weighted Moore-Penrose inverse* of a rectangular matrix.

We generalize concept of generalized algebraic complement, introduced by Moore, Arghiriade, Dragomir and Gabriel. This extension is denoted as weighted generalized algebraic complement.

Moreover, we derive an explicit determinantal representation for the weighted least-squares minimum norm solution of a linear system and prove that this solution lies in the convex hull of the solutions to the square subsystems of the original system.

1. Introduction

Let $\mathbf{C}^n$ be the n-dimensional complex vector space, $\mathbf{C}^{m\times n}$ the set of $m\times n$ complex matrices, and $\mathbf{C}_r^{m\times n}=\{X\in\mathbf{C}^{m\times n}\ :\ \operatorname{rank}(X)=r\}$. We suppose that $A\in\mathbf{C}_r^{m\times n}$, unless indicated otherwise. The adjungate matrix of a square matrix B will be denoted as $\operatorname{adj}(B)$, and its determinant as $|B|$. Conjugate, transposed and conjugate-transposed matrix of A will be denoted as $\overline{A}$, A^T and A^* respectively. Submatrix of A containing rows $\alpha_1,\dots,\alpha_t$ and columns $\beta_1,\dots,\beta_t$ is denoted as $A\begin{bmatrix}\alpha_1\dots\alpha_t\\ \beta_1\dots\beta_t\end{bmatrix}$. Also, minor of a rectangular matrix $A\in\mathbf{C}^{m\times n}$ containing rows $\alpha_1,\dots,\alpha_t$ and columns $\beta_1,\dots,\beta_t$ is denoted as $A\begin{pmatrix}\alpha_1\dots\alpha_t\\ \beta_1\dots\beta_t\end{pmatrix}$ an its algebraic complement is defined as

$$A_{ij}\begin{pmatrix}\alpha_1\ \dots\ \alpha_{p-1}\ \ i\ \ \alpha_{p+1}\ \dots\ \alpha_t\\ \beta_1\ \dots\ \beta_{q-1}\ \ j\ \ \beta_{q+1}\ \dots\ \beta_t\end{pmatrix}=(-1)^{i+j}A\begin{pmatrix}\alpha_1\ \dots\ \alpha_{p-1}\ \alpha_{p+1}\ \dots\ \alpha_t\\ \beta_1\ \dots\ \beta_{q-1}\ \beta_{q+1}\ \dots\ \beta_t\end{pmatrix}.$$

For any $A\in\mathbf{C}^{m\times n}$, $x\in\mathbf{C}^m$, $j\in\{1,\dots,n\}$, $A(j\to x)$ denotes the matrix obtained by replacing the jth column of A with x, and $|A|(j\to x)=|A(j\to x)|$.

Penrose [16] has shown the existence and uniqueness of a solution $X\in\mathbf{C}^{n\times m}$ to the equations

$$(1)\quad AXA=A\,,\quad (2)\quad XAX=X\,,\quad (3)\quad (AX)^*=AX\,,\quad (4)\quad (XA)^*=XA.$$

AMS Subject Classification (1991): 15A09, 15A60, 65F35

For a subset $\mathcal{S}$ of the set $\{1,2,3,4\}$ the set of matrices G obeying the conditions represented in $\mathcal{S}$ will be denoted by $A\{\mathcal{S}\}$. A matrix G in $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and is denoted by $A^{(\mathcal{S})}$. In particular for any $A \in \mathbf{C}^{m\times n}$ the set $A\{1,2,3,4\}$ consists of a single element, the *Moore-Penrose inverse* of A, denoted by $A^{\dagger}$ [2], [17].

In the following theorem general forms of the sets $A\{\mathcal{S}\}$ are described.

THEOREM 1.1 [18] *If $A \in \mathbf{C}_r^{m\times n}$ has a full-rank factorization $A = PQ$, $P \in \mathbf{C}_r^{m\times r}$, $Q \in \mathbf{C}_r^{r\times n}$, $W_1 \in \mathbf{C}^{n\times r}$ and $W_2 \in \mathbf{C}^{r\times m}$ are some matrices such that* $\operatorname{rank}(QW_1) = \operatorname{rank}(W_2P) = \operatorname{rank}(A)$, *then*

$$\begin{aligned} A^{\dagger} &= Q^{\dagger}P^{\dagger} = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* \\ A\{1,2\} &= \{W_1(QW_1)^{-1}(W_2P)^{-1}W_2\} \\ A\{1,2,3\} &= \{W_1(QW_1)^{-1}(P^*P)^{-1}P^*\} \\ A\{1,2,4\} &= \{Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2\}. \end{aligned}$$

Concept of determinant i.e. algebraic complement is intimately related to the concept of generalized inversion of matrices. Determinantal representation of *Moore-Penrose inverse* is studied in [1], [3], [7], [8], [9], [15]. The main result is contained in the following theorem.

THEOREM 1.2 *Element lying on the i-row and j-column of the Moore-Penrose inverse of a given matrix $A \in \mathbf{C}_r^{m\times n}$ can be represented in terms of determinants of square matrices, as follows:*

$$a_{ij}^{(\dagger,r)} = \frac{A_{ji}^{(\dagger,r)}}{\|A\|_r} = \frac{\displaystyle\sum_{\substack{1\le\beta_1<\ldots<\beta_r\le n\\ 1\le\alpha_1<\ldots<\alpha_r\le m}} \overline{A}\begin{pmatrix}\alpha_1 & \ldots & i & \ldots & \alpha_r\\ \beta_1 & \ldots & j & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & i & \ldots & \alpha_r\\ \beta_1 & \ldots & j & \ldots & \beta_r\end{pmatrix}}{\displaystyle\sum_{\substack{1\le\delta_1<\ldots<\delta_r\le n\\ 1\le\gamma_1<\ldots<\gamma_r\le m}} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}\overline{A}\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}}, \quad \begin{pmatrix}1\le i\le n\\ 1\le j\le m\end{pmatrix}.$$

The numerator of this expression represents *generalized algebraic complement* of the order r corresponding to a_{ij}, while the denominator expresses determinantal representation of the norm of A.

Weighted Moore-Penrose inverse is investaged in [2], [6], [12]. The main results are contained in the following three theorems.

THEOREM 1.3 *Let positive-definite (and hermitian) matrices $M \in \mathbf{C}^{m\times m}$ and $N \in \mathbf{C}^{n\times n}$ be given. For any matrix $A \in \mathbf{C}^{m\times n}$ there exists a unique solution $X = A^{\dagger}_{M\bullet,\bullet N} \in A\{1,2\}$ satisfying*

$$(5)\qquad (MAX)^* = MAX \qquad\qquad (6)\qquad (XAN)^* = XAN.$$

Similarly, we use the following notations:

$A^{\dagger}_{M\bullet,N\bullet}$ represents unique solution of the equations (1), (2), and

$$(7)\qquad (MAX)^* = MAX \qquad\qquad (8)\qquad (NXA)^* = NXA;$$

$A^{\dagger}_{\bullet M,N\bullet}$ is unique solution of the equations (1), (2), and

$$(9) \qquad (AXM)^* = AXM \qquad\qquad (10) \qquad (NXA)^* = NXA,$$

while $A^{\dagger}_{\bullet M,\bullet N}$ is unique solution of the equations (1), (2), and

$$(11) \qquad (AXM)^* = AXM \qquad\qquad (12) \qquad (XAN)^* = XAN.$$

THEOREM 1.4 [6] *An equivalent of condition (5) is* $(AXM^{-1})^* = AXM^{-1}$, *while the condition (6) can be expressed in the form* $(N^{-1}XA)^* = N^{-1}XA$.

THEOREM 1.5 [6] *If* $A = PQ$ *is a full rank factorization of* A, *then:*

$$A^{\dagger}_{M\bullet,\bullet N} = (QN)^*(Q(QN)^*)^{-1}((MP)^*P)^{-1}(MP)^*.$$

Using these notions, Theorem 1.4. and Theorem 1.5. the following corollary can be proved.

COROLLARY 1.1 *a)* $A^{\dagger}_{M\bullet,\bullet N} = A^{\dagger}_{\bullet M^{-1},\bullet N} = A^{\dagger}_{M\bullet,N^{-1}\bullet} = A^{\dagger}_{\bullet M^{-1},N^{-1}\bullet}$.

b) $A^{\dagger}_{M\bullet,N\bullet} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^* = A_{M\bullet,\bullet N^{-1}}$;

c) $A^{\dagger}_{\bullet M,N\bullet} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^* = A_{M^{-1}\bullet,\bullet N^{-1}}$;

d) $A^{\dagger}_{\bullet M,\bullet N} = (QN)^*(Q(QN)^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*$.

One of indices of the form $\bullet M^{[-1]}, \bullet N^{[-1]}$; $M^{[-1]}\bullet, \bullet N^{[-1]}$; $\bullet M^{[-1]}, N^{[-1]}\bullet$; $M^{[-1]}\bullet, N^{[-1]}\bullet$, where $M^{[-1]}$ denotes M^{-1} or M and $N^{[-1]}$ denotes N^{-1} or N we formally denote as $\varphi(M,N)$.

In this paper *weighted Moore-Penrose inverse* of a rectangular matrix is presented in terms of her own square minors and square minors of matrix product MAN. This determinantal representation is developed using two different methods. In the first method we develop the determinantal representation of $\{1,2\}$ inverse and *weighted Moore-penrose inverse* is treated as an $\{1,2\}$ inverse. In the second access we generalize concept of *generalized algebraic complement*.

Also, we introduce and investage determinantal representation of *weighted least-squares minimum norm solution* of a linear system.

2. Weighted Moore-Penrose inverse as an $\{1,2\}$ inverse

In the following two Theorems we develop determinantal representation of class of $\{1,2\}$ inverses, and derive determinantal representation of *weighted Moore-Penrose inverse*, which is treated as an $\{1,2\}$ inverse. The determinantal representation of the class of $\{1,2\}$ inverses is a significant result in itself.

THEOREM 2.1 *If* $A = PQ$ *is a full rank factorization of* $A \in \mathbf{C}_r^{m\times n}$ *and* $W_1 \in \mathbf{C}^{n\times r}$, $W_2 \in \mathbf{C}^{r\times m}$ *are some matrices such that* $\mathrm{rank}(QW_1) = \mathrm{rank}(W_2P) = \mathrm{rank}(W_1W_2) = \mathrm{rank}(A)$, *then an element* $a_{ij}^{(1,2)} \in A^{(1,2)}$ *is given by*

$$a_{ij}^{(1,2)} = \frac{\sum\limits_{\substack{1\le\beta_1<\ldots<\beta_r\le n\\ 1\le\alpha_1<\ldots<\alpha_r\le m}} (W_1W_2)^T\begin{pmatrix}\alpha_1 & \ldots & i & \ldots & \alpha_r\\ \beta_1 & \ldots & j & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\le\delta_1<\ldots<\delta_r\le n\\ 1\le\gamma_1<\ldots<\gamma_r\le m}} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}(W_1W_2)^T\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}}.$$

Proof. Starting from $A^{(1,2)} = W_1(QW_1)^{-1}(W_2P)^{-1}W_2$, it is easy to see that $a_{ij}^{(1,2)}$ is equal to

$$\sum_{k=1}^{r}\frac{\sum\limits_{\beta_1<\ldots<\beta_r} W_1^T\begin{pmatrix}1 & \ldots & k & \ldots & r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} Q_{ki}\begin{pmatrix}1 & \ldots & k & \ldots & r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} \sum\limits_{\alpha_1<\ldots<\alpha_r} W_2^T\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ 1 & \ldots & k & \ldots & r\end{pmatrix} P_{jk}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ 1 & \ldots & k & \ldots & r\end{pmatrix}}{\sum\limits_{\delta_1<\ldots<\delta_r} Q\begin{pmatrix}1 & \ldots & r\\ \delta_1 & \ldots & \delta_r\end{pmatrix} W_1^T\begin{pmatrix}1 & \ldots & r\\ \delta_1 & \ldots & \delta_r\end{pmatrix} \sum\limits_{1\le\gamma_1<\ldots<\gamma_r\le r} W_2^T\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ 1 & \ldots & r\end{pmatrix} P\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ 1 & \ldots & r\end{pmatrix}}$$

$$= \frac{\sum\limits_{\substack{\beta_1<\ldots<\beta_r\\ \alpha_1<\ldots<\alpha_n}} (W_1W_2)^T\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}\left[\sum\limits_{k=1}^{r} P_{jk}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ 1 & \ldots & k & \ldots & r\end{pmatrix} Q_{ki}\begin{pmatrix}1 & \ldots & k & \ldots & r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}\right]}{\sum\limits_{\substack{1\le\delta_1<\ldots<\delta_r\le r\\ 1\le\gamma_1<\ldots<\gamma_r\le m}} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}(W_1W_2)^T\begin{pmatrix}\delta_1 & \ldots & \delta_r\\ \gamma_1 & \ldots & \gamma_r\end{pmatrix}}.$$

Using the Cauchy-Binet formula, we can show

$$\sum_{k=1}^{r} P_{jk}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ 1 & \ldots & k & \ldots & r\end{pmatrix} Q_{ki}\begin{pmatrix}1 & \ldots & k & \ldots & r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} = A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}$$

and the proof is completed. ∎

THEOREM 2.2 *Let* $M \in \mathbf{C}^{m\times m}$, $N \in \mathbf{C}^{n\times n}$ *be positive definite, and suppose that* $A = PQ$ *is a full rank factorization of* A*, such that* $\operatorname{rank}(P^*MP) = \operatorname{rank}(QNQ^*) = \operatorname{rank}(MAN) = r$. *Element of the weighted Moore-Penrose inverse* $A^\dagger_{M\bullet,\bullet N}$*, lying on the* i*th row and* j*th column, can be represented in terms of square minors as follows:*

$$(a^\dagger_{M\bullet,\bullet N})_{ij} = \frac{\sum\limits_{\substack{1\le\beta_1<\ldots<\beta_r\le n\\ 1\le\alpha_1<\ldots<\alpha_r\le m}} \overline{(MAN)}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\le\delta_1<\ldots<\delta_r\le n\\ 1\le\gamma_1<\ldots<\gamma_r\le m}} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}\overline{(MAN)}\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}}$$

Proof. According to Theorem 1.1 and Theorem 1.5, *weighted Moore-Penrose inverse* of a matrix A can be obtained as an element from the class of $A\{1,2\}$ inverses satisfying relations $W_1 = (QN)^*$, $W_2 = (MP)^*$. Applying these substitutions in formula which represent determinantal representation of the class of $\{1,2\}$ inverses, the proof can be elementary obtained. ∎

From Theorem 2.2., and Corollary 1.1. it follows:

COROLLARY 2.1 *Let* $M \in \mathbf{C}^{m\times m}$, $N \in \mathbf{C}^{n\times n}$ *be positive definite and* $A = PQ$ *is a full rank factorization of* A. *Then*

$$(A^\dagger_{\varphi(M,N)})_{ij} = \frac{\sum\limits_{\substack{1\le\beta_1<\ldots<\beta_r\le n\\ 1\le\alpha_1<\ldots<\alpha_r\le m}} \overline{(\omega(M,N))}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\le\delta_1<\ldots<\delta_r\le n\\ 1\le\gamma_1<\ldots<\gamma_n\le r}} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}\overline{(\omega(M,N))}\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}},$$

where the expression $\omega(M,N)$ *represents a matrix such that* $\operatorname{rank}(\omega(M,N)) = \operatorname{rank}(A)$ *and*

$$\omega(M,N) = \begin{cases} MAN, & \varphi(M,N) = M\bullet, \bullet N \\ MAN^{-1}, & \varphi(M,N) = M\bullet, N\bullet \\ M^{-1}AN, & \varphi(M,N) = \bullet M, \bullet N \\ M^{-1}AN^{-1}, & \varphi(M,N) = \bullet M, N\bullet \end{cases}.$$

3. Weighted generalized algebraic complement and weighted matrix norm

In this section we define *weighted generalized algebraic complement* and *weighted norm* of rectangular complex matrices, and using these notions we find the known determinantal representation of the *weighted Moore-Penrose inverse*.

DEFINITION 3.1 Weighted norm of $A \in \mathbf{C}_r^{m\times n}$, denoted as $\|A\|^r_{\varphi(M,N)}$ is equal to

$$|\,(MP)^*P\,|\,|\,Q(QN)^*\,|$$

while the weighted adjoint matrix of A, denoted as $\operatorname{adj}\left(A^{(\dagger,r)}_{M\bullet,\bullet N}\right)$, is

$$(QN)^*\operatorname{adj}(Q(QN)^*)\cdot\operatorname{adj}((MP)^*P)(MP)^*.$$

THEOREM 3.1 *Weighted norm of A has the following determinantal representation*

$$\|A\|^r_{\varphi(M,N)} = \sum_{\substack{1\le i_1<\ldots<i_r\le m\\ 1\le j_1<\ldots<j_r\le n}} A\begin{pmatrix} j_1 & \ldots & j_r \\ i_1 & \ldots & i_r\end{pmatrix}\overline{(\omega(M,N))}\begin{pmatrix} j_1 & \ldots & j_r \\ i_1 & \ldots & i_r\end{pmatrix}.$$

Proof. Suppose that $\varphi(M,N) = M\bullet,\bullet N$ and $A = PQ$ is a full rank factorization of A.

$$\|A\|^r_{M\bullet,\bullet N} = |\,(MP)^*P\,|\,|\,Q(QN)^*\,|$$

$$= \left[\sum_{i_1<\ldots<i_r} P\begin{pmatrix} i_1 & \ldots & i_r \\ 1 & \ldots & r\end{pmatrix}\overline{(MP)}\begin{pmatrix} i_1 & \ldots & i_r \\ 1 & \ldots & r\end{pmatrix}\right]\left[\sum_{j_1<\ldots<j_r} Q\begin{pmatrix} 1 & \ldots & r \\ j_1 & \ldots & j_r\end{pmatrix}\overline{(QN)}\begin{pmatrix} 1 & \ldots & r \\ j_1 & \ldots & j_r\end{pmatrix}\right]$$

$$= \sum_{\substack{1\le i_1<\ldots<i_r\le r\\ 1\le j_1<\ldots<j_r\le s}} A\begin{pmatrix} i_1 & \ldots & i_r \\ j_1 & \ldots & j_r\end{pmatrix}\overline{(MAN)}\begin{pmatrix} i_1 & \ldots & i_r \\ j_1 & \ldots & j_r\end{pmatrix}. \quad\blacksquare$$

THEOREM 3.2 *Element lying on ith row and jth column of the weighted adjoint matrix of A, denoted as* $\operatorname{adj}\left(A^{(\dagger,r)}_{M\bullet,\bullet N}\right)_{ij}$ *can be represented in terms of square minors as follows:*

$$\operatorname{adj}\left(A^{(\dagger,r)}_{\varphi(M,N)}\right)_{ij} = \sum_{\substack{1\le\alpha_1<\ldots<\alpha_r\le m\\ 1\le\beta_1<\ldots<\beta_r\le n}} \overline{(\omega(M,N))}\begin{pmatrix} \alpha_1 & \ldots & j & \ldots & \alpha_r \\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix} \alpha_1 & \ldots & j & \ldots & \alpha_r \\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}.$$

Proof. Let $\varphi(M,N) = M\bullet, \bullet N$ and consider a full rank factorization $A = PQ$. Element on ith row and jth column of $(QN)^* \operatorname{adj}(Q(QN)^*)$ is equal to

$$
\begin{aligned}
&\sum_{k=1}^{r} (QN)^*_{ik} (\operatorname{adj}(Q(QN)^*))_{kj} \\
&= \sum_{k=1}^{r} \overline{(QN)}_{ki} \left[(-1)^{k+j} \sum_{j_1<\dots<j_{r-1}} Q \left(\begin{smallmatrix} j_1 & \cdots & \cdots & \cdots & \cdots & j_{r-1} \\ 1 & \dots & j-1 & j+1 & \dots & r \end{smallmatrix}\right) \overline{(QN)} \left(\begin{smallmatrix} j_1 & \cdots & \cdots & \cdots & \cdots & j_{r-1} \\ 1 & \dots & k-1 & k+1 & \dots & r \end{smallmatrix}\right) \right] \\
&= \sum_{j_1<\dots<j_{r-1}} (-1)^j Q \left(\begin{smallmatrix} j_1 & \cdots & \cdots & \cdots & \cdots & j_{r-1} \\ 1 & \dots & j-1 & j+1 & \dots & r \end{smallmatrix}\right) \left[\sum_{k=1}^{r} (-1)^k \overline{(QN)}_{ki} \overline{(QN)} \left(\begin{smallmatrix} j_1 & \cdots & \cdots & \cdots & \cdots & j_{r-1} \\ 1 & \dots & k-1 & k+1 & \dots & r \end{smallmatrix}\right) \right].
\end{aligned}
$$

If i is contained in combination $j_1, \dots, j_{r-1}$, then

$$
\sum_{k=1}^{r} (-1)^k \overline{(QN)}_{ki} \overline{(QN)} \left(\begin{matrix} j_1 & \cdots & \cdots & \cdots & \cdots & j_{r-1} \\ 1 & \dots & k-1 & k+1 & \dots & r \end{matrix}\right) = 0.
$$

If the set $\{j_1, \dots, j_{r-1}\}$ does not contain i, then $i = j_p$ and the system is denoted as $j_1, \dots, j_{p-1}$, $j_{p+1}, \dots, j_r$. Now we get the following representation for $\sum\limits_{k=1}^{r} (QN)^*_{ik} (\operatorname{adj}(Q(QN)^*))_{kj}$:

$$
\begin{aligned}
&\sum_{j_1<\dots<j_{p-1}<j_{p+1}<\dots<j_r} (-1)^j Q \left(\begin{smallmatrix} j_1 & \cdots & j_{p-1} & j_{p+1} & \cdots & j_{r-1} \\ 1 & \dots & j-1 & j+1 & \dots & r \end{smallmatrix}\right) (-1)^p \overline{(QN)} \left(\begin{smallmatrix} j_1 & \cdots & \cdots & i=j_p & \cdots & \cdots & j_r \\ 1 & \dots & k-1 & k & k+1 & \dots & r \end{smallmatrix}\right) \\
&\qquad = \sum_{j_1<\dots<i<\dots<j_r} \overline{(QN)} \left(\begin{smallmatrix} j_1 & \cdots & i & \cdots & j_r \\ 1 & \dots & \dots & \dots & r \end{smallmatrix}\right) Q_{ji} \left(\begin{smallmatrix} j_1 & \cdots & i & \cdots & j_r \\ 1 & \dots & \dots & \dots & r \end{smallmatrix}\right).
\end{aligned}
$$

Similarly, element on ith row and jth column of $\operatorname{adj}((MP)^*P)(MP)^*)$ is equal to

$$
\sum_{1\le\alpha_1<\dots<\alpha_r\le m} \overline{(MP)} \left(\begin{matrix} 1 & \dots & \dots & \dots & r \\ \alpha_1 & \dots & j & \dots & \alpha_r \end{matrix}\right) P_{jk} \left(\begin{matrix} 1 & \dots & \dots & \dots & r \\ \alpha_1 & \dots & j & \dots & \alpha_r \end{matrix}\right).
$$

Now, element lying on the ith row an jth column of *weighted adjoint matrix*, denoted as $\operatorname{adj}(A^{(\dagger,r)}_{M\bullet,\bullet N})_{ij}$ is equal to

$$
\begin{aligned}
&\sum_{k=1}^{r} \left[\sum_{1\le\beta_1<\dots<\beta_r\le n} \overline{(QN)} \left(\begin{smallmatrix} \beta_1 & \dots & i & \dots & \beta_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{smallmatrix}\right) Q_{ki} \left(\begin{smallmatrix} 1 & \dots & \dots & \dots & r \\ \beta_1 & \dots & i & \dots & \beta_r \end{smallmatrix}\right) \right] \\
&\qquad \times \left[\sum_{1\le\alpha_1<\dots<\alpha_r\le m} \overline{(MP)} \left(\begin{smallmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ 1 & \dots & \dots & \dots & r \end{smallmatrix}\right) P_{jk} \left(\begin{smallmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ 1 & \dots & \dots & \dots & r \end{smallmatrix}\right) \right] \\
&= \left[\sum_{\substack{1\alpha_1<\dots<\alpha_r\le m \\ 1\le\beta_1<\dots<\beta_r\le n}} \overline{(MAN)} \left(\begin{smallmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{smallmatrix}\right) \right] \left[\sum_{k=1}^{r} P_{jk} \left(\begin{smallmatrix} 1 & \dots & \dots & \dots & r\alpha_1 & \dots & j & \dots & \alpha_r \\ 1 & \dots & \dots & \dots & r & & & & \end{smallmatrix}\right) Q_{ki} \left(\begin{smallmatrix} 1 & \dots & \dots & \dots & r \\ \beta_1 & \dots & i & \dots & \beta_r \end{smallmatrix}\right) \right] \\
&\qquad = \sum_{\substack{1\le\alpha_1<\dots<\alpha_r\le m \\ 1\le\beta_1<\dots<\beta_r\le n}} \overline{(MAN)} \left(\begin{smallmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{smallmatrix}\right) A_{ji} \left(\begin{smallmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{smallmatrix}\right). \qquad \blacksquare
\end{aligned}
$$

THEOREM 3.3 *Element on the ith row and jth column of the weighted Moore-Penrose inverse is equal to*

$$
(A^{(\dagger,r)}_{\varphi(M,N)})_{ij} \;=\; \frac{\operatorname{adj}(A^{(\dagger,r)}_{\varphi(M,N)})_{ji}}{\|A\|^r_{\varphi(M,N)}}.
$$

Proof. Follows from $A^{\dagger}_{M\bullet,\bullet N} = (QN)^*(Q(QN)^*)^{-1} \cdot ((MP)^*P)^{-1}(MP)^*$ and Corollary 1.1. ∎

Theorem 3.3 is an equivalent of Theorem 2.2.

4. Representation of the weighted Moore-Penrose solution of a system of linear equations

In [5] an explicit determinanatal representation of the *Moore-Penrose solution* of an arbitrary system of linear equations is derived. Using this representation it is proved that the *Moore-Penrose solution* is a *convex combination* of all uniquely solvable partial subsystems. In [4] an equivalent determinantal representation for the *least-squares solution* of an overdetermined linear system is derived. From this fromula it is proved that the *least-squares solution* lies in the *convex hull* of the solutions to the square subsystems of the original system. Also, in [4] this result is extended, and it is proved that this geometric property holds for a more general class of problems which includes the *weighted least-squares* and l_p norm ninimization problems.

In the following theorem we derive determinantal representation of the *weighted Moore-Penrose solution* of a system of linear equations.

THEOREM 4.1 *The ith component of the weighted Moore-Penrose solution $x^{\dagger}_{\varphi(M,N)} = A^{\dagger}_{\varphi(M,N)}z$ of a linear system $Ax = z$, $\quad A \in \mathbf{C}_r^{m\times n}$, $x \in \mathbf{C}^n$, $\quad z \in \mathbf{C}^m$ can be represented in the following determinant representation:*

$$(x^{\dagger}_{\varphi(M,N)})_i = \frac{\sum\limits_{\substack{1\le q_1<\ldots<q_r\le n\\ 1\le p_1<\ldots<p_r\le m}} \overline{(\omega(M,N))\begin{pmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r \end{pmatrix}}\, A\begin{pmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r \end{pmatrix}(i \to {}_p z)}{\|A\|^r_{\varphi(M,N)}},$$

where ${}_p z$ denotes the vector $\{z_{p_1}, \ldots, z_{p_r}\}$.

Proof. If $\varphi(M,N) = M\bullet,\bullet N$ and $A = BC$ is a full-rank factorization of A, then

$$x^{\dagger}_{M\bullet,\bullet N} = (CN)^*(C(CN)^*)^{-1}((MB)^*B)^{-1}(MB)^*z = C^{\dagger}_{M\bullet,\bullet N}B^{\dagger}_{M\bullet,\bullet N}z.$$

In this manner, the starting system splits up into two equivalent systems. First we calculate $y^{\dagger}_{M\bullet,\bullet N} = B^{\dagger}_{M\bullet,\bullet N}z$, $\quad y \in \mathbf{C}^n$. In view of $B^{\dagger}_{M\bullet,\bullet N} = \left((MB)^*B\right)^{-1}(MB)^*$, we get $((MB)^*B)y^{\dagger}_{M\bullet,\bullet N} = (MB)^*z$. The ith component of $y^{\dagger}_{M\bullet,\bullet N}$ is

$$(y^{\dagger}_{M\bullet,\bullet N})_i = \frac{|\,((MB)^*B)(i \to (MB)^*z)\,|}{|\,(MB)^*B\,|} = \frac{|\,(MB)^* \cdot B(i \to z)\,|}{|\,(MB)^*B\,|}, \quad 1 \le i \le r.$$

Applying Cauchy-Binet Theorem, we obtain

$$(y^{\dagger}_{M\bullet,\bullet N})_i = \frac{\sum\limits_{1\le p_1<\ldots<p_r\le m} \overline{(MB)\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r \end{pmatrix}}\, B\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r \end{pmatrix}(i \to {}_p z)}{|\,(MB)^*B\,|}, \quad 1 \le i \le r.$$

Also, using $x^{\dagger}_{M\bullet,\bullet N} = C^{\dagger}_{M\bullet,\bullet N}y^{\dagger}_{M\bullet,\bullet N} = (CN)^*(C(CN)^*)^{-1}y^{\dagger}_{M\bullet,\bullet N}$, it is easy to see that

$$(x^{\dagger}_{M\bullet,\bullet N})_i = \frac{1}{|C(CN)^*|} \cdot \left(\sum_{k=1}^{r} ((CN)^* \operatorname{adj}(C(CN)^*))_{ik} (y^{\dagger}_{M\bullet,\bullet N})_k \right).$$

Element on the ith row and jth column of the matrix $(CN)^* \operatorname{adj}(C(CN)^*)$ is (see Theorem 3.2.):

$$((CN)^* \operatorname{adj}(C(CN)^*))_{ij} = \sum_{1 \le q_1 < \ldots < q_r \le n} \overline{(CN)} \begin{pmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r \end{pmatrix} C_{ji} \begin{pmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r \end{pmatrix}.$$

Now $(x^{\dagger}_{M\bullet,\bullet N})_i$ is equal to

$$\frac{\sum\limits_{k=1}^{r} \sum\limits_{q_1 < \ldots < q_r} \overline{(CN)} \left(\begin{smallmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r \end{smallmatrix} \right) C_{ki} \left(\begin{smallmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r \end{smallmatrix} \right) \sum\limits_{p_1 < \ldots < p_r} \overline{(MB)} \left(\begin{smallmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r \end{smallmatrix} \right) B \left(\begin{smallmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r \end{smallmatrix} \right) (k \to {}_p z)}{\left| C(CN)^* \right| \left| (MB)^* B \right|}$$

$$= \frac{\sum\limits_{\substack{1 \le q_1 < \ldots < q_r \le n \\ 1 \le p_1 < \ldots < p_r \le m}} \overline{(MAN)} \left(\begin{smallmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r \end{smallmatrix} \right) \left[\sum\limits_{k=1}^{r} C_{ki} \left(\begin{smallmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r \end{smallmatrix} \right) B \left(\begin{smallmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r \end{smallmatrix} \right) (k \to {}_p z) \right]}{\|A\|^r_{M\bullet,\bullet N}}.$$

By using Laplace's development on the kth column of the square matrix $B \left(\begin{smallmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r \end{smallmatrix} \right) (k \to {}_p z)$, we get

$$(x^{\dagger}_{M\bullet,\bullet N})_i =$$

$$\frac{\sum\limits_{\substack{1 \le q_1 < \ldots < q_r \le n \\ 1 \le p_1 < \ldots < p_r \le m}} \overline{(MAN)} \left(\begin{smallmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r \end{smallmatrix} \right) \left[\sum\limits_{l=1}^{r} z_{p_l} \sum\limits_{k=1}^{r} C_{ki} \begin{pmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r \end{pmatrix} B_{p_l k} \left(\begin{smallmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r \end{smallmatrix} \right) \right]}{\|A\|^r_{M\bullet,\bullet N}}$$

$$= \frac{\sum\limits_{\substack{1 \le q_1 < \ldots < q_r \le n \\ 1 \le p_1 < \ldots < p_r \le m}} \overline{(MAN)} \left(\begin{smallmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r \end{smallmatrix} \right) \sum\limits_{l=1}^{r} z_{p_l} A_{p_l i} \left(\begin{smallmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r \end{smallmatrix} \right)}{\|A\|^r_{M\bullet,\bullet N}}$$

$$= \frac{\sum\limits_{\substack{1 \le q_1 < \ldots < q_r \le n \\ 1 \le p_1 < \ldots < p_r \le m}} \overline{(MAN)} \left(\begin{smallmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r \end{smallmatrix} \right) A \begin{pmatrix} p_1 & \ldots & p_r \\ q_1 & \ldots & q_r \end{pmatrix} (i \to {}_p z)}{\|A\|^r_{M\bullet,\bullet N}}. \quad \blacksquare$$

As we mentioned above, in [4] it is showed that the *weighted least-squares solution* lies in the *convex hull* of the solutions to the square subsystems of the original system. But, this result includes positive definite diagonal weighted matrices. In the following theorem we generalize this result and prove that arbitrary *weighted Moore-Penrose solution* of a linear system lies in the *convex hull* of the solutions to the square subsystems of the original system.

Theorem 4.2 *The weighted Moore-Penrose solution* $x^{\dagger}_{M\bullet,\bullet N}$ *of system of linear equations* $Ax = z$ *is the convex combination*

$$x^{\dagger}_{M\bullet,\bullet N} = \sum_{\substack{1 \le q_1 < \ldots < q_r \le n \\ 1 \le p_1 < \ldots < p_r \le m}} \beta_p \gamma_q x^{(p,q)}$$

of the solutions of all uniquely solvable r-dimensional subsytems canonically imbedded into $\mathbf{C}^m$, *where*

$$\beta = \sum_{1\le\alpha_1<\ldots<\alpha_r\le m} \overline{(MB)}\begin{pmatrix}\alpha_1 & \ldots & \alpha_r\\ 1 & \ldots & r\end{pmatrix} B\begin{pmatrix}\alpha_1 & \ldots & \alpha_r\\ 1 & \ldots & r\end{pmatrix};$$

$$\gamma = \sum_{1\le\beta_1<\ldots<\beta_r\le n} \overline{(CN)}\begin{pmatrix}1 & \ldots & r\\ \beta_1 & \ldots & \beta_r\end{pmatrix} B\begin{pmatrix}1 & \ldots & r\\ \beta_1 & \ldots & \beta_r\end{pmatrix};$$

$$\beta_p = \frac{1}{\beta}\overline{(MB)}\begin{pmatrix}p_1 & \ldots & p_r\\ 1 & \ldots & r\end{pmatrix} B\begin{pmatrix}p_1 & \ldots & p_r\\ 1 & \ldots & r\end{pmatrix};$$

$$\gamma_q = \frac{1}{\gamma}\overline{(CN)}\begin{pmatrix}1 & \ldots & r\\ q_1 & \ldots & q_r\end{pmatrix} C\begin{pmatrix}1 & \ldots & r\\ q_1 & \ldots & q_r\end{pmatrix}.$$

Proof. According to Theorem 4.1. $(x^{\dagger}_{M\bullet,\bullet N})_i$ has the following determinantal representation

$$\frac{\displaystyle\sum_{\substack{1\le q_1<\ldots<q_r\le n\\ 1\le p_1<\ldots<p_r\le m}} \overline{(MB)}\begin{pmatrix}p_1 & \ldots & p_r\\ 1 & \ldots & r\end{pmatrix}\overline{(CN)}\begin{pmatrix}1 & \ldots & \ldots & \ldots & r\\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix} A\begin{pmatrix}p_1 & \ldots & p_r\\ q_1 & \ldots & q_r\end{pmatrix}(i\to {}_pz)}{\left[\displaystyle\sum_{1\le\alpha_1<\ldots<\alpha_r\le m} P\begin{pmatrix}\alpha_1 & \ldots & \alpha_r\\ 1 & \ldots & r\end{pmatrix}\overline{(MP)}\begin{pmatrix}\alpha_1 & \ldots & \alpha_r\\ 1 & \ldots & r\end{pmatrix}\right]\left[\displaystyle\sum_{1\le\beta_1<\ldots<\beta_r\le n} Q\begin{pmatrix}1 & \ldots & r\\ \beta_1 & \ldots & \beta_r\end{pmatrix}\overline{(QN)}\begin{pmatrix}1 & \ldots & r\\ \beta_1 & \ldots & \beta_r\end{pmatrix}\right]}$$

$$= \sum_{\substack{q_1<\ldots<q_r\\ p_1<\ldots<p_r}} \frac{1}{\beta}\overline{(MB)}\begin{pmatrix}p_1 & \ldots & p_r\\ 1 & \ldots & r\end{pmatrix} B\begin{pmatrix}p_1 & \ldots & p_r\\ 1 & \ldots & r\end{pmatrix}\frac{1}{\gamma}\overline{(CN)}\begin{pmatrix}1 & \ldots & \ldots & \ldots & r\\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix} C\begin{pmatrix}1 & \ldots & \ldots & \ldots & r\\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}\times$$

$$\times\frac{A\begin{pmatrix}p_1 & \ldots & p_r\\ q_1 & \ldots & q_r\end{pmatrix}(i\to {}_pz)}{A\begin{pmatrix}p_1 & \ldots & p_r\\ q_1 & \ldots & q_r\end{pmatrix}}.$$

In the case $A\begin{pmatrix}p_1 & \ldots & p_r\\ q_1 & \ldots & q_r\end{pmatrix}\ne 0$ let $x^{(p,q)}$ be the canonical imbedding of the solution of $A\begin{bmatrix}p_1 & \ldots & p_r\\ q_1 & \ldots & q_r\end{bmatrix} x = {}_pz$ into the m-dimensional space. This means that, according to Cramer's rule, $x^{(p,q)}$ possesses the components

$$x_i^{(p,q)} = \frac{A\begin{pmatrix}p_1 & \ldots & p_r\\ q_1 & \ldots & q_r\end{pmatrix}(i\to {}_pz)}{A\begin{pmatrix}p_1 & \ldots & p_r\\ q_1 & \ldots & q_r\end{pmatrix}}$$

for i contained in combination $1\le q_1<\ldots<q_r\le n$, and $x_i^{(p,q)}=0$ otherwise. In the singular case $A\begin{pmatrix}p_1 & \ldots & p_r\\ q_1 & \ldots & q_r\end{pmatrix}=0$ we define $x^{(p,q)}$ to be the zero vector. Now it is evident that

$$x^{\dagger}_{M\bullet,\bullet N} = \sum_{\substack{1\le q_1<\ldots<q_r\le n\\ 1\le p_1<\ldots<p_r\le m}} \beta_p\gamma_q x^{(p,q)}.$$

Since

$$\sum_{1\le p_1<\ldots<p_r\le m}\beta_p = 1, \qquad \sum_{1\le q_1<\ldots<q_r\le n}\gamma_q = 1$$

the proof is completed. ∎

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(received 14.06.1994.)

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