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ON THE CONVERGENCE OF A THREE-LEVEL VECTOR SOR SCHEME

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Abstract. In this paper we consider a vector alternating directions difference scheme for solving multidimensional wave equation. The scheme reduces to a modified block successive overrelaxation (SOR) algorithm. The stability and the convergence of the scheme are investigated.

As a model problem let us consider the first initial-boundary value problem (IBVP) for the multidimensional wave equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = \Delta u + f\,, \qquad (x,\,t) \in Q = \Omega \times (0,\,T) = (0,\,1)^n \times (0,\,T)\,, \\ &u(x,\,0) = u_0(x)\,, \qquad \frac{\partial u(x,\,0)}{\partial t} = u_1(x)\,, \qquad x \in \Omega\,, \\ &u(x,\,t) = 0, \qquad x \in \Gamma = \partial\Omega\,, \quad t \in (0,\,T)\,. \end{aligned} \tag{1}$$

We assume that the generalized solution of IBVP (1) belongs to the Sobolev space $W_2^s(Q)$, $s \geq 2$ [7]. In this case there exists a trace $u|_{t=t'} \in W_2^{s-1/2}(\Omega) \subset L_2(\Omega)$ for $t' \in [0,\,T]$. We also assume that the solution u can be oddly extended in space variables outside the domain Ω, preserving the Sobolev class.

Let $\overline{\omega}$ be uniform mesh in $\overline{\Omega}$ with step size h. Let us denote $\omega = \overline{\omega} \cap \Omega$, $\gamma = \overline{\omega} \setminus \omega$ and $\omega_i = \omega \cup \{x = (x_1,\,\ldots,\,x_n) \in \gamma \,|\, x_i = 0\}$. Let $\overline{\theta}$ be uniform mesh on $[-\tau/2,\,T]$ with step size τ, and $\theta = \overline{\theta} \cap (0,\,T)$. Finally, let $\overline{Q}_{h\tau} = \overline{\omega} \times \overline{\theta}$. For a function v defined on the mesh $\overline{Q}_{h\tau}$ we define the finite-difference operators v_{x_i}, $v_{\bar{x}_i}$, v_t and $v_{\bar{t}}$ in the usual manner [8]. Let us denote $v = v(x,\,t)$, $\hat{v} = v(x,\,t+\tau)$ and $\check{v} = v(x,\,t-\tau)$.

Let H_h be the set of discrete functions defined on the mesh $\overline{\omega}$, which vanish on γ. Let us denote

$$\Lambda_i\, v = \begin{cases} -v_{x_i\bar{x}_i}\,, & x \in \omega \\ 0\,, & x \in \gamma \end{cases} \qquad \text{and} \qquad \Lambda\, v = \sum_{i=1}^n \Lambda_i\, v\,.$$

The identity operator on H_h will be denoted by I.

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We introduce the following discrete inner product $(v,\,w)_\omega = h^n \sum\limits_{x\in\omega} v(x)\,w(x)$ and the norms

$$\|v\|_\omega = (v,\,v)_\omega^{1/2} = \Big(h^n \sum_{x\in\omega} v^2(x)\Big)^{1/2} \qquad \text{and} \qquad \|v\|_{\omega_i} = \Big(h^n \sum_{x\in\omega_i} v^2(x)\Big)^{1/2} .$$

For a linear, selfadjoint and nonnegative operator A on H_h with $\|v\|_A$ we denote so called "energy" seminorm $\|v\|_A = (A\,v,\,v)_\omega^{1/2}$. In particular

$$\|v\|_{\Lambda_i} = (\Lambda_i\,v,\,v)_\omega^{1/2} = \|v_{x_i}\|_{\omega_i} .$$

With T_i and T_t we denote the Steklov averaging operators in space variables x_i and time variable t (see [4])

$$T_i f(x,\,t) = \frac{1}{h}\int_{x_i-h/2}^{x_i+h/2} f(x_1,\,\ldots\,,\,x_i',\,\ldots\,,\,x_n,\,t)\,dx_i' \;,$$
$$T_t f(x,\,t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} f(x_1,\,\ldots\,,\,x_n,\,t')\,dt' \;.$$

Finally, C will stand for a positive generic constant, independent of h and τ.

We approximate IBVP (1) by the following alternating direction finite-difference scheme (FDS) (see [1], [9])

$$\begin{gathered} v^i_{t\bar t} + \frac{1}{2}\sum_{j=1}^{i-1}(\Lambda_j\,\hat v^j + \Lambda_j\,\check v^j) + \frac{1}{2\,\sigma}\,\Lambda_i\,(\hat v^i + \check v^i) \\ +\Big(1-\frac{1}{\sigma}\Big)\,\Lambda_i\,v^i + \sum_{j=i+1}^{n}\Lambda_j\,v^j = T_1\cdots T_n T_t\,f\,, \qquad t\in\theta\,, \\ v^i\big|_{t=\mp\tau/2} = T_1\cdots T_n\,(u_0 \mp 0.5\,\tau\,u_1)\,, \qquad i=1,\,2,\,\ldots\,,\,n\,. \end{gathered} \tag{2}$$

Here σ is a positive free parameter. Equation (2) can be rewritten in the form

$$\Big(I + \frac{\tau^2}{2\,\sigma}\,\Lambda_i\Big)\,v^i_{t\bar t} + \frac{\tau^2}{2}\sum_{j=1}^{i-1}\Lambda_j\,v^j_{t\bar t} + \sum_{j=1}^{n}\Lambda_j\,v^j = T_1\cdots T_n T_t\,f\,, \qquad t\in\theta\,.$$

FDS (2) is economic since the evaluation of v^i on the next time level reduces to the invertion of the operator $I + \frac{\tau^2}{2\,\sigma}\,\Lambda_i$, represented by threediagonal matrix. It can be treated as a operator (vector) variant of the successive overrelaxation method [3]. A similar FDS, close to Jacobi overrelaxation method (JOR) was considered in [2] and [5].

The errors defined as $z^i = T_1\cdots T_n\,u - v^i$ satisfy FDS

$$\begin{gathered} \Big(I + \frac{\tau^2}{2\,\sigma}\,\Lambda_i\Big)\,z^i_{t\bar t} + \frac{\tau^2}{2}\sum_{j=1}^{i-1}\Lambda_j\,z^j_{t\bar t} + \sum_{j=1}^{n}\Lambda_j\,z^j = \varphi^i\,, \qquad t\in\theta\,, \\ z^i_t\big|_{t=-\tau/2} = \beta\,, \qquad 0.5\,(z^i + \hat z^i)\big|_{t=-\tau/2} = \delta\,, \qquad i=1,\,2,\,\ldots\,,\,n \end{gathered} \tag{3}$$

where

$$\varphi^i = \xi + \sum_{j=1}^{n} \eta^j - \frac{\tau^2}{2} \sum_{j=1}^{i-1} \zeta^j - \frac{\tau^2}{2\,\sigma}\,\zeta^i\,,$$

$$\xi \;= T_1 \cdots T_n \left(u_{t\bar{t}} - T_t\,\frac{\partial^2 u}{\partial t^2} \right),$$

$$\eta^j = T_1 \cdots T_n \left(T_t\,\frac{\partial^2 u}{\partial x_j^2} - u_{x_j \bar{x}_j} \right),$$

$$\zeta^j = \frac{\tau^2}{2}\, T_1 \cdots T_n\, u_{x_j \bar{x}_j t\bar{t}}\,,$$

$$\beta \;= T_1 \cdots T_n \left(T_t\,\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} \right)\Big|_{t=0}\,,$$

$$\delta \;= 0.5\,T_1 \cdots T_n \left(\,u|_{t=-\tau/2} - 2\;u|_{t=0} + u|_{t=\tau/2} \right).$$

To prove the stability and the convergence of FDS (2) we represent equation (3) in the matrix form

$$\begin{gathered} \left[(\mathbf{I} + \frac{\tau^2}{2} \left(\mathbf{L} + \frac{1}{\sigma}\,\mathbf{I} \right) \Lambda \right] \mathbf{z}_{t\bar{t}} + \mathbf{E}\,\Lambda \mathbf{z} = \Phi\,, \qquad t \in \theta\,, \\ \mathbf{z}_t|_{t=-\tau/2} = \mathbf{b}\,, \qquad 0.5\,(\mathbf{z} + \hat{\mathbf{z}})|_{t=-\tau/2} = \mathbf{d}\,, \end{gathered} \tag{4}$$

where $\mathbf{z} = (z^1,\, \ldots\,,\, z^n)^T$, $\Phi = (\varphi^1,\, \ldots\,,\, \varphi^n)^T$, $\mathbf{I} = \operatorname{diag}(I,\, \ldots\,,\, I)$, $\Lambda = \operatorname{diag}(\varLambda_1,\, \ldots\,,\, \varLambda_n)$, $\mathbf{b} = (\beta,\, \ldots\,,\, \beta)^T$, $\mathbf{d} = (\delta,\, \ldots\,,\, \delta)^T$,

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ I & 0 & \ldots & 0 & 0 \\ I & I & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I & I & \ldots & I & 0 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 0 & I & \ldots & I & I \\ 0 & 0 & \ldots & I & I \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & I \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{E} = \mathbf{L} + \mathbf{I} + \mathbf{U}\,.$$

Let us also define the inner product and the associated norm of vector-functions

$$(\mathbf{z},\, \mathbf{w}) = \sum_{i=1}^{n} (z^i,\, w^i)_\omega\,, \qquad \|\mathbf{z}\| = (\mathbf{z},\, \mathbf{z})^{1/2}\,.$$

For a three-level FDS in the form

$$\mathbf{C}\,\mathbf{z}_{t\bar{t}} + \mathbf{A}\,\mathbf{z} = \Psi\,, \tag{5}$$

using the energy method [8] one easily prove the following proposition.

LEMMA 1. *If* $\mathbf{A} = \mathbf{A}^* \geq \mathbf{0}$, *and* $\mathbf{C} - 0.25\,\tau^2\,\mathbf{A} \geq \mathbf{D} = \mathbf{D}^* > \mathbf{0}$ *then FDS* (5) *is stable and the a priori estimate*

$$\max_{t \in \theta} N(\mathbf{z}) \leq N(\mathbf{z})\big|_{t=-\tau/2} + \tau \sum_{t \in \theta} \|\Psi\|_{\mathbf{D}^{-1}}\,,$$

where

$$N^2(\mathbf{z}) = \left\|\mathbf{z}_t\right\|^2_{\mathbf{C}-0.25\,\tau^2\,\mathbf{A}} + \left\|\frac{\mathbf{z}+\hat{\mathbf{z}}}{2}\right\|^2_{\mathbf{A}}$$

holds.

Applying Λ to (4) we obtain a FDS in the canonical form (5), where $\mathbf{A} = \Lambda\,\mathbf{E}\,\Lambda = \mathbf{A}^* \geq \mathbf{0}$, $\mathbf{C} = \Lambda + \frac{\tau^2}{2}\,\Lambda\,\mathbf{L}\,\Lambda + \frac{\tau^2}{2\,\sigma}\,\Lambda^2$ and $\Psi = \Lambda\Phi$. One can easily acknowledge that

$$\big((\mathbf{C}-0.25\,\tau^2\,\mathbf{A})\,\mathbf{z},\,\mathbf{z}\big) = (\Lambda\,\mathbf{z},\,\mathbf{z}) + \tau^2\,\frac{2-\sigma}{4\,\sigma}\,(\Lambda\,\mathbf{z},\,\Lambda\,\mathbf{z})\,,$$

which means that for $0<\sigma<2$, $\mathbf{C}-0.25\,\tau^2\,\mathbf{A} \geq \Lambda > \mathbf{0}$, and, consequently, FDS (5) is stable.

From lemma 1 we obtain the a priori estimate

$$\max_{t\in\theta} N(\mathbf{z}) \leq N(\mathbf{z})\big|_{t=-\tau/2} + \tau \sum_{t\in\theta} \|\Psi\|_{\Lambda^{-1}}\,. \tag{6}$$

Further

$$\begin{aligned}
N^2(\mathbf{z}) &= \left\|\mathbf{z}_t\right\|^2_{\mathbf{C}-0.25\,\tau^2\,\mathbf{A}} + \left\|\frac{\mathbf{z}+\hat{\mathbf{z}}}{2}\right\|^2_{\mathbf{A}} \geq \left\|\mathbf{z}_t\right\|^2_{\Lambda} + \left\|\Lambda\,\frac{\mathbf{z}+\hat{\mathbf{z}}}{2}\right\|^2_{\mathbf{E}} \\
&= \sum_{i=1}^{n} \left\|z^i_t\right\|^2_{\Lambda_i} + \left\|\sum_{i=1}^{n} \Lambda_i\,\frac{z^i+\hat{z}^i}{2}\right\|^2_{\omega} \equiv \left\|\mathbf{z}\right\|^2_2\,, \\
N^2(\mathbf{z})\big|_{t=-\tau/2} &= \|\mathbf{b}\|^2_{\mathbf{C}-0.25\,\tau^2\,\mathbf{A}} + \|\mathbf{d}\|^2_{\mathbf{A}} \\
&= \sum_{i=1}^{n} \left(\|\beta\|^2_{\Lambda_i} + \tau^2\,\frac{2-\sigma}{4\,\sigma}\,\|\Lambda_i\,\beta\|^2_{\omega}\right) + \left\|\sum_{i=1}^{n} \Lambda_i\,\delta\right\|^2_{\omega}\,, \\
\left\|\Psi\right\|_{\Lambda^{-1}} &= \left\|\Phi\right\|_{\Lambda} = \left(\sum_{i=1}^{n} \left\|\varphi^i\right\|^2_{\Lambda_i}\right)^{1/2}.
\end{aligned}$$

Substituting in (6), when $\tau \asymp h$ (i.e. $C_1\,h \leq \tau \leq C_2\,h$), we obtain

$$\max_{t\in\theta} \|\mathbf{z}\|_2 \leq C \sum_{i=1}^{n} \left(\left\|\beta_{x_i}\right\|_{\omega_i} + \left\|\delta_{x_i\bar{x}_i}\right\|_{\omega} + \tau \sum_{t\in\theta} \left\|\varphi^i_{x_i}\right\|_{\omega_i}\right). \tag{7}$$

To prove the convergence of FDS (2) we must estimate the terms $\varphi^i_{x_i} = \xi_{x_i} + \sum_{j=1}^{n} \eta^j_{x_i} - \frac{\tau^2}{2} \sum_{j=1}^{i-1} \zeta^j_{x_i} - \frac{\tau^2}{2\,\sigma}\,\zeta^i_{x_i}$, β_{x_i} and $\delta_{x_i\bar{x}_i}$. The value of ξ_{x_i} in the node $(x,\,t) \in \omega_i \times \theta$ is a bounded linear functional of $u \in W^s_2(e)$, where $e = \prod_{l=1}^{n}(x_l-2h,\,x_l+2h) \times (t-\tau,\,t+\tau)$ and $s \geq 2$. Moreover, $\xi^j_{x_i}$ vanishes on the polynomials of the fourth degree. Using the Bramble-Hilbert lemma and the methodology proposed in [6] and developed in [4], for $\tau \asymp h$, we obtain

$$|\xi_{x_i}| \leq C\,h^{s-4-n/2}\,|u|_{W^s_2(e)}\,, \qquad 3 \leq s \leq 5\,.$$

From here, by summation over the meshes ω_i and θ, it follows

$$\tau \sum_{t\in\theta} \|\xi_{x_i}\|_{\omega_i} \le C\, h^{s-3}\, \|u\|_{W_2^s(Q)}\,, \qquad 3 \le s \le 5\,.$$

In the same manner we can estimate $\eta^j_{x_i}$, $\zeta^j_{x_i}$, β_{x_i} and $\delta_{x_i\bar{x}_i}$. From these estimates and the inequality (7) we obtain the following convergence rate estimate for FDS (2):

$$\max_{t\in\theta} \|\mathbf{z}\|_2 \le C\, h^{s-3}\, \|u\|_{W_2^s(Q)}\,, \qquad 3 \le s \le 5\,. \tag{8}$$

Another group of convergence rate estimates can be obtained in the following way. Denoting

$$z = \Lambda^{-1} \sum_{i=1}^{n} \Lambda_i\, z^i\,,$$

from (3) we obtain

$$\Big(I + \frac{\tau^2}{2\,\sigma}\,\Lambda_i\Big)\, z^i_{t\bar{t}} + \frac{\tau^2}{2} \sum_{j=1}^{i-1} \Lambda_j\, z^j_{t\bar{t}} + \Lambda\, z = \varphi^i\,.$$

After solving in $z^i_{t\bar{t}}$, applying the operator Λ_i and summing over i we obtain

$$\begin{gathered} \mathbf{z}_{t\bar{t}} + \mathbf{A}\,\mathbf{z} = \Psi\,, \qquad t \in \theta\,, \\ \mathbf{z}_t\big|_{t=-\tau/2} = \beta\,, \qquad 0.5\,(\mathbf{z} + \hat{\mathbf{z}})\big|_{t=-\tau/2} = \delta\,, \end{gathered} \tag{9}$$

where

$$\mathbf{A} = \sum_{i=1}^{n} A_i\,, \qquad \Psi = \Lambda^{-1} \sum_{i=1}^{n} \hat{A}_i\, \varphi^i\,.$$

$$A_i = \Lambda_i \Big(I + \frac{\tau^2}{2\,\sigma}\,\Lambda_i\Big)^{-1} \prod_{j=1}^{i-1} \Big(I + \frac{\tau^2}{2\,\sigma}\,\Lambda_j\Big)^{-1} \Big[I + \frac{\tau^2}{2}\Big(\frac{1}{\sigma} - 1\Big)\Lambda_j\Big]\,,$$

$$\hat{A}_i = \Lambda_i \Big(I + \frac{\tau^2}{2\,\sigma}\,\Lambda_i\Big)^{-1} \prod_{j=i+1}^{n} \Big(I + \frac{\tau^2}{2\,\sigma}\,\Lambda_j\Big)^{-1} \Big[I + \frac{\tau^2}{2}\Big(\frac{1}{\sigma} - 1\Big)\Lambda_j\Big]\,.$$

For $0 < \sigma < 2$ we have

$$-\Lambda_i \le A_i = A_i^* \le \Lambda_i\,, \qquad -\Lambda_i \le \hat{A}_i = \hat{A}_i^* \le \Lambda_i\,, \qquad 0 < \mathbf{A} = \mathbf{A}^* \le \Lambda \qquad \text{and}$$

$$\begin{aligned} I - \frac{\tau^2}{4}\,\mathbf{A} &= \frac{1}{2} \prod_{j=1}^{n} \Big(I + \frac{\tau^2}{2\,\sigma}\,\Lambda_j\Big)^{-1} \Big\{ \prod_{j=1}^{n} \Big(I + \frac{\tau^2}{2\,\sigma}\,\Lambda_j\Big) + \prod_{j=1}^{n} \Big[I + \frac{\tau^2}{2}\Big(\frac{1}{\sigma} - 1\Big)\Lambda_j\Big] \Big\} \\ &\ge \prod_{j=1}^{n} \Big(I + \frac{\tau^2}{2\,\sigma}\,\Lambda_j\Big)^{-1} > 0\,. \end{aligned}$$

In such a way, accordingly to lemma 1, FDS (9) is unconditionally stable. In the case when $\tau \asymp h$ we also have

$$I - 0.25\,\tau^2\,\mathbf{A} \geq c\,I\,, \qquad \mathbf{A} \geq c\,\Lambda\,,$$

where c is a positive constant.

Using these relations and lemma 1 we obtain the a priori estimate

$$\begin{aligned} \max_{t\in\theta} \|z\|_1 &\equiv \max_{t\in\theta} \left(\left\| z_t \right\|_\omega^2 + \left\| \frac{z+\hat z}{2} \right\|_\Lambda^2 \right)^{1/2} \\ &\leq C \left(\|\beta\|_\omega + \sum_{i=1}^n \left\| \delta_{x_i} \right\|_{\omega_i} + \tau \sum_{t\in\theta} \sum_{i=1}^n \|\varphi^i\|_\omega \right). \end{aligned} \tag{10}$$

Similarly, applying operator $\mathbf{A}^{k-1}$ $(k = 2, 3, \dots)$ to (9) and repeating the same procedure, we obtain

$$\begin{aligned} \max_{t\in\theta} \|z\|_k &\equiv \max_{t\in\theta} \left(\left\| z_t \right\|_{\Lambda^{k-1}}^2 + \left\| \frac{z+\hat z}{2} \right\|_{\Lambda^k}^2 \right)^{1/2} \\ &\leq C \left(\|\beta\|_{\Lambda^{k-1}} + \left\| \delta \right\|_{\Lambda^k} + \tau \sum_{t\in\theta} \sum_{i=1}^n \|\varphi^i\|_{\Lambda^{k-1}} \right). \end{aligned} \tag{11}$$

In such a way, the problem of deriving the convergence rate estimate for FDS (9), or (2), is now reduced to estimation of the right hand side terms in (10) and (11). Using the Bramble-Hilbert lemma, in the same manner as in the previous case, from (10–11) we obtain

$$\max_{t\in\theta} \|z\|_k \leq C\,h^{s-k-1}\,\|u\|_{W_2^s(Q)}\,, \qquad k+1 \leq s \leq k+3\,; \qquad k = 1, 2, \dots \tag{12}$$

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