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L^p-ESSENTIAL SPECTRAL THEORY OF ORDINARY DIFFERENTIAL OPERATORS WITH ALMOST CONSTANT COEFFICIENTS

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Abstract. In this paper investigation is conducted of various essential spectra of minimal, maximal and intermediate ordinary differential operators in scale of Lebesque spaces $L^p(a,\infty)$, $1 \le p \le \infty$, obtained by means of relatively small perturbations of differential operators with constant coefficients of order n by differential operators of the same order, which generalizes the results [1–3]. This makes it possible to prove the new analogons of the classical Weyl theorem of invariance of essential spectrum as well as to obtain the precise formulas for calculating essential spectra of various classes of ordinary differential operators in Lebesque spaces L^p. In contemporary mathematical literature a few assertions are known as Weyl's theorem (see, for example, survey [4]). The classical Weyl theorem states that if A and B are self-adjoint and $A-B$ is compact then $\sigma_e(A) = \sigma_e(B)$, where σ_e is the essential spectrum of an operator. Generalization of Weyl theorem on various essential spectra for closed operators in Banach spaces and special classes of perturbations is dealt with in papers [5–7].

Let T be a closed linear operator densely defined on a complex Banach space. Essential spectra of an operator T could be defined as complements in a complex plane $\mathbf{C}$ of sets defined by various Fredholm properties of family of operators $T-\lambda I$:

$$\sigma_{ek}(T) := \mathbf{C} \setminus \Delta_k(T), \quad k = 1,2,3,4,5,$$
$$\sigma_{e2}^{+}(T) := \mathbf{C} \setminus \Phi^{+}(T) \quad \text{and} \quad \sigma_{e2}^{-}(T) := \mathbf{C} \setminus \Phi^{-}(T),$$

where $\Delta_1(T) := \{\lambda \in \mathbf{C} : \overline{R(T-\lambda I)} = R(T-\lambda I)\}$, $\Phi^{+}(T) := \{\lambda \in \Delta_1(T) : nul(T-\lambda I) < \infty\}$, $\Phi^{-}(T) := \{\lambda \in \Delta_1(T) : def(T-\lambda I) < \infty\}$, $\Delta_2(T) := \Phi^{+}(T) \cup \Phi^{-}(T) = s - \Phi(T)$, $\Delta_3(T) := \Phi^{+}(T) \cap \Phi^{-}(T) = \Phi(T)$, $\Delta_4(T) := \{\lambda \in \Delta_3(T) : ind(T-\lambda I) = 0\}$, $\Delta_5(T) := \{\lambda \in \Delta_4(T) : \text{a deleted neighbourhood of } \lambda \text{ lies in the resolvent set } \rho(T)\}$.

Each of the sets $\sigma_{ek}(T), k = \overline{1,5}$, $\sigma_{e2}^{+}(T)$ and $\sigma_{e2}^{-}(T)$ has been reffered to as the essential spectrum of T accoding to (1) Goldberg, (2) Kato, (2^+) Wolf, (2^-) Gustafson-Weidmann, (3) Fredholm, (4) Weyl or Schechter, (5) Browder. It is clear

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that $\sigma_{ek}(T) \subseteq \sigma_{el}(T)$ for $k \leq l$ and $\sigma_{e2}(T) \subseteq \sigma_{e2}^{\pm}(T) \subseteq \sigma_{e3}(T)$, where the inclusion might be proper. The essential spectra $\sigma_{ek}(T),\ k = 1, 2, 2^{\pm}, 3, 4, 5$, can be described by other equivalent means [8–11].

The basis of the theory of the essential spectrum σ_{e1} for ordinary differential operators in L^p spaces is due to Rota [12]. Balslev and Gamelin [13] investigated the Fredholm essential spectrum σ_{e3} of ordinary differential operators in spaces $L^p, 1 < p < \infty$, and generalization of these results for σ_{e1} in L^p, $1 \leq p \leq \infty$, is dealt with in Goldberg's monograph [14].

Let us consider a formal differential expression

$$\tau := \sum_{k=0}^{n} a_k(t) D^k, \quad a \leq t < \infty, \quad -\infty < a < \infty, \tag{1}$$

where $a_k(t)$ are complex valued functions such that $a_k(t) \in C^k[a, \infty)$, $a_n(t) \neq 0$, $1/a_n, a_k \in L^\infty(a, \infty)$, $0 \leq k \leq n$, and $D := d/dt$. Denote by $T(\tau, p, [a, \infty))$ a *maximal operator* corresponding to $(\tau, p, [a, \infty))$ which is defined on $L^p(a, \infty)$ as follows:

$$D[T(\tau, p, [a, \infty))] := \{f : f^{(n-1)} \in AC_{loc}[a, \infty);\ f, \tau f \in L^p(a, \infty)\},$$

where $AC_{loc}[a, \infty)$ is the set of complex valued functions f, absolutely continuous on each compact subinterval from $[a, \infty)$ and

$$T(\tau, p, [a, \infty))f := \tau f \quad \text{for} \quad f \in D[T(\tau, p, [a, \infty))].$$

We denote by $T_0(\tau, p, [a, \infty))$ a *minimal operator* defined on $L^p(a, \infty)$ for $1 \leq p < \infty$ as closure of restriction of the maximal operator $T(\tau, p, [a, \infty))$ on the set of functions from $D[T(\tau, p, [a, \infty))]$, having compact support in (a, ∞), and for $1 < p \leq \infty$ defined by a Banach conjugate $T'(\tau^*, p', [a, \infty))$, where τ^* is the formally conjugated differential operation $\tau^* g := \sum_{k=0}^{n} (-1)^k D^k (a_k g)$, and $1/p + 1/p' = 1$ if $1 < p < \infty$; $p' = \infty$ if $p = 1$; $p' = 1$ if $p = \infty$. Various properties of essential spectra of minimal and maximal ordinary differential operators are investigated in [15–17].

Theorem 1. *Let $S(\tau, p, [a, \infty)), -\infty < a < \infty$, be a closed linear differential operator in $L^p(a, \infty), 1 \leq p \leq \infty$, which is an extension of minimal operator $T_0(\tau, p, [a, \infty))$ and a restriction of maximal operator $T(\tau, p, [a, \infty))$ generated by differential operation τ (1) with smooth coefficients $a_k(t)$, $1/a_n \in L^\infty(a, \infty)$, $0 \leq k \leq n$,*

$$T_0(\tau, p, [a, \infty)) \subseteq S(\tau, p, [a, \infty)) \subseteq T(\tau, p, [a, \infty)).$$

Then for any $b \in (a, \infty)$ and five versions of essential spectra of differential operators $S(\tau, p, [a, \infty))$ and $S(\tau, p, [b, \infty))$ the following equalities hold:

$$\sigma_{ek}[S(\tau, p, [a, \infty))] = \sigma_{ek}[S(\tau, p, [b, \infty))], \quad k = 1, 2, 2^{\pm}, 3. \tag{2}$$

For Weyl essential spectrum σ_{e4} of minimal and maximal differential operators the following equalities hold:

$$\begin{aligned} \sigma_{e4}[T_0(\tau, p, [a, \infty))] &= \sigma_{e4}[T_0(\tau, p, [b, \infty))], \\ \sigma_{e4}[T(\tau, p, [a, \infty))] &= \sigma_{e4}[T(\tau, p, [b, \infty))]. \end{aligned} \tag{3}$$

Proof. We will consider some key points of the proof. To check the formulas (2) it suffices to consider an equality for Fredholm essential spectrum σ_{e3} as essential spectra $\sigma_{ek}, k = 1,2,3,$ and $\sigma_{e2}^{\pm}$ coincide for minimal $T_0(\tau,p,[a,\infty))$ and maximal $T(\tau,p,[a,\infty))$ operators [15]. The operator $T_{0b}(\tau,p,[a,\infty))$ for which $D[T_{0b}(\tau,p,[a,\infty))] := \{f : f^{(n-1)} \in AC_{loc}[a,\infty);\ f^{(j)}(a) = f^{(j)}(b) = 0,$ $j = \overline{0,n-1};\ f,\tau f \in L^p(a,\infty)\}$ and $T_{0b}(\tau,p,[a,\infty)) := \tau f$, in the direct sum of Banach spaces $L^p(a,\infty) = L^p(a,b) \oplus L^p(b,\infty)$ can be written as a decomposition $T_{0b}(\tau,p,[a,\infty)) = T_0(\tau,p,[a,b]) \oplus T_0(\tau,p,[b,\infty))$. This presentation implies the equality for Fredholm essential spectra

$$\sigma_{e3}[T_{0b}(\tau,p,[a,\infty))] = \sigma_{e3}[T_0(\tau,p,[a,b])] \cup \sigma_{e3}[T_0(\tau,p,[b,\infty))]. \tag{4}$$

Since $\sigma_{e3}[T_0(\tau,p,[a,b])] = \emptyset$ and minimal operator $T_0(\tau,p,[a,\infty))$ is an n-dimensional extension of the operator $T_{0b}(\tau,p,[a,\infty))$, the equality (4) implies $\sigma_{e3}[T_0(\tau,p,[a,\infty))] = \sigma_{e3}[T_0(\tau,p,[b,\infty))]$.

To prove the equality (3) we will introduce an additional operator $A_b(\tau,p,[a,b])$ defined on $L^p(a,b)$ for which $D[A_b(\tau,p,[a,b])] := \{f : f^{(n-1)} \in AC_{loc}[a,b];$ $f^{(j)}(b) = 0, j = \overline{0,n-1}\}$ and $A_b(\tau,p,[a,b])f := \tau f$. We define in the direct sum of Banach spaces $L^p(a,\infty) = L^p(a,b) \oplus L^p(b,\infty)$ an operator $T_b(\tau,p,[a,\infty)) :=$ $A_b(\tau,p,[a,b]) \oplus T_0(\tau,p,[b,\infty))$. The identical operator I is A_b-compact since the bounded inverse A_b^{-1} is compact and from $\rho(A_b) \neq \emptyset$ we have $\sigma_{e4}[A_b(\tau,p,[a,b])] =$ $\emptyset$. Therefore the following equality holds

$$\sigma_{e4}[T_b(\tau,p,[a,\infty))] \ = \ \sigma_{e4}[T_0(\tau,p,[b,\infty))]. \tag{5}$$

We would like to note that the operator $T_b(\tau,p,[a,\infty))$ is an n-dimensional extension of the operator $T_{0b}(\tau,p,[a,\infty))$. On the other hand, the minimal operator $T_0(\tau,p,[a,\infty))$ is an n-dimentional extension of the operator $T_{0b}(\tau,p,[a,\infty))$ and hence $ind[T_b(\tau,p,[a,\infty)) - \lambda I] = ind[T_0(\tau,p,[a,\infty)) - \lambda I]$. Therefore, the correlation (5) implies the equality (3) for Weyl essential spectrum of minimal operators. The theorem is proved. ∎

We denote by $B(\nu,p,[a,\infty))$ (respectively $B_0(\nu,p,[a,\infty))$) for $-\infty < a < \infty$ a maximal (minimal) differential operator generated in $L^p(a,\infty)$, $1 \le p < \infty$, by the formal differential operation

$$\nu \ := \ \sum_{k=0}^{n-1} b_k(t) D^k, \quad a \le t < \infty, \tag{6}$$

where complex valued functions $b_k \in C^k[a,\infty)$, $0 \le k \le n-1$, and by $T(\tau+\nu,p,[a,\infty))$ (respectively $T_0(\tau+\nu,p,[a,\infty))$) a maximal (minimal) operator generated in $L^p(a,\infty)$, $-\infty < a < \infty, 1 \le p < \infty$, by the formal differential operation $\tau+\nu$, where τ and ν is defined by the formulas (1) and (6). The operators $B(\nu,p,[a,\infty))$, $B_0(\nu,p,[a,\infty))$ and $T(\tau+\nu,p,[a,\infty))$, $T_0(\tau+\nu,p,[a,\infty))$ are defined likewise the maximal and minimal differential operator generated by the operation τ.

Theorem 2. *The maximal differential operator $B(\nu,p,[a,\infty))$ (the minimal differential operator $B_0(\nu,p,[a,\infty))$) generated by ν (6) in $L^p(a,\infty)$ is $T(\tau,p,[a,\infty))$-bounded (respectively $T_0(\tau,p,[a,\infty))$-bounded) for the differential operation τ (1), $-\infty<a<\infty$ and $1\le p<\infty$ if $b_k\in L^p_{loc}(a,\infty)$ and*

$$\sup_{m\le s<\infty}\int_s^{s+1}|b_k(t)|^p dt\to 0 \quad as \quad m\to\infty,\quad 0\le k\le n-1. \tag{7}$$

In case of fairly large $a\in(0,\infty)$ for maximal and minimal differential operators considered the following equalities hold

$$T(\tau+\nu,p,[a,\infty))=T(\tau,p,[a,\infty))+B(\nu,p,[a,\infty)), \tag{8}$$

$$T_0(\tau+\nu,p,[a,\infty))=T_0(\tau,p,[a,\infty))+B_0(\nu,p,[a,\infty)), \tag{9}$$

and a relative bound of differential operators $B(\nu,p,[a,\infty))$ and $B_0(\nu,p,[a,\infty))$ is strictly less than unity.

Proof. Denote for simplicity $T(\tau+\nu):=T(\tau+\nu,p,[a,\infty))$, $T(\tau):=T(\tau,p,[a,\infty))$ and $B(\nu):=B(\nu,p,[a,\infty))$. We will prove the equality (8) for maximal operators. It suffices to check the equality for domains $D[T(\tau+\nu)]=D[T(\tau)]$. The condition (7) of the theorem implies that $\forall\varepsilon>0$ there is a number $a\in(0,\infty)$ such that

$$\int_s^{s+1}|b_k(t)|^p\,dt<\varepsilon^p \quad \text{for} \quad [s,s+1]\subset[a,\infty),\ 0\le k\le n-1. \tag{10}$$

The estimates (10) for coefficients of the perturbating differential operation ν (6) imply the relative boundedness of the operator $B(\nu)$ in comparison with $T(\tau)$.

For functions $f\in W_p^n(a,\infty):=\{f:\ f^{(n-1)}\in AC_{loc}[a,\infty),\ f^{(i)}\in L^p[a,\infty),\ 0\le i\le n\}$, $1\le p<\infty$, and $0\le k\le n-1$ the following inequlity holds

$$\|b_kf^{(k)}\|_p^p\ \le\ C\Big(\|f^{(k+1)}\|_p^p+\|f^{(k)}\|_p^p\Big)\sup_{s\in[a,\infty)}\int_s^{s+1}|b_k(t)|^p\,dt, \tag{11}$$

where $\|\cdot\|_p$ is the norm in $L^p(a,\infty)$ [17].

Under conditions on coefficients $a_k(t)$ of the differential operation τ (1) there exists a constant K, which depends on p,n, the length of an interval I as well as the maximum of the numbers $\|1/a_n\|_{\infty,I}$ and $\|a_k\|_{\infty,I}, 0\le k\le n-1$, such that for $f\in D[T(\tau,p,I)]$ the following inequality holds [14]

$$\|f^{(k)}\|_{p,I}^p\ \le\ K\Big(\|\tau f\|_{p,I}^p+\|f\|_{p,I}^p\Big),\quad 0\le k\le n,\ 1\le p<\infty. \tag{12}$$

Due to the estimate (12) the inequalities (10) and (11) immediately imply that there are a constant M and $\varepsilon=\varepsilon(a)>0$ such that for functions $f\in D[T(\tau)]$ and $1\le p<\infty$ the following inequality holds

$$\|\nu f\|_p\ \le\ \sum_{k=0}^{n-1}\|b_kf^{(k)}\|_p\ \le\ \varepsilon M\Big(\|\tau f\|_p+\|f\|_p\Big). \tag{13}$$

If $f \in D[T(\tau)]$ then by definition $\tau f \in L^p(a,\infty)$, therefore the inequality (13) implies $\nu f \in L^p(a,\infty)$ whence $f \in D[T(\tau+\nu)]$. Thus we have proved the inclusion $D[T(\tau)] \subseteq D[T(\tau+\nu)]$. Now we are going to prove the inverse inclusion $D[T(\tau+\nu)] \subseteq D[T(\tau)]$. Let $f \in D[T(\tau+\nu)]$; then $(\tau+\nu)f \in L^p(a,\infty)$ and $f^{(k)}, 0 \le k \le n-1$, are continuous functions on each compact interval $J \subset [a,\infty)$ of length not less than 1, therefore $\nu f \in L^p(J)$, hence $\tau f = (\tau+\nu)f - \nu f \in L^p(J)$. The inequality $\|\tau f\|_{p,J} \le \|(\tau+\nu)f\|_{p,J} + \|\nu f\|_{p,J}$ and the inequality (13), which also holds for the norm $\|\cdot\|_{p,J}$ in space $L^p(J)$, imply the estimate (for $0 < \varepsilon M < 1$)

$$\|\nu f\|_{p,J} \le \frac{\varepsilon M}{1-\varepsilon M}\Big(\|(\tau+\nu)f\|_p + \|f\|_p\Big). \tag{14}$$

Since this inequality holds for any subinterval $J \subset [a,\infty)$ we have $\nu f \in L^p(a,\infty)$, hence $\tau f = (\tau+\nu)f - \nu f \in L^p(a,\infty)$, i.e. $f \in D[T(\tau)]$, therefore $D[T(\tau+\nu)] = D[T(\tau)]$, whence the equality (8). The equality (9) can be proved like the similar assertion of theorem 1 of paper [17]. The theorem is proved. ∎

Using theorem 2 and theorem on essential spectra of maximal and minimal ordinary differential operators with constant coefficients [16] one can find precise formulas for essential spectra of differential operators with almost constant coefficients, i.e. differential operators with variable coefficients tending to constant ones at infinity.

Let us consider a formal differential operation of the type

$$\mu := \tau + \nu = \sum_{k=0}^{n} a_k D^k + \sum_{k=0}^{n-1} b_k(t) D^k, \quad a \le t < \infty, \tag{15}$$

where a_k are complex numbers, τ is a differential operation of type (1) with constant coefficients and $b_k(t)$ are complex valued functions such that $b_k \in C^k(a,\infty)$, $k = \overline{0,n}$.

Theorem 3. *Let the coefficients $b_k(t)$, $0 \le k \le n-1$ in the differential operation μ (15) satisfy the integral conditions* (7)*. Then for essential spectra of minimal $T_0(\mu,p,[a,\infty))$, maximal $T(\mu,p,[a,\infty))$ operators generated by μ (15) in $L^p(a,\infty)$, $-\infty < a < \infty, 1 \le p < \infty$, and for the closed differential operator $S(\mu,p,[a,\infty))$, which is an extension of minimal and a restriction of maximal operators, as well as for similar operators defined by the differential operations τ (1) with constant coefficients and ν (6), the following equalities hold, which are the generalizations of Weyl theorem:*

$$\sigma_{ek}[S(\mu,p,[a,\infty))] = \sigma_{ek}[S(\tau,p,[a,\infty)) + S(\nu,p,[a,\infty))] = \sigma_{ek}[S(\tau,p,[a,\infty))],$$

$k = 1, 2, 2^{\pm}, 3,$

$$\sigma_{ek}[T_0(\mu,p,[a,\infty))] = \sigma_{ek}[T_0(\tau,p,[a,\infty)) + T_0(\nu,p,[a,\infty))] = \sigma_{ek}[T_0(\tau,p,[a,\infty))],$$

$k = 4, 5,$

$$\sigma_{ek}[T(\mu,p,[a,\infty))] = \sigma_{ek}[T(\tau,p,[a,\infty)) + T(\nu,p,[a,\infty))] = \sigma_{ek}[T(\tau,p,[a,\infty))],$$

$k = 4, 5.$

Proof. Denote by $T(\mu) := T(\mu, p, [a,\infty))$, $T_0(\mu) := T_0(\mu, p, [a,\infty))$ and by $T(\tau+\nu), T(\tau), B(\nu)$ (respectively $T_0(\tau+\nu), T_0(\tau), B_0(\nu)$) the maximal (minimal) operators defined while proving theorem 2.

Note that the inequality (12) for norms of derivatives also holds for a differential operation $\tau - \lambda,\ \lambda \in \mathbf{C}$. Therefore for functions $f \in D[T(\tau)] = D[T(\tau+\nu)]$ in corresponding Lebesque spaces $L^p(a,\infty),\ 1 \le p < \infty$, the inequalities (13) and (14) imply that there exists such $a \in (0,\infty)$ and, due to the conditions (7) on coefficients b_k, such fairly small $\varepsilon = \varepsilon(a), 0 < \varepsilon < 1$, for which the following inequalities hold

$$\begin{aligned} \|\nu f\|_p &\le \varepsilon\big(\|(\tau-\lambda)f\|_p + \|f\|_p\big), \\ \|\nu f\|_p &\le \varepsilon\big(\|(\tau+\nu-\lambda)f\|_p + \|f\|_p\big). \end{aligned}$$

These estimates as well as the following equalities for maximal operators $T(\tau+\nu) - \lambda I = (T(\tau) - \lambda I) + B(\nu)$ (see formula (8)) and $T(\tau) - \lambda I = (T(\tau) + B(\nu) - \lambda I) - B(\nu) = (T(\tau+\nu) - \lambda I) - B(\nu)$ imply that the operator $T(\tau+\nu) - \lambda I$ can be considered as a relatively small perturbation of the operator $T(\tau) - \lambda I$ and vice versa. Hence applying the theorem of stability of the index of closed semi-Fredholm operators [11] we have $\lambda \in \Phi^+(T(\tau))$ if and only if $\lambda \in \Phi^+(T(\tau) + B(\nu))$ and for indices the following equality holds $ind(T(\tau) - \lambda I) = ind(T(\tau) + B(\nu) - \lambda I) = ind(T(\tau+\nu) - \lambda I)$. Therefore for essential spectra of the maximal operator $T(\tau+\nu)$ the following equalities hold

$$\sigma_{ek}[T(\tau+\nu)] = \sigma_{ek}[T(\tau)],\ k = \overline{1,4}, \quad \sigma^{\pm}_{e2}[T(\tau+\nu)] = \sigma^{\pm}_{e2}[T(\tau)]. \tag{16}$$

Arguing in the same way, using the equality (9) for the minimal operators $T_0(\tau)$ and $B_0(\nu)$, one can show that for essential spectra of the minimal operator $T_0(\tau+\nu)$ the corresponding equalities hold

$$\sigma_{ek}[T_0(\tau+\nu)] = \sigma_{ek}[T_0(\tau)],\ k = \overline{1,4}, \quad \sigma^{\pm}_{e2}[T_0(\tau+\nu)] = \sigma^{\pm}_{e2}[T_0(\tau)]. \tag{17}$$

The theorem is proved. ∎

Corollary 1. *Essential spectra of differential operators generated by the operation μ (15) and defined in theorem 3 can be calculated by the formulas:*

$$\sigma_{ek}[S(\mu,p,[a,\infty))] = \sigma^{\pm}_{e2}[S(\mu,p,[a,\infty))] = \{P(\lambda) : Re\lambda = 0\},\ k = \overline{1,3}, \tag{18}$$

$$\sigma_{ek}[T_0(\mu,p,[a,\infty))] = \sigma[T_0(\mu,p,[a,\infty))] = \{P(\lambda) : Re\lambda \ge 0\},\ k = 4,5, \tag{19}$$

$$\sigma_{ek}[T(\mu,p,[a,\infty))] = \sigma[T(\mu,p,[a,\infty))] = \{P(\lambda) : Re\lambda \le 0\},\ k = 4,5, \tag{20}$$

where P is a polynomial corresponding the differential operation with constant coefficients τ (1)

$$P(t) := \sum_{k=0}^{n} a_k t^k.$$

Proof. In theorem 2 of paper [16] the precise formulas for finding all essential spectra of maximal and minimal differential operators with constant coefficients were obtained. Therefore the equalities (16), (17) and theorem 3 imply the corresponding assertions in case of formal differential operation $\mu = \tau = \nu$ (15) for essential spectra $\sigma_{ek},\ k = \overline{1,5}$, and $\sigma_{e2}^{\pm}$ of minimal $T_0(\mu)$ and maximal $T(\mu)$ perturbated differential operators in Lebesque spaces $L^p(a,\infty)$ and corresponding $a \in (0,\infty)$. The remaining equalities for intermediate differential operators $S(\mu,p,[a,\infty))$ follow from theorem 1 of paper [15]. Finally the formulas (18)–(20) of all essential spectra of considered differential operators in Lebesque spaces $L^p(a,\infty),\ 1 \le p < \infty$, for any $a \in (-\infty,\infty)$ follow from theorem 1. The corollary is proved. ■

Let us consider a differential operation μ of more general type, namely, as a perturbation of τ by a differential operation of the same order n, i.e.

$$\overline{\mu} \ := \ \sum_{k=0}^{n}(a_k + b_k(t))D^k, \quad a \le t < \infty, \tag{21}$$

where a_k are complex numbers and complex valued functions of real argument $b_k \in C^k(a,\infty)$, for $k = \overline{0,n}$.

Corollary 2. *Let for coefficients of the differential operation $\overline{\mu}$ (21) the following conditions hold $b_n, 1/(a_n + b_n) \in L^\infty(a,\infty)$ and coefficients $b_k(t),\ 0 \le k \le n$, satisfy conditions of tending to 0 at infinity (7). Then for the minimal $T_0(\overline{\mu},p,[a,\infty))$, maximal $T(\overline{\mu},p,[a,\infty))$ and intermediate $S(\overline{\mu},p,[a,\infty))$ differential operators generated in $L^p(a,\infty)$, $-\infty < a < \infty, 1 \le p < \infty$, by the differential operation $\overline{\mu}$ (21) the following generalizations of the classical Weyl theorem hold:*

$$\begin{aligned}
\sigma_{ek}[S(\overline{\mu},p,[a,\infty))] &= \sigma_{ek}[S(\tau,p,[a,\infty))], \quad k = 1,2,2^{\pm},3,\\
\sigma_{ek}[T_0(\overline{\mu},p,[a,\infty))] &= \sigma_{ek}[T_0(\tau,p,[a,\infty))], \quad k = 4,5,\\
\sigma_{ek}[T(\overline{\mu},p,[a,\infty))] &= \sigma_{ek}[T(\tau,p,[a,\infty))], \quad k = 4,5.
\end{aligned}$$

Besides, for differential operators defined by the operations $\overline{\mu}$ (21) the formulas for essential spectra (18)–(20) hold.

Proof. Since theorem 2 holds for the differential operation τ (1) with variable coefficients, considering the conditions on the coefficients $b_k(t),\ 0 \le k \le n$, (7) it suffices to investigate the particular case of the differential operation $\overline{\mu}$ (21) with the coefficients $b_k(t) = 0,\ 0 \le k \le n-1$, the operation of the type

$$\overline{\mu} := \tau + b_n(t)D^n \ = \ (a_n + b_n(t))D^n + \sum_{k=0}^{n-1} a_k D^k. \tag{22}$$

In other words to prove the assertion of the theorem for differential operators with $\overline{\mu}$ (22) and then to consider the general differential operation $\overline{\mu}$ (21) as a relatively small perturbation of the differential operation (22). To do so it suffices to present the differential operation $\overline{\mu}$ (22) in the form

$$\overline{\mu} - \lambda \ = \ \Big(1 + \frac{b_n(t)}{a_n}\Big)(\tau - \lambda) - \sum_{k=0}^{n-1} \frac{b_n(t)}{a_n} a_k D^k + \frac{b_n(t)}{a_n}\lambda. \quad ■$$

Corollary 3. *For essential spectra of minimal $T_0(\overline{\mu},\infty,[a,\infty))$, maximal $T(\overline{\mu},\infty,[a,\infty))$ and intermediate $S(\overline{\mu},\infty,[a,\infty))$ differential operators generated by the formal differential operation $\overline{\mu}$ (21) in space $L^\infty(a,\infty)$, $-\infty < a < \infty$, with coefficients satisfying the conditions of corollary 2 and with their derivatives satisfying*

$$\sup_{m\le s<\infty}\int_s^{s+1}|b_k^{(i)}(t)|^p dt \to 0 \quad as \quad m\to\infty, \quad 0\le i\le k,\ 0\le k\le n,$$

the formulas (18)–(20) hold for differential operators in case $p=\infty$.

To prove this assertion it suffices to apply corollary 2 to differential operators generated in space $L^1(a,\infty)$ by the formal conjugated differential operation

$$(\overline{\mu})^* f := \sum_{k=0}^{n}(-1)^k D^k\big((a_k+b_k(t))f\big),$$

and then using the formulas of duality (see, for instance, [14,15]) proceed to differential operators defined by the operation $\overline{\mu}$ (21) in space $L^\infty(a,\infty)$.

Theorem 3 and corollary 1,2,3 generalize the results of [13,14] for Fredholm and Goldberg essential spectra of maximal operators in space $L^p(0,\infty)$ as well as the results of [10,18] for various essential spectra of ordinary differential operators in Hilbert space. We would like to note books [19,20] on localization of essential spectrum of ordinary self-adjoint differential operators with variable coefficients.

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