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# HYPERBOLIC REALIZATIONS OF TILINGS BY ZHUK SIMPLICES

**Milica Stojanović**

**Abstract.** One possibility to classify hyperbolic sapce groups is to look for their fundamental domains. For simplicial domains are combinatorialy classified face pairing identifications, but the space of realization is not known. In this paper two series of fundamental simplices are investigated, which have three equivalence classes for edges and two for vertices. Three edges in the first class belong to the same face and vertices of that face are in the same class. Those simplices are hyperbolic, mainly with outer vertices. If so, then truncated simplex tilings are also investigated and classified with their metric data and other conditions.

## 1. Introduction

Hyperbolic space groups are isometry groups, acting discontinuously on the hyperbolic 3-space with compact fundamental domain. In aim of classifying them one may look for their fundamental domains. Face pairing identifications of a given polyhedron give us generators and relations for a space group by Poincaré theorem [1], [2], [5].

The simplest fundamental domains are simplices and truncated simplices by polar planes of vertices when they lie out of the absolute. There are 64 combinatorially different face pairings of fundamental simplices [12], [13], [8], furthermore 35 solid transitive non-fundamental simplex identifications [8]. I. K. Zhuk [12], [13] has classified Euclidean and hyperbolic fundamental simplices of finite volume up to congruence. Some completing cases are discussed in [4], [7], [10], [11]. Algorithmic procedure is given by E. Molnár and I. Prok [7]. In [8] and [9] the authors summarize all these results, arranging identified simplices into 32 families. Each of them is characterized by the so-called maximal series of simplex tilings. Besides spherical, Euclidean, hyperbolic realizations there exist also other metric realizations in 3-dimensional simply connected homogeneous Riemannian spaces, moreover, metrically non-realizable topological simplex tilings occur as well.

Two simplices, investigated in this paper, were described and considered in special cases with other methods for first time by I. K. Zhuk [12], [13]. They have three

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equivalence classes for edges: $a\{A_0A_1\}$, $b\{A_1A_2, A_1A_3\}$, $c\{A_0A_2, A_2A_3, A_3A_0\}$ and two for vertices: $\{A_1\}$ resp. $\{A_0, A_2, A_3\}$ (Fig. 1.a, 2.a).

Sum of dihedral angles around edges in the same equivalence class is always of the form $\pi/\nu$. That is the reason we have three parameters $a$, $b$ and $c$. I. K. Zhuk obtained in [12], [13] only values of parameters for which simplices are either spherical of hyperbolic with finite vertices. E. Molnár, I. Prok and J. Szirmai have extended these results in [9] for some parameters, to the Nil and $\widetilde{SL_2\mathbf{R}}$ geometries (see also [6]). Some cases of simplices in hyperbolic space with vertices out of absolute, are also mentioned in [10], [11]. In this paper we have classified all cases of these simplices which are realizable in spaces of constant curvature including hyperbolic simplices with vertices which are not finite. With previous investigations this provides complete result for all values of parameters.

When vertices are out of the absolute, the simplex is not compact and then we truncate it with polar planes of the vertices. The new compact polyhedron obtained in that way is fundamental domain of some larger group. It has new triangular faces whose pairing gives new generators. Dihedral angles around new edges are $\pi/2$. That means there are four congruent polyhedra around them in the fundamental space filling.

The geometrical results of this paper will be given in section 4 and the metrical ones in section 5 in Theorems 2 and 3.

I started with investigations in this field during my studies in Budapest, under supervision of Prof. Emil Molnár.

## 2. Projective metrics, spherical and hyperbolic spaces

The projective 3-space $P^3$ can be introduced in the usual way by the 4-dimensional real vector space $V^4$ and its dual space $\mathcal{V}_4^*$ of linear forms. Then the 1-dimensional subspaces of $V^4$ (or the 3-subspaces of $\mathcal{V}_4^*$) represent the points of $P^3$, the 1-subspaces of $\mathcal{V}_4^*$ (or the 3-subspaces of $V^4$) represent the planes of $P^3$ and 2-subspaces of $V^4$ (or of $\mathcal{V}_4^*$) represent the straight lines of $P^3$. The point $X(x)$ and the plane $\alpha(a)$ are incident iff $xa = 0$, ($x \in V^4 \setminus \{0\}$, $a \in \mathcal{V}_4^* \setminus \{0\}$). If $\{e_i\}$ is a basis in $V^4$ and $\{e^j\}$ is its dual basis in $\mathcal{V}_4^*$, i.e. $e_ie^j = \delta_i^j$ (the Kronecker symbol), then the form $a = e^ja_j$ takes the value $xa = x^ia_i$ on the vector $x = x^ie_i$. We use the summation convention for the same upper and lower indices.

We can take basis $\{b^i\}$ in $\mathcal{V}_4^*$ in such a way to represent planes containing simplex faces opposite to the vertices $A_i$, respectively. Projective metric in $P^3$ can be introduced by giving a bilinear form

$$\langle\, ;\, \rangle : \mathcal{V}_4^* \times \mathcal{V}_4^* \to \mathbf{R},\ \langle b^iu_i; b^jv_j\rangle = u_ib^{ij}v_j,$$

where $(\langle b^i; b^j\rangle) = (b^{ij})$ is a Schläfli matrix. Vectors $a_j$ of the dual basis $\{a_j\}$ in $V^4$, defined by $a_jb^i = \delta_j^i$ represent the vertices $A_j$ of the same simplex. The induced bilinear form

$$\langle\, ;\, \rangle : V^4 \times V^4 \to \mathbf{R},\ \langle x^ia_i; y^ja_j\rangle = x^ia_{ij}y^j$$

is defined by the matrix $(\langle a_i; a_j\rangle) = a_{ij}$ inverse to $(b^{ij})$.

We assume that the bilinear form $\langle\, ;\, \rangle$ by $(b^{ij})$ has either signature $(+,+,+,-)$ which characterizes the hyperbolic metric, or $(+,+,+,+)$, this will be the elliptic (spherical) metric. Euclidean geometry would be described by signature $(+,+,+,0)$, that will not occur in our considerations.

It is well-known that the bilinear form induces the distance and the angle measure of the 3-space. Let $X(x)$ and $Y(y)$ be two points in the projective space $P^3$. Then their distance $d(x,y)$ is determined by

$$\cos(d(x,y)) = \frac{\langle x;y\rangle}{\sqrt{\langle x;x\rangle\langle y;y\rangle}} \quad \text{and} \quad \mathrm{ch}(d(x,y)) = -\frac{\langle x;y\rangle}{\sqrt{\langle x;x\rangle\langle y;y\rangle}} \tag{1}$$

for elliptic and hyperbolic case, respectively.

The next lemma will be used in section 5.

Lemma 1. *For any $(r+1)$-minor determinant of a regular matrix $(a_{ij})$ and complementary $(n-r)$-minor of its inverse $(b^{ij})$ the following equality holds*

$$\begin{pmatrix} a_{i_0 j_0} & \dots & a_{i_0 j_r} \\ \vdots & & \vdots \\ a_{i_r j_0} & \dots & a_{i_r j_r} \end{pmatrix} = \det(a_{ij}) \begin{pmatrix} b^{i_{r+1} j_{r+1}} & \dots & b^{i_{r+1} j_n} \\ \vdots & & \vdots \\ b^{i_n j_{r+1}} & \dots & b^{i_n j_n} \end{pmatrix} \sigma.$$

*Here* $\sigma = \mathrm{sign}(i_0,\dots,i_r,i_{r+1},\dots,i_n)\cdot\mathrm{sign}(j_0,\dots,j_r,j_{r+1},\dots,j_n)$ *denotes the sign product of the corresponding permutations of the elements* $0,1,\dots,n$.

## 3. Construction of discontinuously acting isometry groups

Identifications on the simplex $\mathcal{T}$ are face pairings by isometries, satisfying the following conditions

a) For each face $f_{g^{-1}}$ of $\mathcal{T}$ there is another face $f_g$ and an identifying isometry $g$ of the space $S^3(H^3)$, which maps $f_{g^{-1}}$ onto $f_g$ and $\mathcal{T}$ onto $\mathcal{T}^g \not\cong \mathcal{T}$, the neighbor of $\mathcal{T}$ along $f_g$.

b) The isometry $g^{-1}$ maps the face $f_g$ onto $f_{g^{-1}}$ and $\mathcal{T}$ onto $\mathcal{T}^{g^{-1}}$, joining the simplex $\mathcal{T}$ along $f_{g^{-1}}$.

The face pairing identifications of $\mathcal{T}$ generate an isometry group $G$.

These generators induce subdivision of the edges into oriented segments such that a segment does not contain two equivalent points in its interior. An equivalence class consisting of edge segments $e_1, e_2, \dots, e_r$ with dihedral angles $\varepsilon(e_1), \varepsilon(e_2), \dots, \varepsilon(e_r)$, respectively, is defined by the following algorithm

$$(e_1, f_{g_1^{-1}}) \xrightarrow{g_1} (e_2, f_{g_1});\ (e_2, f_{g_2^{-1}}) \xrightarrow{g_2} (e_3, f_{g_2});\ \dots;\ (e_r, f_{g_r^{-1}}) \xrightarrow{g_r} (e_1, f_{g_r}) \tag{2}$$

where the symbols are not necessarily distinct.

In other words the segment $e_1$ is successively surrounded by simplices

$$\mathcal{T}, \mathcal{T}^{g_1^{-1}}, \mathcal{T}^{g_2^{-1}g_1^{-1}}, \dots, \mathcal{T}^{g_r^{-1}\cdots g_2^{-1}g_1^{-1}}$$

which fill an angular region of measure $2\pi/\nu$. If plane reflection $m_i = g_i$ occurs then each edge segment comes two times in (2) and so

$$\varepsilon(e_1) + \cdots + \varepsilon(e_r) = \pi/\nu, \tag{3}$$

otherwise

$$\varepsilon(e_1) + \cdots + \varepsilon(e_r) = 2\pi/\nu. \tag{4}$$

The cycle transformation $c = g_1 g_2 \cdots g_r$ belonging to the edge segment class $\{e_i\}$ is a rotation, say, of order $\nu$. Thus we have the cycle relation

$$(g_1 g_2 \cdots g_r)^\nu = 1 \tag{5}$$

in the second case and an analogous in the first case.

c) Assume that (3) or (4) holds for the dihedral angles at $\{e_1\}$ in each segment equivalence class.

We need the specified Poincaré theorem:

Theorem 1. *Let $\mathcal{T}$ be a simplex, or a truncated simplex in a space $\mathcal{S}^3$ of constant curvature and $G$ be the group generated by the face identifications, satisfying conditions a)–c). Then $G$ is discontinuously acting group on $\mathcal{S}^3$, $\mathcal{T}$ is a fundamental domain for $G$ and the cycle relations of type (5) for every equivalence class of edge segments form a complete set of relations for $G$, if we also add the relations $g_i^2 = 1$ to the occasional involutive generators $g_i = g_i^{-1}$.*

## 4. Zhuk simplices and their isometry groups

a) Isometries which identify faces of simplex $\mathcal{T}_1$ (Fig. 1. a) are

$$r_1 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_3 & A_2 \end{pmatrix}; \quad r_2 : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_3 & A_2 & A_0 \end{pmatrix}; \quad r : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_0 & A_1 & A_3 \end{pmatrix}.$$

By notations in [8], [9] it is representing simplices from Family 30 with groups of simplex tilings $\Gamma_{39}(a; 2b; 6c)$. In the first edge equivalence class denoted by $a$ is the edge $A_0A_1$, in the class $b$ are the edges $A_1A_2$ and $A_1A_3$, and in the class $c$ lie $A_0A_2$, $A_0A_3$ and $A_2A_3$. The face pairing isometries divide edges in class $c$ into two equivalent oriented segments. Vertices $A_0$, $A_2$ and $A_3$ are in one and $A_1$ is in another equivalence class of vertices. Relations for the isometry group are obtained by Theorem 1 and the presentation is

$$G(\mathcal{T}_1, a, b, c) = (r_1, r_2, r - (r)^a = (rr_1)^b = (r_1 r_2 r r_2 r^{-1} r_2)^c$$
$$= r_1^2 = r_2^2 = 1;\ a \geq 3, b \geq 2, c \geq 1).$$

Considering vertex figures on a 2-dimensional surface around the vertices, we can obtain a fundamental domain for the stabilizer group $G_{A_1}$ of vertex $A_1$ and e.g. $G_{A_3}$ of vertex $A_3$. Transformation $r_1$ is mapping vertex figure $\mathcal{T}_{A_3}$ onto $\mathcal{T}_{A_2}^{r_1}$ and $r_2$ is mapping $\mathcal{T}_{A_3}$ onto $\mathcal{T}_{A_0}^{r_2}$. That means that $\mathcal{T}_{A_3}$ and $\mathcal{T}_{A_2}^{r_1}$ have a joint edge corresponding to the joint face $f_{r_1}$ of the simplices $\mathcal{T}_1$ and $\mathcal{T}_1^{r_1}$ and similarly,

Fig. 1

$\mathcal{T}_{A_3}$ and $\mathcal{T}_{A_0}^{r_2}$ have a joint edge corresponding to $f_{r_2}$. One of the domains for $G_{A_3}$ (Fig. 1.b) is

$$\mathcal{P}_{A_3} := \mathcal{T}_{A_2}^{r_1} \cup \mathcal{T}_{A_3} \cup \mathcal{T}_{A_0}^{r_2}.$$

In the diagram for $\mathcal{P}_{A_3}$ and $\mathcal{P}_{A_1}$ the minus sign in notations $a^-$, $b^-$ and $c^-$ means that edges in these classes are directed to vertices according to vertex figure (plus means opposite direction).

When parameters $a, b, c$ are such that simplex $\mathcal{T}_1$ is hyperbolic and that the vertices either in the first or in the second equivalence class are out of the absolute, it is possible to truncate the simplex by polar planes of the vertices. Then we get a compact polyhedron denoted by $\mathcal{O}_1$. If we equip $\mathcal{O}_1$ with additional face pairing isometries, it will be a fundamental domain for the group $G(\mathcal{O}_1, a, b)$ which will be a supergroup for $G(\mathcal{T}_1, a, b)$. Trivial group extension with plane reflections in polar planes of the outer vertices is always possible. In the case of vertices $A_0, A_2, A_3$ the new relations, which are necessary to add to group $G(\mathcal{T}_1, a, b)$ to obtain supergroup in this way, are (Fig. 1.c)

$$\begin{aligned} m_2 r_1 m_3 r_1 = (m_2 r_2)^2 = m_0 r_2 m_3 r_2 = m_2 r m_3 r^{-1} = m_0 r m_0 r^{-1} \\ = m_0^2 = m_2^2 = m_3^2 = 1 \end{aligned}$$

where $m_i$ is plane reflection corresponding to the vertex $A_i$. There are no more possibilities for face pairing isometries for this equivalence class (see [4] and [10]).

Fig. 2

For vertex $A_1$ these relations with trivial extension are (Fig. 1.d)

$$(m_1r_1)^2 = m_1rm_1r^{-1} = m_1^2 = 1.$$

For $A_1$ it is also possible to equip the new triangular face with half-turn $r_3$ and then the new relations are (Fig. 1.e)

$$(r_1r_3)^2 = (rr_3)^2 = r_3^2 = 1.$$

If $a \neq 2b$ then $\mathcal{T}_1$ and $\mathcal{O}_1$ are maximal (without further symmetries) (see [8], [9]).

b) Notations for simplex $\mathcal{T}_2$ and its isometry group in [8], [9] are Family 26 and $\Gamma_{29}(a;4b;12c)$, respectively. Face pairing isometries of $\mathcal{T}_2$ (Fig. 2.a) are

$$m : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}; \quad r_1 : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_3 & A_2 & A_0 \end{pmatrix}; \quad r : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_0 & A_1 & A_3 \end{pmatrix}.$$

Division of edges and vertices into equivalence classes is similar as for simplex $\mathcal{T}_1$ and the isometry group has the presentation

$$\begin{aligned} G(\mathcal{T}_2, a, b, c) &= (m, r_1, r - (r)^a = (rmr^{-1}m)^{b_1} = (r_1rr_1r^{-1}r_1mr_1rr_1r^{-1}r_1m)^{c_1} \\ &= r_1^2 = m^2 = 1;\ a \geq 3, b_1 \geq 1, c_1 \geq 1). \end{aligned}$$

For $2b_1 = b$ and $2c_1 = c$ simplex $\mathcal{T}_2$ has the same metrical properties as the simplex $\mathcal{T}_1$ with even $b$ and $c$.

Fundamental domain for the stabilizer group $G_{A_3}$ of vertex $A_3$, say, contains vertex figure around that vertex. Since $r_1$ is mapping vertex figure $\mathcal{T}_{A_3}$ onto $\mathcal{T}_{A_0}^{r_1}$ and $r^{-1}$ is mapping $\mathcal{T}_{A_3}$ onto $\mathcal{T}_{A_2}^{r^{-1}}$, one domain of $G_{A_3}$ (Fig. 2.b) is

$$\mathcal{P}_{A_3} := \mathcal{T}_{A_0}^{r_1} \cup \mathcal{T}_{A_3} \cup \mathcal{T}_{A_2}^{r^{-1}}.$$

If simplex $\mathcal{T}_2$ is hyperbolic with vertices outside of absolute, we have analogous cases as at $\mathcal{T}_1$. For outer $A_0, A_2, A_3$ we consider the polar planes and corresponding plane reflections $m_0, m_2, m_3$ (Fig. 2.c). We get the new relations

$$(m_2m)^2 = (m_3m)^2 = m_0r_1m_3r_1 = (m_2r_1)^2 = m_2rm_3r^{-1} = m_0rm_0r^{-1}$$
$$= m_0^2 = m_2^2 = m_3^2 = 1.$$

For outer $\mathrm{A}_1$ we have two possibilities to equip the polar planes with identification either with plane reflection $m_1$ (Fig. 2.d) to get new relations

$$(m_1m)^2 = m_1rm_1r^{-1} = m_1^2 = 1,$$

or with half-turn $r_2$ (Fig. 2.e) and new relations

$$(r_2m)^2 = (rr_2)^2 = r_2^2 = 1.$$

The groups of tessellations with $\mathcal{T}_2$ and $\mathcal{O}_2$ are maximal, if $a \neq 4b$.

### 5. Realization of Zhuk simplices in spaces of constant curvature

Schläfli matrix for simplices $\mathcal{T}_1$ and $\mathcal{T}_2$ are the same

$$B = (b_{ij}) = \begin{bmatrix} 1 & r & s & s \\ r & 1 & q & q \\ s & q & 1 & p \\ s & q & p & 1 \end{bmatrix}, \tag{6}$$

where $p = -\cos 2\pi/a$, $q = -\cos\gamma$, $r = -\cos(\pi/c - 2\gamma)$, $s = -\cos\pi/b$, with the natural parameters $a \geq 3$, $b \geq 2$, $c \geq 1$ and $0 < \gamma < \pi/2c$. In agreement with mentioned in sect. 4 for simplex $\mathcal{T}_2$ parameters $b$ and $c$ can take only even values ($2b_1 = b, 2c_1 = c$), thus we deal with simplex $\mathcal{T}_1$ that is more general.

Symmetries of simplex $\mathcal{T}_1$ are giving us a necessary condition $d(A_0, A_2) = d(A_2, A_3)$. Using the inverse matrix $(a_{ij})$ in (1) we can express any side length (or its function) by the angles, so

$$\frac{a_{02}^2}{a_{00}a_{22}} = \frac{a_{23}^2}{a_{22}a_{33}} \quad \text{or} \quad \frac{a_{00}a_{22} - a_{02}^2}{a_{00}a_{22}} = \frac{a_{22}a_{33} - a_{23}^2}{a_{22}a_{33}}.$$

Applying our lemma, we get

$$a_{33}(b_{11}b_{33} - b_{13}^2) = a_{00}(b_{00}b_{11} - b_{01}^2)$$

and with notations as in (6),

$$f(\gamma) = 0 \tag{7}$$

for $f(\gamma) = (1-p)(1+p-2q^2)(1-r^2) - (1-q^2-r^2-s^2+2rsq)(1-q^2)$. Angle $\gamma$ can take its values from interval $(0, \pi/2c)$.

Lemma 2. *Equation (7) has unique solution for* $(a,b,c)$, *except* $a \geq 6$, $b = 2$, $c = 1$. *These cases are discussed in [6] and [9].*

*Proof.* A) Note that it holds $f(0) < 0$ and $f(\pi/2c) > 0$ for $c \geq 2$ and all values of parameters $a$ and $b$. It is possible to prove that then $f'(\gamma) > 0$. So, for $c \geq 2$, (7) has the unique solution.

B) If $c = 1$ then $r = 2q^2 - 1$ and $f(\gamma) = (1-q^2)f_1(\gamma)$ where

$$f_1(\gamma) = -4q^4(1-2p) - 4q^3 s + q^2(1-4p^2) + 2qs + s^2.$$

Since $1 - q^2 > 0$ (except for $q^2 = 1$ i.e. for $\gamma = 0$) we may consider $f_1(\gamma)$ instead of $f(\gamma)$.

a) In the case $b = 2$ it is $f_1(\gamma) \equiv 0$ for $a = 3$ and $f_1(\gamma) = 0 \Leftrightarrow q^2 = \frac{1+2p}{4}$ for $a \geq 4$. In the first case $(a,b,c) = (3,2,1)$, we get spherical simplex with dihedral angles $\pi/2$ and $\pi/4$ at edges $A_2A_3$ and $A_0A_2$ ($A_0A_3$), respectively. Second case is true only for $a = 4$ and $a = 5$.

b) If $b \geq 3$ and $a = 3$ then $f_1(0) < 0$, $f_1(\pi/2) > 0$ and $f_1'(\gamma) = 2q's(1-6q^2)$ where $q' = \sin\gamma$. Since $f_1'(\gamma)$ changes sign only for $q = -1/\sqrt{6}$ (i.e. $\gamma = \arccos(1/\sqrt{6})$), the equation $f_1(\gamma) = 0$ holds and so (7) has unique solution (with $\gamma \in (0, \arccos(1/\sqrt{6}))$.

c) For $b \geq 3$ and $a \geq 4$ again $f_1(0) < 0$, $f_1(\pi/2) > 0$, and $f_1'(\gamma) = 2q' f_2(\gamma)$, with $f_2(\gamma) = q(1-2p)(1+2p-8q^2) + s(1-6q^2)$. Since $2q' > 0$, sign of $f_1'(\gamma)$ is the same as of $f_2(\gamma)$. Function $f_2(\gamma)$ changes sign: $f_2(0) > 0$, $f_2(\pi/2) < 0$ and

$$f_2'(\gamma) = q'(-24q^2(1-2p) - 12qs + 1 - 4p^2).$$

It is possible to check that $f_2'(\gamma) < 0$ for $a \geq 6$ (since $q < 0$) and $f_2'(\gamma)$ changes sign only for $a = 4$ and $a = 5$. So, (similarly as in b)) $f_2(\gamma)$ (i.e. $f_1'(\gamma)$) changes sign only once. That means that (7) has the unique solution. ∎

Theorem 2. *Simplex* $\mathcal{T}_1$ *is spherical when* $(a,b,c)$ *takes* $(3,2,1)$, $(4,2,1)$, $(5,2,1)$, $(3,3,1)$. *Simplex is hyperbolic for* $(a,b,1)$, $a \geq 3$, $b \geq 3$ *except* $(3,3,1)$, *and for* $(a,b,c)$, $a \geq 3$, $b \geq 2$, $c \geq 2$. *Simplex* $\mathcal{T}_2$ *is hyperbolic for all parameters.*

Remark. In [8], [9] it is proved that simplex tiling by $\mathcal{T}_1$ is realizable in Nil space for $(6,2,1)$ and for $(a,2,1)$, $a \geq 7$ in the universal covering space of $SL_2(\mathbf{R})$.

*Proof.* Signature of $B$ in (6), obtained by finding the eigenvalues, is $(+,+,+,-)$, $(+,+,+,0)$ or $(+,+,+,+)$ whenever necessary condition is satisfied. Then we have simplex in hyperbolic, Euclidean or spherical space, respectively. The last critical sign is also possible to get as the sign of $\det B$. In our case $\det B = (1-p)Q$, with $Q = (1+p)(1-r^2) + 4qrs - 2(q^2+s^2)$. Since $1 - p > 0$ it is enough to consider sign of $Q$.

A) For $c \geq 2$, $q^2$, $r^2$ are in interval $(1/2, 1)$ and $p \leq 1/2$, $s \leq 0$ hold. Then $Q < O$, so these are hyperbolic cases.

B) a) If $c = 1$ and $b = 2$ then $q^2 = \frac{1+2p}{4}$ (for $a \in \{3,4,5\}$) and so $Q = q^2(-2p^2 + p + 1) > 0$. These are spherical cases.

b) If $c = 1$, $b \geq 3$ and $a \geq 4$ it is easy to prove that $Q < 0$, so we have hyperbolic cases.

c) For $c = 1$, $a = 3$ and $b \geq 3$ it is $Q = -2f_1(\gamma) + 2q^2(2 - 3q^2)$ ($f_1(\gamma)$ is defined in Lemma 2.). According to Lemma 2, $f_1(\gamma) = 0$ has to be satisfied. Thus, sign of $Q$ depends on sign of $2 - 3q^2$. Since $f_1(\arccos\sqrt{2/3})$ is positive for $a = 3$ and negative for $a \geq 4$, we have spherical and hyperbolic simplex, respectively.

Since parameters $b = 2b_1$, $c = 2c_1$ for simplex $\mathcal{T}_2$ are even, then case B never appears thus $\mathcal{T}_2$ is always hyperbolic. ■

For hyperbolic simplices it is interesting to investigate the cases when the vertices are proper, or they lie on the absolute or out of the absolute. Therefore, we need any submatrix $B_{ii}$ of $B$ in (6), corresponding to the vertex $A_i$, which we obtain by excluding $i$-th row and $i$-th column. Since vertices $A_0$, $A_2$ and $A_3$ are in the same equivalence class it is enough to consider only $B_{00}$ for vertex $A_0$ and $B_{11}$ for $A_1$.

Equivalently, as a 2-dimensional situation, we obtain well known simpler criteria in

THEOREM 3. *a) The vertex $A_1$ is proper ($>$), lies on the absolute ($=$) or out of it ($<$) if*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \gtreqless \frac{1}{2}.$$

*b) The vertices $A_0, A_2, A_3$ are proper ($>$), they are on the absolute ($=$) or out of it ($<$) if*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \gtreqless \frac{3}{2}.$$

We see that $A_0, A_2, A_3$ are outer vertices for any $c \geq 2$, since $a \geq 3$, $b \geq 2$ hold by our starting assumption.

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University of Belgrade, Faculty of Organizational Sciences, Jove Ilića 154, YU-11040 Belgrade