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# BESOV SPACES ON BOUNDED SYMMETRIC DOMAINS

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**Abstract.** We define and study a class of holomorphic Besov type spaces $B^p$, $0 < p < 1$, on bounded symmetric domains $\Omega$. We show that the dual of holomorphic Besov space $B^p$, $0 < p < 1$, on bounded symmetric domain $\Omega$ can be identified with the Bloch space $\mathcal{B}^\infty$.

## 1. Introduction

Let $\Omega$ be an irreducible symmetric domain in $C^n$ in its Harish-Chandra realization. In [11] and [13] K. Zhu defined and studied a class of holomorphic Besov-type spaces $B^p$ on $\Omega$ for $1 \le p \le \infty$. In [4] analogous holomorphic Besov spaces $B^p$ are defined for $0 < p < 1$ and some of the results presented in [11] and [13] are extended to the case $0 < p < 1$. The main purpose of this paper is to show that the dual of $B^p$, $0 < p < 1$, can be indentified with the Bloch space $\mathcal{B}^\infty$.

It is well known [5] that the domain $\Omega$ is uniquely determined (up to a biholomorphic mapping among standard irreducible bounded symmetric domains) by three analytic invariants; $r$, $a$ and $b$, all of which are nonnegative integers. The invariant $r$ is called the rank of $\Omega$, which is of course always positive. See [5] for the definition of $a$ and $b$. We shall make extensive use of the following invariant of $\Omega$: $N = a(r-1) + b + 2$.

Let $\nu$ be Lebesgue measure on $\Omega$ normalized so that $\nu(\Omega) = 1$. For $0 < p < \infty$ the Bergman space $L^p_a(\Omega)$ is the closed subspace of $L^p(\Omega, d\nu)$ consisting of holomorphic functions. The Bergman projection $P$ (namely, the orthogonal projection from $L^2(\Omega, d\nu)$ onto $L^2_a(\Omega)$) is an integral operator

$$Pf(z) = \int_\Omega K(z,w) f(w)\, d\nu(w), \qquad z \in \Omega, \quad f \in L^2(\Omega, d\nu).$$

By [3] there exists a polynomial $h(z,w)$ in z and $\bar{w}$ such that the Bergman kernel of $\Omega$ is given by

$$K(z,w) = h(z,w)^{-N}, \qquad z, w \in \Omega.$$

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Throughout this paper we assume $\alpha$ is a real number satisfying $\alpha > -1$. Let $c_\alpha$ be a positive normalizing constant such that the measure $d\nu_\alpha(z) = c_\alpha h(z,z)^\alpha d\nu(z)$ has total mass 1 on $\Omega$.

Let $H(\Omega)$ be the space of all holomorphic functions in $\Omega$. We equip $H(\Omega)$ with the topology of uniform convergence on compact sets. In [13] it is shown that the operator

$$D^{m,\alpha}\colon H(\Omega) \to H(\Omega), \qquad m \geq 0, \quad \alpha > -1,$$

defined by

$$D^{m,\alpha} f(z) = \lim_{r\to 1} \int_\Omega \frac{f(rw)\, d\nu_\alpha(w)}{h(z,w)^{N+\alpha+m}}, \quad f \in H(\Omega),$$

is continuous and invertible on $H(\Omega)$. The inverse of $D^{m,\alpha}$ admits the following integral representation:

$$D_{m,\alpha} f(z) = c_{m+\alpha} \lim_{r\to 1} \int_\Omega \frac{h(w,w)^{m+\alpha} f(rw)\, d\nu(w)}{h(z,w)^{N+\alpha}},\ f \in H(\Omega),\ z \in \Omega.$$

We note that if

$$f \in L_a^{1,\alpha}(\Omega) = L^1(\Omega, d\nu_\alpha) \cap H(\Omega)$$

then

$$D^{m,\alpha} f(z) = \int_\Omega \frac{f(w)\, d\nu_\alpha(w)}{h(z,w)^{N+\alpha+m}}.$$

The above formula extends the domain of $D^{m,\alpha}$ to $L^1(\Omega, d\nu_\alpha)$. We write

$$V_{m,\alpha} f(z) = h(z,z)^m D^{m,\alpha} f(z), \text{ for } f \in L^1(\Omega, d\nu_\alpha), \quad m \geq 0, \quad \alpha > -1,$$

and

$$E_{m,\alpha} f(z) = h(z,z)^m D^{m,\alpha} f(z), \quad m \geq 0, \quad \alpha > -1, \quad \text{for} \quad f \in H(\Omega).$$

Thus, if $f \in L_a^{1,\alpha}(\Omega)$, then $E_{m,\alpha} f = V_{m,\alpha} f$.

We begin with a result from [4].

Theorem 1.1. *Let $k, m > \frac{N-1}{p}$ and $\alpha, \beta > \max\{\frac{a(r-1)}{2p} - N, -1\}$. If $0 < p \leq 1$ and $f \in H(\Omega)$ then $\int_\Omega |E_{m,\alpha} f(z)|^p\, d\tau(z) < \infty$ if and only if $\int_\Omega |E_{k,\beta} f(z)|^p\, d\tau(z) < \infty$, where $d\tau(z) = h(z,z)^{-N}\, d\nu(z)$ is the Möbious invariant measure on $\Omega$.*

Recall that the holomorphic Besov space $B^p$, $1 \leq p \leq \infty$ consists of functions in $H(\Omega)$ such that $E_{N,0} f$ is in $L^p(\Omega, d\tau)$ (see [11] and [13]). For any irreducible bounded symmetric domain $\Omega$ we have $\frac{a(r-1)}{2} - N \leq -2$. Thus, a holomorphic function $f$ in $\Omega$ belongs to $B^1$ if and only if $E_{m,\alpha} f$ is in $L^1(\Omega, d\tau)$ for some (any) $m > N-1$ and some (any) $\alpha > -1$. This is also proved in [13], Theorem 4, by a different method.

Definition 1.2. For $0 < p \leq 1$ the holomorphic Besov space $B^p = B^p(\Omega)$ consists of holomorphic functions $f \in H(\Omega)$ such that

$$\|f\|_{B^p} = |f(0)| + \|E_{m,\alpha} f\|_{L^p(\Omega, d\tau)} < \infty,$$

for some (any) $m > \frac{N-1}{p}$ and $\alpha > \max\{\frac{a(r-1)}{2p} - N, -1\}$.

For every $z$ in $\Omega$ let $E_r(z)$ be the closed Bergman metric ball with center $z$ and radius $r > 0$, i.e.,

$$E_r(z) = \{w : \beta(z,w) \leq r\},$$

where $\beta(\cdot,\cdot)$ is the Bergman metric on $\Omega$.

For a complex measurable function $f$ on $B$ we define

$$M_{\infty,r} = \operatorname{esssup}\{\, |f(w)| :\ w \in E_r(z)\,\}$$

and

$$M_{p,r}f(z) = \left[\frac{1}{\tau(r)}\int_{E_r(z)} |f(w)|^p \, d\tau(w)\right]^{\frac{1}{p}}, \qquad 0 < p < \infty,$$

where $\tau(r) = \tau(E_r(z))$.

For $0 < p,q \leq \infty$, we define $L_r^{p,q}(\Omega, d\tau)$ to be the space of all measurable functions $f$ on $\Omega$ for which

$$\|f\|_{L_r^{p,q}(\Omega,d\tau)} = \|M_{p,r}f\|_{L^q(\Omega,d\tau)} < \infty.$$

Since the definition is independent of $r$, $0 < r < 1$, we will write $L^{p,q}(\Omega, d\tau)$ instead of $L_r^{p,q}(\Omega, d\tau)$ (see [1]).

We let $P_\alpha$ denote the orthogonal projection from $L^2(\Omega, d\nu_\alpha)$ onto $L_a^{2,\alpha}(\Omega) = L^2(\Omega, d\nu_\alpha) \cap H(\Omega)$. It can be shown that (see [9], for instance)

$$P_\alpha f(z) = \int_\Omega \frac{f(w)\, d\nu_\alpha(w)}{h(z,w)^{N+\alpha}}, \quad z \in \Omega,\ f \in L^2(\Omega, d\nu_\alpha).$$

The above formula extends the domain of $P_\alpha$ to $L^1(\Omega, d\nu_\alpha)$. Note that $P_\alpha f = D^{0,\alpha} f$ for $f \in L^1(\Omega, d\nu_\alpha)$.

If $1 \leq p \leq \infty$ and $\alpha > -1$ then $B^p = P_\alpha L^p(\Omega, d\tau)$, see [13]. In [4] it is shown that the analytic Besov space $B^p$, $0 < p < 1$, can be naturally embedded as a complemented subspace of $L^{1,p}(\Omega, d\tau)$ by a topological embedding

$$E_{m,\alpha}\colon B^p \to L^{1,p}(\Omega, d\tau).$$

It is also shown that $E_{m,\alpha} \circ P_\alpha$ is projection on this embedded copy and that $B^p = P_\alpha L^{1,p}(\Omega, d\tau)$.

More precicely the following theorem is proven.

Theorem 1.3. *Let* $0 < p < 1$. *Then for any* $\alpha > \max\{\frac{a(r-1)}{2p} - N, -1\}$,

$$P_\alpha\colon L^{1,p}(\Omega, d\tau) \to B^p$$

*is a continuous linear map. Moreover if* $m > \frac{N-1}{p}$ *and* $\alpha > \max\{\frac{a(r-1)}{2p} - N, -1\}$ *then*

$$E_{m,\alpha}\colon B^p \to L^{1,p}(\Omega, d\tau)$$

*is a topological embedding.*

Now we apply Theorem 1.3. to obtain a result about duality.

THEOREM 1.4. *Let $0 < p < 1$, $m > \frac{N-1}{p}$ and $\alpha = m - N$. The integral pairing*

$$\langle f, g\rangle_\tau = \int_\Omega E_{m,\alpha} f(z)\, \overline{E_{m,\alpha} g(z)}\, d\tau(z)$$

*induces the following duality $(B^p)^\star = B^\infty$.*

## 2. Duality

A linear functional $\lambda$ on $B^p$, $0 < p < 1$, is said to be bounded if

$$\|\lambda\| = \sup\{|\lambda(f)| : \|f\|_{B^p} \le 1\} < \infty.$$

The dual space of $B^p$, denoted $(B^p)^\star$, is then the space of all bounded linear functionals on $B^p$. In [13] it is shown that each $(L_a^{p,\alpha})^\star$, $0 < p \le 1$, can be identified with $B^\infty$ via volume integral pairing

$$\langle f, g\rangle_\beta = \lim_{r\to 1} \int_\Omega f(rz) g(z)\, d\nu_\beta(z), \quad \text{where} \quad \beta = \frac{N+\alpha}{p} - N.$$

Our Theorem 1.4 shows that $(B^p)^\star$ can also be identified with $B^\infty$, but via a different integral pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\tau$.

*Proof of Theorem 1.4.* First, assume that $g$ is a function in $B^\infty$. Then

$$\|g\|_{B^\infty} \ge \sup_{z\in\Omega} h(z,z)^m |D^{m,\alpha} g(z)| = \sup_{z\in\Omega} |E_{m,\alpha} g(z)|, \quad \text{(see [13])}.$$

We show that $g$ gives rise to a bounded linear functional on $B^p$ under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\tau$. By Theorem 1.3 if $f \in B^p$ then

$$\|f\|_{B^p} \ge \|E_{m,\alpha} f\|_{L^{1,p}(\Omega,d\tau)} \ge C\|E_{m,\alpha} f\|_{L^1(\Omega,d\tau)}.$$

Thus if $f \in B^p$, then we have

$$|\langle f, g\rangle_\tau| \le \sup_{z\in\Omega} |E_{m,\alpha} g(z)| \|E_{m,\alpha} f\|_{L^1(\Omega,d\tau)} \le C\|f\|_{B^p}\|g\|_{B^\infty}.$$

Conversely, assume that $\lambda$ is a bounded linear functional on $B^p$; we show that $\lambda$ arises from a function in $B^\infty$. Since $E_{m,\alpha}$ is a topological embedding of $B^p$ into $L^{1,p}(\Omega, d\tau)$, $\lambda \circ E_{m,\alpha}^{-1}$ extends to a bounded linear functional on $L^1(\Omega, d\tau)$. Thus there exists a function $\varphi \in L^\infty(\Omega, d\tau)$ such that

$$\lambda \circ E_{m,\alpha}^{-1}(\psi) = \int_\Omega \psi(z)\overline{\varphi(z)}\, d\tau(z), \qquad \psi \in L^1(\Omega, d\tau).$$

When $f \in B^p$, $E_{m,\alpha} f \in L^{1,p}(\Omega, d\tau) \subset L^1(\Omega, d\tau)$. Therefore

$$\lambda(f) = \int_\Omega E_{m,\alpha} f(z)\overline{\varphi(z)}\, d\tau(z) = \int_\Omega V_{m,\alpha} f(z)\overline{\varphi(z)}\, d\tau(z), \quad f \in B^p.$$

As was noted in Introduction $E_{m,\alpha}f = V_{m,\alpha}f$ if $f \in L_a^{1,\alpha}(\Omega)$. Since $B^p \subset B^\infty$, we have that $\int_\Omega |f(\xi)| h(\xi,\xi)^\alpha \, d\nu(\xi) < \infty$. (See the remark following Lemma 7 in [11] ). Let $h = P_\alpha \varphi$. Then $h \in B^\infty$ by Theorem 6 ([13])

$$\begin{aligned} E_{m,\alpha}h(z) = V_{m,\alpha}h(z) &= h(z,z)^m D^{m,\alpha} P_\alpha \varphi(z) = h(z,z)^m D^{m,\alpha}(D^{0,\alpha}\varphi)(z) \\ &= h(z,z)^m D^{m,\alpha}\varphi(z) = V_{m,\alpha}\varphi(z). \end{aligned}$$

To finish the proof of Theorem 1.4 it remains to show that

$$\langle V_{m,\alpha}f, V_{m,\alpha}\varphi\rangle_\tau = \langle V_{m,\alpha}^2 f, \varphi\rangle_\tau \quad \text{and that } \; c_{m+\alpha} V_{m,\alpha}^2 f = c_\alpha V_{m,\alpha} f.$$

This follows easily from Fubini's theorem and the reproducing property of $P_\alpha$. Note that $\alpha = m - N$. We leave the details to the interested reader. Thus,

$$\lambda(f) = \int_\Omega V_{m,\alpha}f(z)\overline{V_{m,\alpha}g(z)}\, d\tau(z) = \int_\Omega E_{m,\alpha}f(z)\overline{E_{m,\alpha}g(z)}\, d\tau(z)$$

for all $f \in B^p$, where $g = c_{m+\alpha}c_\alpha^{-1} h \in B^\infty$. ■

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