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CHARACTERIZATION OF REGULAR SEMIRINGS

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Abstract. The main purpose of this paper is to establish some necessary and sufficient conditions for a semiring to be regular, in terms of its k-ideals.

1. Introduction

A semiring is a non-empty set R equipped with two binary operations, called addition, $+$, and multiplication, denoted by juxtaposition such that R is multiplicatively a semigroup and additively a commutative semigroup and that the multiplication is distributive with respect to the addition both from the left and from the right. An element denoted by 0 is called the zero of R if $a+0=a$ and $a0=0a=0$ for all $a \in R$. A semiring is said to be regular in the sense of von Neumann (cf. [1]) if for every element $a \in R$, there exist some x, $y \in R$ such that $a+axa=aya$. A k-ideal I of a semiring R is an ideal such that, if $a \in I$ and $x \in R$ and $a+x \in I$ then $x \in I$.

DEFINITION 1.1. Let I be a sebsemiring of a semiring R. Then $\overline{I} = \{\, a \in R \mid a+x \in I \text{ for some } x \in I \,\}$ is called k-closure of I.

It is easy to check that, if I is an ideal of R, then $\overline{I}$ is a k-ideal. In fact, it is the smallest k-ideal containing I and $I=\overline{I}$ if and only if I is a k-ideal. We now state teh following theorem which was proved in [2].

THEOREM 1.2. *A semiring R is regular if and only if $A \cap B = \overline{AB}$ holds for every right k-ideal A and left k-ideal B of R.*

2. Semiring $\mathcal{I}(\boldsymbol{R})$

Let R be a semiring and $\mathcal{I}(R)$ be the set of all k-ideals of R. In $\mathcal{I}(R)$ we define the following operations of "addition" denoted by $\oplus$ and "multiplication" denoted by $\circ$; for any $I, J \in \mathcal{I}(R)$, $I \oplus J = \overline{I+J}$, $I \circ J = \overline{IJ}$, where IJ is the set consisting of all finite sums of the form $\sum_{i=1}^{n} a_i b_i$, $n \in \mathbf{N}$, with $a_i \in I$ and $b_i \in J$. Through a

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lengthy but routine calculation it can be shown that $(\mathcal{I}(R), \oplus, \circ)$ is a semiring. If R be taken as multiplicatively commutative, then so will be $\mathcal{I}(R)$.

We now prove the following theorem.

THEOREM 2.1. *A multiplicatively commutative semiring with zero,* $0 \neq 1$, *is regular if and only if* $\mathcal{I}(R)$, *as defined above, is regular.*

Proof. Let R be a multiplicatively commutative regular semiring. Then for any two k-ideals I and J we can write from Theorem 1.2 that $I \cap J = \overline{IJ}$. Hence $I \circ J = \overline{IJ} = I \cap J$. To prove that $\mathcal{I}(R)$ is regular, let $I \in \mathcal{I}(R)$, we see that $I \oplus I \circ R \circ I = I \circ R \circ I$ holds. In fact, $I \oplus I \circ R \circ I = I \oplus I \cap R \cap I = I \oplus I = \overline{I + I} = \overline{I} = I$ and $I \circ R \circ I = I \cap R \cap I = I$, proving the regularity of $\mathcal{I}(R)$. Conversely, let $\mathcal{I}(R)$ be regular. We want to prove that R is so. Let $a \in R$. We consider the principal k-ideal generated by a, i.e. $\overline{Ra}$ (since R is commutative and $1 \in R$). By regularity of $\mathcal{I}(R)$, we get that there exist $A, B \in \mathcal{I}(R)$ such that

$$\overline{Ra} \oplus \overline{Ra} \circ A \circ \overline{Ra} = \overline{Ra} \circ B \circ \overline{Ra}.$$

We see that $a \in \overline{Ra} \oplus \overline{Ra} \circ A \circ \overline{Ra}$. Hence $a \in \overline{Ra} \circ B \circ \overline{Ra}$, i.e. $a \in \overline{\overline{\overline{Ra}\,B}\,\overline{Ra}}$.

Now, $\overline{\overline{\overline{Ra}\,B}\,\overline{Ra}} \subseteq \overline{\overline{\overline{Ra}}\,\overline{\overline{Ra}}}$ $[\because \overline{\overline{Ra}\,B} \subseteq \overline{\overline{Ra}} = \overline{Ra}] = \overline{\overline{aR}\,\overline{Ra}}$ [by commutativity]. Hence, $a \in \overline{\overline{aR}\,\overline{Ra}}$, so that $a + \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{m} p_i q_i$, where $x_i, p_i \in \overline{aR}$ and $y_i, q_i \in \overline{Ra}$ for all $i = 1, 2, \ldots, n$ or m, as the case may be. We have

$$x_i + at_i = as_i, \quad p_i + at'_i = as'_i, \quad y_i + r_i a = r'_i a \tag{α}$$

for some t_i, s_i, t'_i, s'_i, r_i, $r'_i \in R$ with $i = 1, 2, \ldots, n$ or m as the case may be. From (α) we derive

$$x_i y_i + x_i r_i a + at_i y_i + at_i r_i a = as_i(y_i + r_i a) = as_i r'_i a$$

and $at_i y_i + at_i r_i a = at_i r'_i a$, hence $x_i y_i + as_i r_i a + at_i r'_i a = as_i r'_i a + at_i r_i a$. Thus we have

$$x_i y_i + au_i a = av_i a$$

for $u_i = s_i r_i + t_i r'_i$ and $v_i = s_i r'_i + t_i r_i$. Similarly we can get $p_i q_i + au'_i a = av'_i a$ for some $u'_i, v'_i \in R$.

Therefore $a + axa = aya$, for $x = \sum_{i=1}^{n} v_i + \sum_{i=1}^{m} u'_i$, $y = \sum_{i=1}^{n} u_i + \sum_{i=1}^{m} v'_i$. hence R is regular. ■

3. Semiprime k-ideal

DEFINITION 3.1. A k-ideal I of a semiring R is said to be semiprime if and only if $I = \sqrt{I}$, where $\sqrt{I} = \{ a \in R \mid a^n \in I$ for some positive integer $n \}$.

THEOREM 3.2. *A commutative semiring* R *is regular if and only if every* k*-ideal of* R *is semiprime.*

Proof. Let R be a regular semiring and I be any k-ideal of R. Since $I \subseteq \sqrt{I}$ always holds, it will be sufficient to prove the reverse inclusion only. Let $0 \neq a \in \sqrt{I}$.

Then $a^n \in I$, for some positive integer n. Since also $a^{n+1} \in I$, we can assume that n is odd. By regularity of a, there exist $x, y \in R$ such that $a + axa = aya$, i.e. (by commutativity of R)

$$a + a^2x = a^2y, \tag{1}$$

i.e. $a^2 + a^3x = a^3y$, i.e. $a^2x + a^3x^2 = a^3yx$, i.e. $a + a^2x + a^3x^2 = a + a^3yx$, i.e. (by (1)) $a^2y + a^3x^2 = a + a^3yx$, i.e. $a^2y + a^3xy + a^3x^2 = a + a^3xy + a^3yx$, i.e.

$$a^3y^2 + a^3x^2 = a + a^3(xy + yx). \tag{2}$$

On multiplying both sides of the relation (2) by a repeatedly we get

$$a^n(x^2 + y^2) = a^n(xy + yx) + a^{n-2}. \tag{3}$$

Now, as I is an ideal and $a^n \in I$ we get $a^n(x^2 + y^2) \in I$ and $a^n(xy + yx) \in I$. Again as I is a k-ideal we get from (3) $a^{n-2} \in I$. Repeating this process enough number of times ultimately we have $a \in I$. Consequently $\sqrt{I} \subseteq I$ so that $I = \sqrt{I}$.

Going in the other direction, let us now assume that R is a commutative semiring in which every k-ideal is semiprime, i.e. $I = \sqrt{I}$. Now, given any $0 \neq a \in R$, we consider the k-ideal $\overline{Ra^2}$. As we know that $a^3 \in \overline{Ra^2}$ and every k-ideal is semiprime, we get

$$a \in \sqrt{\overline{Ra^2}} = \overline{Ra^2},$$

i.e. $a + xa^2 = ya^2$ for some $x, y \in R$, i.e. $a + axa = aya$ (by commutativity on R). Hence R is regular. ■

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