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NORM INEQUALITY FOR THE CLASS OF SELF-ADJOINT ABSOLUTE VALUE GENERALIZED DERIVATIONS

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Abstract. We prove that for all $0 \leq \alpha \leq 2/3$
$$\||A|^\alpha X - X|B|^\alpha\| \leq 2^{2-\alpha}\|X\|^{1-\alpha}\|AX - XB\|^\alpha,$$
for all bounded Hilbert space operators $A = A^*$, $B = B^*$ and X, as well as
$$\||A|^\alpha - |B|^\alpha\| \leq 2^{2-\alpha}\|A - B\|^\alpha,$$
for arbitrary bounded A and B.

Let H be a complex, infinite dimensional Hilbert space, $B(H)$ the algebra of all bounded linear operators on H and let $\|\cdot\|$ stands for the norm in $B(H)$. The following theorem compares a class of the absolute value generalized derivations on $B(H)$, induced by a pair of self-adjoint operators.

THEOREM 1. *For all* $0 \leq \alpha \leq 2/3$ *we have*
$$\||A|^\alpha X - X|B|^\alpha\| \leq 2^{2-\alpha}\|X\|^{1-\alpha}\|AX - XB\|^\alpha,$$
for bounded Hilbert space operators $A = A^*$, $B = B^*$ *and* X.

Proof. Let $A = U|A|$ and $B = V|B|$ be polar decompositions of A and B, with unitary $U = U^*$ and $V = V^*$, $|A| = \sqrt{A^*A}$ and $|B| = \sqrt{B^*B}$. Thus
$$\begin{aligned}\||A|^\alpha X - X|B|^\alpha\| &= \\ &\|U|A|^{\frac{\alpha}{2}}\left(U|A|^{\frac{\alpha}{2}}X - XV|B|^{\frac{\alpha}{2}}\right) + \left(U|A|^{\frac{\alpha}{2}}X - XV|B|^{\frac{\alpha}{2}}\right)V|B|^{\frac{\alpha}{2}}\| \\ &\leq 2\|U|A|^{\frac{\alpha}{2}}X - XV|B|^{\frac{\alpha}{2}}\|^{\frac{1-\alpha}{1-\alpha/2}} \times \\ &\left\|\frac{|A|^{1-\frac{\alpha}{2}}\left(U|A|^{\frac{\alpha}{2}}X - XV|B|^{\frac{\alpha}{2}}\right) + \left(U|A|^{\frac{\alpha}{2}}X - XV|B|^{\frac{\alpha}{2}}\right)|B|^{1-\frac{\alpha}{2}}}{2}\right\|^{\frac{\alpha/2}{1-\alpha/2}},\end{aligned} \tag{1}$$

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by Corollary 2.2 of [2] applied to $U|A|^{\frac{\alpha}{2}}X - XV|B|^{\frac{\alpha}{2}}$ instead of X and $r = \frac{2-\alpha}{\alpha} \geq 2$. As $\alpha/2 \leq 1/3$, then an application of Theorem 3.1 of [1] for $p = 2/\alpha \geq 3$ shows that

$$\|U|A|^{\frac{\alpha}{2}}X - XV|B|^{\frac{\alpha}{2}}\| \leq \|2X\|^{1-\frac{\alpha}{2}}\|AX - XB\|^{\frac{\alpha}{2}}. \tag{2}$$

Also, we have

$$\begin{aligned} &\||A|^{1-\frac{\alpha}{2}}(U|A|^{\frac{\alpha}{2}}X - XV|B|^{\frac{\alpha}{2}}) + (U|A|^{\frac{\alpha}{2}}X - XV|B|^{\frac{\alpha}{2}})|B|^{1-\frac{\alpha}{2}}\|/2 \\ &\quad = \|AX - XB + (U|A|^{\frac{\alpha}{2}}X|B|^{1-\frac{\alpha}{2}} - |A|^{1-\frac{\alpha}{2}}XV|B|^{\frac{\alpha}{2}})\|/2 \\ &\quad \leq \|AX - XB\|, \end{aligned} \tag{3}$$

by Lemma 3.2 of [1] applied for $p = 1$ and $s = \frac{\alpha}{2}$. Now, according to (2) and (3), (1) finally gives

$$\begin{aligned} \||A|^{\alpha}X - X|B|^{\alpha}\| &\leq 2\|2X\|^{(1-\frac{\alpha}{2})\frac{1-\alpha}{1-\alpha/2}}\|AX - XB\|^{\frac{\alpha}{2}\frac{1-\alpha}{1-\alpha/2}+\frac{\alpha/2}{1-\alpha/2}} \\ &= 2^{2-\alpha}\|X\|^{1-\alpha}\|AX - XB\|^{\alpha}. \quad \blacksquare \end{aligned} \tag{4}$$

This theorem also enables us to derive the following perturbation result for a class of the absolute value map in $B(H)$.

THEOREM 2. *For all $0 \leq \alpha \leq 2/3$ we have*

$$\||A|^{\alpha} - |B|^{\alpha}\| \leq 2^{2-\alpha}\|A - B\|^{\alpha}, \tag{5}$$

for arbitrary bounded Hilbert space operators A and B.

Proof. Define $C = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix}$ as operators acting on $H \oplus H$. A straightforward calculation shows that $C = C^*$, $D = D^*$, $|C|^{\alpha} = \begin{bmatrix} |A|^{\alpha} & 0 \\ 0 & |A^*|^{\alpha} \end{bmatrix}$ and $|D|^{\alpha} = \begin{bmatrix} |B|^{\alpha} & 0 \\ 0 & |B^*|^{\alpha} \end{bmatrix}$. Also

$$\|C - D\| = \max\{\|A - B\|, \|A^* - B^*\|\} = \|A - B\|$$

and

$$\||A|^{\alpha} - |B|^{\alpha}\| \leq \max\{\||A|^{\alpha} - |B|^{\alpha}\|, \||A^*|^{\alpha} - |B^*|^{\alpha}\|\} = \||C|^{\alpha} - |D|^{\alpha}\|.$$

An application of the preceeding theorem to self-adjoint C and D and $X = I$ gives

$$\||A|^{\alpha} - |B|^{\alpha}\| \leq \||C|^{\alpha} - |D|^{\alpha}\| \leq 2^{2-\alpha}\|C - D\|^{\alpha} = 2^{2-\alpha}\|A - B\|^{\alpha},$$

completing the proof. ■

REFERENCES

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