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ON A THEOREM OF KY-FAN TYPE

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Abstract. We prove a statement of the Ky-Fan type which makes it possible in some situations to find the second term of the singular values asymptotics of a sum of two operators.

Finding the first term in the asymptotic expansion of singular values of some classes of operators [1] is very convenient using the following theorem of Ky-Fan [2; Theorem 2.3, p. 52]:

THEOREM 1. *If A and B are compact operators on a Hilbert space $\mathcal{H}$ such that for some $a>0$, $r>0$*

$$\lim_{n\to\infty} n^r s_n(A)=a, \qquad \lim_{n\to\infty} n^r s_n(B)=0,$$

then $\lim_{n\to\infty} n^r s_n(A+B)=a$.

Here, $s_n(T)=\lambda_n((T^*T)^{1/2})$. $\big(\lambda_n((T^*T)^{1/2})\big)$ denote the non increasing sequence of the eigenvalues of $(T^*T)^{1/2}$.

Finding the higher terms in the singular values asymptotics of a sum of two operators cannot be done using the previous result. The following statement of the Ky-Fan type makes it possible in some situations to find the second term of the singular values asymptotics of a sum of two operators.

THEOREM 2. *If A and B are compact operators on a Hilbert space $\mathcal{H}$ such that*

$$\begin{aligned} &s_n(A)=a_1 n^{-\alpha_1}+a_2 n^{-\alpha_2}+o(n^{-\alpha_2}) \qquad (a_1>0,\ a_2\in \mathbf{R})\\ &s_n(B)=o(n^{-\beta_1}) \end{aligned} \tag{1}$$

and $0<\alpha_1<\alpha_2<\alpha_1+1$, $\beta_1>\alpha_2/(1+\alpha_1-\alpha_2)$, *then*

$$s_n(A+B)=a_1 n^{-\alpha_1}+a_2 n^{-\alpha_2}+o(n^{-\alpha_2}). \tag{2}$$

Proof. Since $0<\alpha_1<\alpha_2<\alpha_1+1$, from $\beta_1>\alpha_2/(1+\alpha_1-\alpha_2)$ it follows

$$\frac{\beta_1-\alpha_2}{\alpha_2}>\frac{\alpha_2-\alpha_1}{\alpha_1+1-\alpha_2}.$$

Let θ be some number such that

$$\frac{\beta_1-\alpha_2}{\alpha_2}>\theta>\frac{\alpha_2-\alpha_1}{\alpha_1+1-\alpha_2} \tag{3}$$

and let $k=k(n)=[n^{\frac{\theta}{1+\theta}}]-1$, $m=m(n)=[n^{\frac{1}{1+\theta}}]$ ($[x]$ is the integer part of x). Since every integer n can be represented in the form

$$n=(k+1)m+j,\quad j=j(n)\in\mathbf{N},\quad 0\leqslant j\leqslant n^{\frac{1}{1+\theta}}+n^{\frac{\theta}{1+\theta}},$$

then from the properties of the singular values of a sum of two operators [2] it follows

$$s_n(A+B)\leqslant s_{km+j}(A)+s_{m+1}(B)$$

and consequently

$$\begin{aligned} &n^{\alpha_2}\left(s_n(A+B)-\frac{a_1}{n^{\alpha_2}}-\frac{a_2}{n^{\alpha_2}}\right)\\ &\leqslant\left(\frac{n}{km+j}\right)^{\alpha_2}(km+j)^{\alpha_2}\left[s_{km+j}(A)-\frac{a_1}{(km+j)^{\alpha_1}}-\frac{a_2}{(km+j)^{\alpha_2}}\right]+n^{\alpha_2}s_{m+1}(B)\\ &\qquad+a_1n^{\alpha_2}\left[\frac{1}{(km+j)^{\alpha_1}}-\frac{1}{n^{\alpha_1}}\right]+a_2n^{\alpha_2}\left[\frac{1}{(km+j)^{\alpha_2}}-\frac{1}{n^{\alpha_2}}\right].\end{aligned}$$

Now, we prove that

$$\varlimsup_{n\to\infty}n^{\alpha_2}\left[s_n(A+B)-\frac{a_1}{n^{\alpha_1}}-\frac{a_2}{n^{\alpha_2}}\right]\leqslant 0. \tag{4}$$

We observe that $k=k(n)\sim m^{\theta}=m(n)^{\theta}$ and $\lim_{n\to\infty}\frac{j(n)}{n}=0$ ($=\lim_{n\to\infty}\frac{j}{km}$). Since $n=(k+1)m+j\to\infty$ (when $m=m(n)\to\infty$) then from (1) (because $(n/(km+j))^{\alpha_2}\leqslant 2^{\alpha_2}$) it follows

$$\lim_{n\to\infty}\left(\frac{n}{km+j}\right)^{\alpha_2}(km+j)^{\alpha_2}\left[s_{km+j}(A)-\frac{a_1}{(km+j)^{\alpha_1}}-\frac{a_2}{(km+j)^{\alpha_2}}\right]=0.$$

It is enough to prove that

$$\begin{aligned}\lim_{n\to\infty}((k+1)m+j)^{\alpha_2}s_m(B)=0,\\ \lim_{n\to\infty}((k+1)m+j)^{\alpha_2}\left[\frac{1}{(km+j)^{\alpha_1}}-\frac{1}{n^{\alpha_1}}\right]=0,\\ \lim_{n\to\infty}n^{\alpha_2}\left[\frac{1}{(km+j)^{\alpha_2}}-\frac{1}{n^{\alpha_2}}\right]=0.\end{aligned} \tag{5}$$

Since

$$\begin{aligned}((k+1)m+j)^{\alpha_2}s_m(B)\sim\frac{m^{\theta\alpha_2}}{m^{\beta_1-\alpha_2}}m^{\beta_1}s_m(B),\\ ((k+1)m+j)^{\alpha_2}\left[\frac{1}{(km+j)^{\alpha_1}}-\frac{1}{((k+1)m+j)^{\alpha_1}}\right]\sim\alpha_1\frac{m^{\alpha_2-\alpha_1}}{m^{\theta(\alpha_1+1-\alpha_2)}},\\ ((k+1)m+j)^{\alpha_2}\left[\frac{1}{(km+j)^{\alpha_2}}-\frac{1}{((k+1)m+j)^{\alpha_2}}\right]\sim\frac{\alpha_2}{m^{\theta}}\quad(n\to\infty)\end{aligned}$$

then (5) follows from (1) and (3).

From $A = A + B - B$ it follows

$$s_{km+j}(A+B) \geqslant s_{(k+1)m+j}(A) - s_{m+1}(B).$$

Similarly, starting from the previous inequality, we get

$$\varliminf_{n\to\infty} n^{\alpha_2}(s_n(A+B) - a_1 n^{-\alpha_1} - a_2 n^{-\alpha_2}) \geqslant 0. \tag{6}$$

From (4) and (6) the statement of Theorem 2 follows. ■

Remark. The proof is carried out without any change if we start from the more general assumptions

$$s_n(A) = \frac{L_1(n)}{n^{\alpha_1}} + \frac{L_2(n)}{n^{\alpha_2}} + o\left(\frac{L_2(n)}{n^{\alpha_2}}\right),$$
$$s_n(B) = o\left(\frac{L_3(n)}{n^{\beta_1}}\right),$$

where $0 < \alpha_1 < \alpha_2 < \alpha_1 + 1$, $\beta_1 > \alpha_2/(1 + \alpha_1 - \alpha_2)$ and L_i $(i = 1, 2, 3)$ are slowly varying functions. Then

$$s_n(A+B) = \frac{L_1(n)}{n^{\alpha_1}} + \frac{L_2(n)}{n^{\alpha_2}} + o\left(\frac{L_2(n)}{n^{\alpha_2}}\right).$$

We give an example how Theorem 2 can be applied. By $\int_{-\pi}^{\pi} R(x,y)\ \cdot\ dy$ we denote operator acting on $L^2(-\pi,\pi)$ with the kernel $R(\cdot\,,\cdot)$.

Let $H(x,y) = k(x-y) + K(x,y)$, where the function k is even, periodic with the period 2π, such that

$$\left|\int_{-\pi}^{\pi} k(t)\cos nt\, dt\right| = a_1 n^{-\alpha_1} + a_2 n^{-\alpha_2} + o(n^{-\alpha_2}) \quad (\alpha_1 < a_2 < \alpha_1 + 1, a_1 > 0). \tag{7}$$

Theorem 3. *If for some* $r > \left[\dfrac{\alpha_2}{1+\alpha_1-\alpha_2}\right]$ *the function* $\partial^r K/\partial y^r$ *is bounded on* $[-\pi,\pi]^2$, *then*

$$s_n\left(\int_{-\pi}^{\pi} H(x,y)\ \cdot\ dy\right) = a_1 n^{-\alpha_1} + a_2 n^{-\alpha_2} + o(n^{-\alpha_2}). \tag{8}$$

Proof. It is well known that the even periodic function k with period 2π generates an integral operator on $L^2(-\pi,\pi)$ whose singular values are $|\int_{-\pi}^{\pi} k(t)\cos nt\, dt|$. Since $\partial^r K/\partial y^r$ is a bounded function on $[-\pi,\pi]^2$, then by Krein theorem [2; p. 157] we have

$$s_n\left(\int_{-\pi}^{\pi} K(x,y)\ \cdot\ dy\right) = o\left(n^{-r-\frac{1}{2}}\right).$$

From Theorem 2, previous asymptotic formula and (7) we get (8). ■

REFERENCES

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