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ASYMPTOTIC BEHAVIOUR OF A SEQUENCE DEFINED BY ITERATION

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Abstract. One considers some results in [1] and [2] concerning asymptotic behaviour of recurrently defined sequences and one gives more general and more precise form to them, especially by obtaining new terms in asymptotic expansion.

Introduction. In this paper we consider asymptotic behaviour of sequences $(x_n)_{n\in N}$ given by the recurrent formula $x_n = f(x_{n-1})$. We extend the following result (see problem I174 [1]).

THEOREM P. *Let* $f:(0,\alpha)\to(0,\alpha)$, *where* $\alpha>0$, *be a continuous function such that* $0<f(x)<x$ *for every* $x\in(0,\alpha)$ *and* $f(x)=x-ax^k+bx^{k+p}+o(x^{k+p})$, *when* $x\to+0$, *where* $k>1$, p, a *and* b *are positive numbers. Let* $x_0\in(0,\alpha)$ *and* $x_n=f(x_{n-1})$, $n\geqslant 1$. *Then* $x_n\sim((k-1)an)^{-\frac{1}{(k-1)}}$.

In the special case of the sequence given by $x_n = x_{n-1}-x_{n-1}^2$, $0<x_0<1$ it was proved in [2] that $x_n\sim\frac{1}{n}-\frac{\ln n}{n^2}$. Thus the second term in asymptotic development of this sequence is determined.

Now it is natural to ask if we can determine the second term in asymptotic development of the sequence given in Theorem P.

Let us first prove Theorem P. Our version of this proof is somewhat different from its original form, and the idea and structure of this version was starting point and inspiration for our further consideration.

Proof. Since $0<f(x)<x$ for every $x\in(0,\alpha)$, the sequence x_n is decreasing so that there exists $\lim_{n\to+\infty}x_n$. It can be proved by standard procedure that $\lim_{n\to+\infty}x_n=0$. We have

$$x_n=f(x_{n-1})=x_{n-1}-ax_{n-1}^k+bx_{n-1}^{k+p}+o(x_{n-1}^{k+p}).$$

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Since $x_n > 0$, we can write

$$\frac{1}{x_n^{k-1}} = \frac{1}{f(x_{n-1})^{k-1}} = \frac{(1 - ax_{n-1}^{k-1} + bx_{n-1}^{k-1+p} + o(x_{n-1}^{k-1+p}))^{-(k-1)}}{x_{n-1}^{k-1}}.$$

Since $x_n \to 0$ when $n \to +\infty$, it follows that $-ax_{n-1}^{k-1} + bx_{n-1}^{k-1+p} + o(x_{n-1}^{k-1+p}) \to 0$ when $n \to +\infty$. Using the asymptotic formula $(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + o(x^2)$, $x \to 0$, we get

$$\frac{1}{x_n^{k-1}} = \frac{1}{x_{n-1}^{k-1}} + (k-1)a - b(k-1)x_{n-1}^p + a^2\binom{k}{2}x_{n-1}^{k-1} + o(x_{n-1}^p) + o(x_{n-1}^{k-1}). \quad (1)$$

Hence,

$$\frac{1}{x_n^{k-1}} = \frac{1}{x_{n-1}^{k-1}} + (k-1)a + O(x_{n-1}^s) \quad (2)$$

for $n \geqslant n_0$, natural number n_0 being fixed and $s = \min(p, k-1)$.

It follows from (2) that

$$\frac{1}{x_n^{k-1}} = \frac{1}{x_{n_0}^{k-1}} + (k-1)a(n-n_0) + O(x_{n_0}^s) + \cdots + O(x_{n-1}^s) \quad (3)$$

and hence

$$\frac{1}{nx_n^{k-1}} = \frac{1}{nx_{n_0}^{k-1}} + (k-1)a\left(1 - \frac{n_0}{n}\right) + \frac{O(x_{n_0}^s) + \cdots + O(x_{n-1}^s)}{n}. \quad (4)$$

By Stoltz theorem, we obtain $\lim_{n\to+\infty} \frac{1}{n}[O(x_{n_0}^s) + \cdots + O(x_{n-1}^s)] = 0$.

Now it follows from (4) that $\lim_{n\to+\infty} 1/(nx_{n-1}^{k-1}) = (k-1)a$.

Finally, we obtain $x_n \sim ((k-1)an)^{-\frac{1}{k-1}}$. ∎

Remark. An inspection of the proof shows that the condition $b > 0$ is not essential.

In order to improve our estimate we are going to consider the following cases.

Case 1. Let $p > k-1$. Then it follows from (1) that

$$\frac{1}{x_n^{k-1}} = \frac{1}{x_{n-1}^{k-1}} + a(k-1) + a^2\frac{k(k-1)}{2}x_{n-1}^{k-1} + o(x_{n-1}^{k-1}). \quad (5)$$

Using the substitution $b_n = x_n^{k-1}$, we can write (5) in the form of

$$\frac{1}{b_n} = \frac{1}{b_{n-1}} + a(k-1) + a^2\binom{k}{2}b_{n-1} + o(b_{n-1}). \quad (6)$$

By (6), we obtain

$$\frac{1}{b_n} = \frac{1}{b_{n_0}} + a(k-1)(n-n_0) + a^2\binom{k}{2}\left(\sum_{i=n_0}^{n-1} b_i\right) + o(b_{n_0}) + \cdots + o(b_{n-1}) \quad (7)$$

for $n \geqslant n_0$.

From Theorem P it follows that $b_n = \frac{1}{(k-1)an} + o(\frac{1}{n})$, and so

$$\frac{1}{b_n} = \frac{1}{b_{n_0}} + a(k-1)(n-n_0) + a^2\binom{k}{2}\sum_{i=n_0}^{n-1}\left(\frac{1}{(k-1)ai} + o\left(\frac{1}{i}\right)\right) + o(b_{n_0}) + \cdots + o(b_{n-1}).$$

By Euler's formula, $1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n = \gamma_n \to \gamma \in \mathbb{R}$ $(n \to +\infty)$ and since $o(b_i) = o(\frac{1}{i})$ we have

$$\frac{1}{b_n} = \frac{1}{b_{n_0}} + a(k-1)(n-n_0) + a^2\binom{k}{2}\left(\frac{1}{(k-1)a}\left(\ln\left(\frac{n-1}{n_0-1}\right) + \gamma_{n-1} - \gamma_{n_0-1}\right) + \sum_{i=n_0}^{n-1} o\left(\frac{1}{i}\right)\right).$$

Hence

$$\frac{1}{b_n} = c(n_0) + a(k-1)n + a^2\binom{k}{2}\left(\frac{1}{(k-1)a}\left(\ln n + \gamma_{n-1} - \frac{1}{n}\right) + \sum_{i=n_0}^{n} o\left(\frac{1}{i}\right)\right),$$

where $c(n_0)$ is a constant depending of n_0.

Since $\lim_{n\to+\infty}\sum_{i=n_0}^{n-1} o(\frac{1}{i})/\ln n = 0$, it follows that $\frac{1}{b_n} = a(k-1)n + \frac{ak}{2}\ln n + o(\ln n)$. Hence

$$x_n = b_n^{\frac{1}{k-1}} = \left(a(k-1)n + \frac{ak}{2}\ln n + o(\ln n)\right)^{-\frac{1}{k-1}}$$
$$= \frac{1}{(a(k-1)n)^{\frac{1}{k-1}}} - \frac{k}{2(k-1)^{\frac{2k-1}{k-1}}a^{\frac{1}{k-1}}}\frac{\ln n}{n^{\frac{k}{k-1}}} + o\left(\frac{\ln n}{n^{\frac{k}{k-1}}}\right).$$

Case 2. Let $p = k-1$. Now instead of (6), we have

$$\frac{1}{b_n} = \frac{1}{b_{n-1}} + (k-1)a + b_{n-1}\left(\binom{k}{2}a^2 - (k-1)b\right) + o(b_{n-1}),$$

and so the final asymptotic formula of the preceeding case in this case becomes

$$x_n = \frac{1}{(n(k-1)a)^{\frac{1}{k-1}}} - \frac{ka^2-2b}{2(a(k-1))^2}\frac{\ln n}{((k-1)a)^{\frac{1}{k-1}}n^{\frac{k}{k-1}}} + o\left(\frac{\ln n}{n^{\frac{k}{k-1}}}\right).$$

Case 3. Let $p < k-1$. In the same way as in the cases 1 and 2, we obtain:

$$\frac{1}{b_n} = \frac{1}{b_{n-1}} + (k-1)a - b(k-1)\,b_{n-1}^{\frac{p}{k-1}} + o(b_{n-1}^{\frac{p}{k-1}})$$
$$\frac{1}{b_n} = \frac{1}{b_{n_2}} + (k-1)a(n-n_2) - b(k-1)\sum_{i=n_2}^{n-1} b_i^{\frac{p}{k-1}} + \sum_{i=n_2}^{n-1} o(b_i^{\frac{p}{k-1}}).$$

Hence

$$\frac{1}{b_n} = \frac{1}{b_{n_2}} + (n-n_2)(k-1)a - b(k-1)\sum_{i=n_2}^{n-1}\frac{1}{(a(k-1)i)^{\frac{p}{k-1}}} + \sum_{i=n_2}^{n-1} o\left(\frac{1}{i^{\frac{p}{k-1}}}\right)$$

$$= c(n_2) + n(k-1)a - \frac{b(k-1)}{(a(k-1))^{\frac{p}{k-1}}}\sum_{i=n_2}^{n-1}\frac{1}{i^{\frac{p}{k-1}}} + \sum_{i=n_2}^{n-1} o\left(\frac{1}{i^{\frac{p}{k-1}}}\right). \tag{8}$$

We will use the known asymptotic formula

$$\sum_{i=1}^{n-1}\frac{1}{i^s} \sim \frac{n^{1-s}}{1-s}. \tag{9}$$

Combining (8) and (9) with $s = p/(k-1)$, we get

$$\frac{1}{b_n} = n(k-1)a - \frac{(k-1)^2 b}{(a(k-1))^{\frac{p}{k-1}}}\frac{1}{(k-1-p)}n^{\frac{k-1-p}{k-1}} + o(n^{\frac{k-1-p}{k-1}}).$$

Hence

$x_n =$

$$= (n(k-1)a)^{-\frac{1}{k-1}}\left(1 - \frac{(k-1)b}{a(k-1-p)(a(k-1))^{\frac{p}{k-1}}}\frac{1}{n^{\frac{p}{k-1}}} + o\left(\frac{1}{n^{\frac{p}{k-1}}}\right)\right)^{-\frac{1}{(k-1)}}$$

$$= \frac{1}{(n(k-1)a)^{\frac{1}{k-1}}} + \frac{b}{a(k-1-p)(a(k-1))^{\frac{p+1}{k-1}}}\frac{1}{n^{\frac{p+1}{k-1}}} + o\left(\frac{1}{n^{\frac{p+1}{k-1}}}\right).$$

Remark. If $b = 0$ then we need not investigate the relation between p and $k-1$, because it is easy to verify that this case was considered in Case 1.

We state the results obtained in the cases 1, 2 and 3 in the following theorem.

Theorem 1. *Under the same condition as in Theorem P we can conclude*

a) If $p > k-1$ then

$$x_n = \frac{1}{(a(k-1)n)^{\frac{1}{k-1}}} - \frac{k}{2(k-1)^{\frac{2k-1}{k-1}}a^{\frac{1}{k-1}}}\frac{\ln n}{n^{\frac{k}{k-1}}} + o\left(\frac{\ln n}{n^{\frac{k}{k-1}}}\right).$$

b) If $p = k-1$ then

$$x_n = \frac{1}{(a(k-1)n)^{\frac{1}{k-1}}} - \frac{ka^2-2b}{2(a(k-1))^2}\frac{\ln n}{((k-1)a)^{\frac{1}{k-1}}n^{\frac{k}{k-1}}} + o\left(\frac{\ln n}{n^{\frac{k}{k-1}}}\right).$$

c) If $p < k-1$ and $b \neq 0$ then

$$x_n = \frac{1}{(a(k-1)n)^{\frac{1}{k-1}}} + \frac{b}{a(k-1-p)(a(k-1))^{\frac{p+1}{k-1}}} \frac{1}{n^{\frac{p+1}{k-1}}} + o\left(\frac{1}{n^{\frac{p+1}{k-1}}}\right).$$

EXAMPLE 1. Let $f(x) = \sin x$ and $\alpha = \pi$. Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$ we see that $a = 1/6$, $b = 1/120$, $k = 3$, $p = 2$. Since $k-1 = p$, by the Case 2. we obtain

$$x_n = \frac{\sqrt{3}}{\sqrt{n}} - 3\frac{\sqrt{3}}{10}\frac{\ln n}{n^{3/2}} + o\left(\frac{\ln n}{n^{3/2}}\right).$$

EXAMPLE 2. Let $f(x) = \ln(1+x)$, and $\alpha = +\infty$. Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$ we see that $a = 1/2$, $b = 1/3$, $k = 2$, $p = 1$. Again we have as in the Example 1 $k-1 = p$, and by the Case 2.

$$x_n = \frac{2}{n} + \frac{2}{3}\frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right).$$

EXAMPLE 3. Let $f(x) = x - x^2$, and $\alpha = 1$. Then $a = 1$, $b = 0$, $k = 2$. Since $b = 0$ we obtain

$$x_n = \frac{1}{n} - \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right).$$

Clearly, we can try to determine other terms in asymptotic development of our sequence. The following theorem gives us a formula which contains three terms in asymptotic developments of the sequence x_n in Case 1:

THEOREM 2. *Let $p > k-1$ and let all others conditions of Theorem P be satisfied. Then*

$$x_n = \frac{1}{(a(k-1)n)^{\frac{1}{k-1}}} - \frac{k}{2a^{\frac{1}{k-1}}(k-1)^{\frac{2k-1}{k-1}}} \frac{\ln n}{n^{\frac{k}{k-1}}} + \frac{d}{n^{\frac{k}{k-1}}} + o\left(\frac{1}{n^{\frac{k}{k-1}}}\right)$$

where d is a constant depending on x_0.

Proof. We can repeat the procedure from Theorem 1. By Case 1 we have

$$\frac{1}{b_n} = a(k-1)n + \frac{ak}{2}\ln n + o(\ln n).$$

Hence

$$b_n = \frac{1}{a(k-1)n} - \frac{k}{2a(k-1)^2}\frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right). \tag{10}$$

Using standard substitution $b_n = x_n^{k-1}$ in (1) we get

$$\frac{1}{b_n} = \frac{1}{b_{n-1}} + a(k-1) + a^2\binom{k}{2}b_{n-1} + O(b_{n-1}^{l/(k-1)}), \quad \frac{l}{k-1} = \frac{\min(p, 2(k-1))}{k-1} > 1. \tag{11}$$

By (10), we obtain

$$\frac{1}{b_n} = c + a(k-1)n$$
$$+ a^2 \binom{k}{2} \sum_{i=n_0}^{n-1} \left(\frac{1}{a(k-1)i} - \frac{k}{2a(k-1)^2} \frac{\ln i}{i^2} + o\left(\frac{\ln i}{i^2}\right)\right) + \sum_{i=n_0}^{n-1} O\left(\frac{1}{i^{\frac{l}{k-1}}}\right)$$

where c is a constant depending on x_0 and n_0, hence

$$\frac{1}{b_n} = a(k-1)n + \frac{ak}{2} \ln n + c_n.$$

The sequence c_n has a finite limit since the following two series are convergent

$$\sum_{i=n_0}^{+\infty} \frac{\ln i}{i^2}, \qquad \sum_{i=n_0}^{+\infty} O\left(\frac{1}{i^{\frac{l}{k-1}}}\right).$$

If we put $c_n = c(x_0) + d_n$, where $c(x_0) = \lim_{n\to+\infty} c_n$ we have

$$b_n = \frac{1}{a(k-1)n} - \frac{k}{2a(k-1)^2}\frac{\ln n}{n^2} - \frac{c(x_0)}{a(k-1)n^2} + o\left(\frac{1}{n^2}\right). \tag{12}$$

Finally, by (12), we obtain

$$x_n = b_n^{\frac{1}{k-1}} = \frac{1}{(a(k-1)n)^{\frac{1}{k-1}}} - \frac{k}{2a^{\frac{1}{k-1}}(k-1)^{\frac{2k-1}{k-1}}}\frac{\ln n}{n^{\frac{k}{k-1}}} + \frac{d(x_0)}{n^{\frac{k}{k-1}}} + o\left(\frac{1}{n^{\frac{k}{k-1}}}\right)$$

where $d(x_0) = -c(x_0)/a^{\frac{1}{k-1}}(k-1)^{\frac{k}{k-1}}$. ■

Of course, we can obtain in the same manner an analogous formula for other two cases. This task we leave to the reader.

We finish this paper by determination of first five terms in the asymptotic development of the sequence given by $x_n = x_{n-1} - x_{n-1}^2$, $0 < x_0 < 1$. This sequence satisfies the following relation

$$\frac{1}{x_n} = \frac{1}{x_{n-1}} + 1 + x_{n-1} + x_{n-1}^2 + o(x_{n-1}^2). \tag{13}$$

Using Theorem 2 we get

$$1/x_n = n + \ln n + c(x_0) + d_n, \tag{14}$$

where $\lim_{n\to+\infty} d_n = 0$.

By (13) and (14), we have

$$d_n = d_{n-1} - \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right),$$

hence

$$d_n = \sum_{k=n}^{+\infty}(d_k - d_{k+1}) = \sum_{k=n}^{+\infty}\left(\frac{\ln k}{k^2} + o\left(\frac{\ln k}{k^2}\right)\right) \sim \sum_{k=n}^{+\infty}\frac{\ln k}{k^2} \sim \frac{\ln n}{n}$$

and

$$1/x_n = n + \ln n + c(x_0) + \frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right).$$

The last equality gives us the desired asymptotic development, i.e.

$$x_n = \frac{1}{n} - \frac{\ln n}{n^2} - \frac{c(x_0)}{n^2} + \frac{\ln^2 n}{n^3} + (2c(x_0) - 1)\frac{\ln n}{n^3} + o\left(\frac{\ln n^2}{n^3}\right).$$

Of course, if we had a better asymptotic evaluation of $f(x)$, we could obtain better asymptotic formula for x_n.

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