

GENERAL REPRESENTATIONS OF PSEUDOINVERSES

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Abstract. In this paper we investigate general representations of various classes of generalized inverses for bounded operators over Hilbert and Banach spaces. These representations are expressed by means of the full-rank decomposition of bounded operators and adequately selected operators.

1. Introduction

Let \mathcal{X}_1 and \mathcal{X}_2 denote arbitrary Banach spaces and $B(\mathcal{X}_1, \mathcal{X}_2)$ denote the set of all bounded operators from \mathcal{X}_1 into \mathcal{X}_2 . For an arbitrary operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$, we use $\mathcal{N}(A)$ to denote its kernel, and $\mathcal{R}(A)$ to denote its image.

For $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ we say that an operator $X \in B(\mathcal{X}_2, \mathcal{X}_1)$ is a generalized inverse of A , provided that some of the following equations are satisfied:

$$(1) \quad AXA = A, \quad (2) \quad XAX = X$$

If X satisfies the equation (1), then X is called a g -inverse of A . If X satisfies the equations (1) and (2), then it is called a reflexive g -inverse of A .

It is well-known that an operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ has a g -inverse if and only if $\mathcal{R}(A)$ is closed, and $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, are complemented subspaces of \mathcal{X}_1 and \mathcal{X}_2 .

The notion of the full-rank decomposition for complex matrices is well-known and frequently used. Recall the definition of the full rank factorization for a bounded operator acting on Banach spaces from [2] and [3]:

Let $A \in B(\mathcal{X}_1, \mathcal{X}_2)$. If there exist: a Banach space \mathcal{X}_3 and operators $Q \in B(\mathcal{X}_1, \mathcal{X}_3)$ and $P \in B(\mathcal{X}_3, \mathcal{X}_2)$, such that P is left invertible, Q is right invertible and

$$A = PQ, \tag{1.1}$$

then we say that (1.1) is the full-rank decomposition of A .

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It is well-known that an operator $A \in B(\mathcal{X}_1, \mathcal{X}_2)$ has the full-rank decomposition, if and only if A is g -invertible. In this case \mathcal{X}_3 is isomorphic to $\mathcal{R}(A)$, and $\mathcal{R}(A) = \mathcal{R}(P)$ [3].

We say that $A \in B(\mathcal{X})$ has the Drazin inverse, if there exists an operator $A^D \in B(\mathcal{X})$, such that A^D satisfies the equation (2) and the equations

$$(1^k) \quad A^{k+1}A^D = A^k, \quad (5) \quad A^D A = AA^D,$$

for some non-negative integer k . Let us mention that the Drazin inverse, if it exists, is unique. The smallest k in the previous definition is called the index of A and denoted by $\text{ind}(A)$. In the case $\text{ind}(A) = 1$ the Drazin inverse is known as the group inverse of A , denoted by $A^\#$.

Recall that $\text{asc}(A)$ (respectively $\text{des}(A)$), the ascent (respectively descent) of A , is the smallest non-negative integer n , such that $\mathcal{N}(A^n) = \mathcal{N}(A^{n+1})$ (respectively $\mathcal{R}(A^n) = \mathcal{R}(A^{n+1})$). If no such n exists, then $\text{asc}(A) = \infty$ (respectively $\text{des}(A) = \infty$) [4]. It is well-known that A has the Drazin inverse, if and only if the ascent and descent of A are finite (hence, equal to $\text{ind}(A)$) [3], [4].

In the case when \mathcal{H}_1 and \mathcal{H}_2 are Hilbert spaces, it is well-known that an operator $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ has a g -inverse if and only if $\mathcal{R}(A)$ is closed. Among the equations (1), (2) we also consider the following equations in X :

$$(3) \quad (AX)^* = AX, \quad (4) \quad (XA)^* = XA.$$

For a subset \mathcal{S} of the set $\{1, 2, 3, 4\}$, the set of operators obeying the conditions contained in \mathcal{S} is denoted by $A\{\mathcal{S}\}$. An operator in $A\{\mathcal{S}\}$ is called an \mathcal{S} -inverse of A and is denoted by $A^{(\mathcal{S})}$. If $\mathcal{R}(A)$ is closed, the set $A\{1, 2, 3, 4\}$ consists of a single element, the Moore-Penrose inverse of A , denoted by A^\dagger .

We also consider the following equations, which define the weighted Moore-Penrose inverse:

$$\begin{array}{ll} (3M) & (MAX)^* = MAX \\ (3M') & (AX)^*M = MAX \end{array} \quad \begin{array}{ll} (4N) & (NXA)^* = NXA; \\ (4N') & (XA)^*N = NXA, \end{array}$$

where $M \in B(\mathcal{H}_2)$, $N \in B(\mathcal{H}_1)$ are positive or invertible. Any solution of the equations (1), (2), (3M) and (4N), when it exists, will be denoted by $A_{M,N}^\dagger$. Similarly, any solution of the equations (1), (2), (3M') and (4N'), when it exists, will be denoted by $A_{M',N'}^\dagger$.

We investigate general representations and conditions for the existence of generalized inverses of bounded linear operators on Hilbert spaces, arising from the factorization (1.1). As a related result we investigate some representations of a generalized inverse $A_{T,S}^{(2)}$. Obtained representations are generalizations of the analogous results available in the literature for matrices. We also introduce a general representation and conditions for the existence of the Drazin inverse of a bounded operator on a Banach space. These representations are based on the full-rank decomposition of A^l , where $l \geq \text{ind}(A)$. Such an approach in representation of the Drazin inverse is not employed before even for complex matrices.

2. Results

Firstly, we investigate general representations of $\{1, 2\}$ -inverses, $\{1, 2, 3\}$, $\{1, 2, 4\}$ -inverses, the Moore-Penrose and the weighted Moore-Penrose for operators in arbitrary Hilbert spaces.

We shall frequently use the following observation. If $S \in B(\mathcal{H}_1, \mathcal{H}_2)$ is onto, then SS^* is invertible and S^\dagger is the right inverse of S . Analogously, if $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ is one-to-one with closed range, then T^*T is invertible and T^\dagger is the left inverse of T .

In the beginning, we state an analogy of the well-known result from [10, pp. 20, 28].

LEMMA 2.1. *If $A = PQ$ is the full-rank decomposition of $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ according to (1.1), then:*

- (a) *Any right inverse of Q can be represented in the following form: $Q_r^{-1} = W_1(QW_1)^{-1}$, for an arbitrary operator $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$ such that QW_1 is invertible.*
- (b) *Any left inverse of P can be represented in the following form: $P_l^{-1} = (W_2P)^{-1}W_2$, for an arbitrary operator $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$ such that W_2P is invertible.*
- (c) *Any reflexive generalized inverse X of A has the form $X = Q_r^{-1}P_l^{-1}$ for an arbitrary right inverse Q_r^{-1} of Q and an arbitrary left inverse P_l^{-1} of P .*

In the literature there are known general representations for various classes of generalized inverses, for the set of complex matrices. The general representation of $\{1, 2\}$ inverses for matrices is investigated in [9] and [10, pp. 20, 28]. The general representations of $\{1, 2, 3\}$ and $\{1, 2, 4\}$ inverses for matrices are investigated in [9]. In [6] there is given a general representation and conditions for the existence of the group inverse for a given complex matrix. The general representation of the Moore-Penrose inverse is given in [2], for arbitrary Hilbert spaces.

In the following theorem we give general representations of $\{1, 2\}$, $\{1, 2, 3\}$ and $\{1, 2, 4\}$ inverses for an arbitrary bounded operator on Hilbert spaces. As a consequence we obtain the known representation of the Moore-Penrose inverse from [2].

THEOREM 2.1. *Let $A = PQ$ be a full-rank decomposition of $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ according to (1.1). Then:*

- (a) *$X \in A\{1, 2\}$ if and only if there exist operators $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$ and $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, such that QW_1 and W_2P are invertible in $B(\mathcal{H}_3)$. In such a case, X possesses the following general representation*

$$X = W_1(QW_1)^{-1}(W_2P)^{-1}W_2 \quad (2.1)$$

- (b) *$X \in A\{1, 2, 3\}$ if and only if there exists an operator $W_1 \in B(\mathcal{H}_3, \mathcal{H}_1)$, such that QW_1 is invertible in $B(\mathcal{H}_3)$. In the case when it exists, a general representation for X is as follows:*

$$X = W_1(QW_1)^{-1}(P^*P)^{-1}P^*. \quad (2.2)$$

- (c) $X \in A\{1, 2, 4\}$ if and only if there exists an operator $W_2 \in B(\mathcal{H}_2, \mathcal{H}_3)$, such that W_2P is invertible in $B(\mathcal{H}_3)$. In this case

$$X = Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2.$$

- (d) $A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^*$.

Proof. (a) Follows from Lemma 2.1.

(b) If X has the form (2.2), then it is easy to verify $X \in A\{1, 2, 3\}$. We need to prove that the form (2.2) holds for all $\{1, 2, 3\}$ inverses of A . Indeed, if $X \in A\{1, 2, 3\}$, then $X = Q_r^{-1}P_l^{-1}$, and from the equation (3) it follows that $(PP_l^{-1})^* = PP_l^{-1}$. Thus $P^*PP_l^{-1} = P^*$. Operator P^*P is invertible, so that $P_l^{-1} = (P^*P)^{-1}P^*$. The right inverse of Q retains the general form $Q_r^{-1} = W_1(QW_1)^{-1}$ from Lemma 2.1. Consequently,

$$X = W_1(QW_1)^{-1}(P^*P)^{-1}P^*.$$

The proof of the statement (c) is similar as the proof of (b). Also, (d) follows from (b) and (c). For the part (d) see also [2]. ■

Now, we shall consider the weighted Moore-Penrose inverse under the various hypothesis. The weighted Moore-Penrose inverse is investigated in [1], [7] and [10] for the set of complex matrices and in [8] for matrices over an integral domain. If M and N are positive, then $A_{M,N}^\dagger$ and $A_{M',N'}^\dagger$ always exist [1], [10], and $A_{M,N}^\dagger = A_{M',N'}^\dagger$. In [8] and [7] it is derived a representation and conditions for the existence of $A_{M,N}^\dagger$ and $A_{M',N'}^\dagger$, respectively, under the more general assumptions that the matrices M and N are invertible (not necessary positive).

THEOREM 2.2. *Let $A = PQ$ be a full-rank decomposition of A according to (1.1). Then:*

- (a) *If $M \in B(\mathcal{H}_2)$ and $N \in B(\mathcal{H}_1)$ are invertible operators, then $A_{M,N}^\dagger$ exists if and only if P^*MP and $QN^{-1}Q^*$ are invertible selfadjoint operators. In that case*

$$\begin{aligned} A_{M,N}^\dagger &= N^{-1}Q^*(QN^{-1}Q^*)^{-1}(P^*MP)^{-1}P^*M^* \\ &= N^{-1}Q^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^*. \end{aligned} \quad (2.3)$$

- (b) *Let $M \in B(\mathcal{H}_2)$ and $N \in B(\mathcal{H}_1)$ be invertible operators, such that $QN^{-1}Q^*$ is left invertible and P^*MP right invertible. Then $A_{M',N'}^\dagger$ exists if and only if $QN^{-1}Q^*$ and P^*MP are invertible and*

$$\begin{aligned} E &= N^{-1}Q^*(QN^{-1}Q^*)^{-1} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}, \\ F &= (P^*MP)^{-1}P^*M = ((MP)^*P)^{-1}(MP)^*. \end{aligned} \quad (2.4)$$

In this case is $A_{M',N'}^\dagger = Q_r^{-1}P_l^{-1}$, where $Q_r^{-1} = E$ and $P_l^{-1} = F$.

- (c) *If $M \in B(\mathcal{H}_2)$ and $N \in B(\mathcal{H}_1)$ are positive and invertible operators, then*

$$\begin{aligned} A_{M,N}^\dagger &= A_{M',N'}^\dagger = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^* \\ &= N^{-1}Q^*(QN^{-1}Q^*)^{-1}(P^*MP)^{-1}P^*M. \end{aligned} \quad (2.5)$$

Proof. (a) If M and N are invertible operators and $A_{M,N}^\dagger$ exists, using the principles from [8], from (3M) and (4N) we get

$$P^*MPP_l^{-1} = P^*M^*, \quad Q_r^{-1}Q(N^{-1})^*Q^* = N^{-1}Q^*, \quad (2.6)$$

$$P^*MP = P^*M^*P, \quad QN^{-1}Q^* = Q(N^{-1})^*Q^*. \quad (2.7)$$

From (2.7) we conclude that P^*MP and $QN^{-1}Q^*$ are selfadjoint operators. We now prove that $P^*MP = P^*M^*P$ and $QN^{-1}Q^* = Q(N^{-1})^*Q^*$ are invertible. Indeed, from (1) and (3M) we get the following equation (see [8]):

$$(QXM^{-1}X^*Q^*)(P^*M^*P) = I.$$

This means that P^*M^*P is left invertible. Also, since P^*M^*P is selfadjoint, we conclude that P^*M^*P is invertible. Similarly, (1) and (4N) imply the following

$$(QN^{-1}Q^*)(P^*X^*N^*XP) = I,$$

which means that $QN^{-1}Q^*$ is right invertible, so it is also invertible.

Using invertibility of $QN^{-1}Q^*$ and P^*M^*P , from (2.6) it follows

$$P_l^{-1} = (P^*M^*P)^{-1}(MP)^*, \quad Q_r^{-1} = N^{-1}Q^*(Q(N^{-1})^*Q^*)^{-1}. \quad (2.8)$$

Now, the representations (2.3) follows from $A_{M,N}^\dagger = Q_r^{-1}P_l^{-1}$, (2.8) and (2.7).

(b) Suppose that $M \in B(\mathcal{H}_2)$ and $N \in B(\mathcal{H}_1)$ are invertible operators, such that $QN^{-1}Q^*$ is left invertible and P^*MP right invertible and $A_{M',N'}^\dagger$ exists. From the equations (1) and (3M') in the same way as in [7] we get $(QXM^{-1}X^*Q^*)(P^*MP) = I$, which means that P^*MP is left invertible. Similarly, from (1) and (4N') we obtain $(QN^{-1}Q^*)(P^*X^*N^*XP) = I$, which implies the right invertibility of $QN^{-1}Q^*$. According to the assumptions, we conclude that P^*MP and $QN^{-1}Q^*$ are invertible. The identities (2.4) can be proved using the method from [7].

On the other hand, if P^*MP and $QN^{-1}Q^*$ are invertible and (2.4) holds, one can verify that $Q_r^{-1}P_l^{-1}$ (where $Q_r^{-1} = E$ and

$$P_l^{-1} = F) \text{ satisfies the equations which define } A_{M',N'}^\dagger.$$

(c) Firstly, we prove that $Q(QN^{-1})^*$ and $(MP)^*P$ are positive and invertible in $B(\mathcal{H}_3)$. If $x \in \mathcal{H}_3$ and $\|x\| = 1$, then

$$(Q(QN^{-1})^*x, x) = (N^{-1}Q^*x, Q^*x) > 0.$$

Suppose that $\inf_{\|x\|=1} (Q(QN^{-1})^*x, x) = 0$. Then there exists a sequence of unit vectors $(x_n)_n$ in \mathcal{H}_3 , such that $\lim_n (N^{-1}Q^*x_n, Q^*x_n) = 0$. Since N^{-1} is positive and invertible, it follows that there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$, such that $\lim_k Q^*x_{n_k} = 0$. Now, it follows that Q^* is not one-to-one with closed range, so Q is not onto. We get the contradiction, so $Q(QN^{-1})^*$ is positive and invertible

in $B(\mathcal{H}_3)$. Analogously, we can prove that $(MP)^*P$ is positive and invertible in $B(\mathcal{H}_3)$.

The rest of the proof follows from parts (a) and (b). ■

Now, we consider $\{2\}$ -generalized inverses with prescribed range and kernel. Fundamental results for matrices can be found in [1] and [5].

Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $X \in B(\mathcal{H}_2, \mathcal{H}_1)$ be a $\{2\}$ -inverse of A , such that $\mathcal{R}(X) = T$ is a closed subspace of \mathcal{H}_1 and $\mathcal{N}(X) = S$ is a closed subspace of \mathcal{H}_2 . Then we write $X = A_{T,S}^{(2)}$. For given closed subspaces T of \mathcal{H}_1 and S of \mathcal{H}_2 , it is a natural question when $A_{T,S}^{(2)}$ exists? The answer in the case of arbitrary Hilbert spaces is given in the following theorem.

We state the following elementary result.

LEMMA 2.2. *Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$, T and S be closed subspaces of \mathcal{H}_1 and \mathcal{H}_2 respectively. Then the following statements are equivalent:*

- (a) A has a $\{2\}$ -inverse $X \in B(\mathcal{H}_2, \mathcal{H}_1)$ such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$;
- (b) $A: T \rightarrow A(T)$ is invertible and $A(T) \oplus S = \mathcal{H}_2$.

In the case when (a) or (b) holds, X is unique and is denoted by $A_{T,S}^{(2)}$.

Now, we generalize the result from [5].

THEOREM 2.3. *Suppose that A, T and S satisfy the condition (a) or (b) from Lemma 2.2 and let $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ be such that $\mathcal{R}(Y) = T$ and $\mathcal{N}(Y) = S$. If there exists a Hilbert space \mathcal{H}_3 and a left invertible operator $E \in B(\mathcal{H}_3, \mathcal{H}_1)$ such that $\mathcal{R}(E) = T$, then*

$$W = E^*YAE \in B(\mathcal{H}_3)$$

is invertible in $B(\mathcal{H}_3)$ and

$$A_{T,S}^{(2)} = EW^{-1}E^*Y.$$

Proof. Notice that $E: \mathcal{H}_3 \rightarrow T$ is invertible, $A: T \rightarrow AT$ is invertible and $Y: AT \rightarrow T$ is invertible. Since $\mathcal{N}(E^*)^\perp = \mathcal{R}(E) = T$, it follows that $E^*: T \rightarrow \mathcal{H}_3$ is invertible, so W is invertible. Now, it is easy to verify that $EW^{-1}E^*Y$ is a $\{2\}$ -inverse of A . Also, $\mathcal{N}(Y) = S$, $W^{-1}E^*Y: AT \rightarrow \mathcal{H}_3$ is invertible and $\mathcal{R}(EW^{-1}E^*Y) = \mathcal{R}(E) = T$, $\mathcal{N}(EW^{-1}E^*Y) = S$, so

$$A_{T,S}^{(2)} = EW^{-1}E^*Y. \quad \blacksquare$$

THEOREM 2.4. *Let $A: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $X: \mathcal{H}_2 \rightarrow \mathcal{H}_1$. Then $X \in A\{2\}$ if and only if there exist Hilbert spaces $\mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5$ and operators*

$$C \in B(\mathcal{H}_4, \mathcal{H}_1), D \in B(\mathcal{H}_2, \mathcal{H}_3), W_1 \in B(\mathcal{H}_5, \mathcal{H}_4), W_2 \in B(\mathcal{H}_3, \mathcal{H}_5),$$

such that DAC is g -invertible and W_2DACW_1 is invertible. In this case:

$$X = CW_1(W_2DACW_1)^{-1}W_2D. \quad (2.9)$$

Proof. If X possesses the form (2.9), it is not difficult to verify $X \in A\{2\}$. On the other hand, using the method from [10], it is easy to verify that $X \in A\{2\}$ if and only if there exist operators C and D , such that

$$X = C(DAC)^{(1,2)}D, \quad C \in B(\mathcal{H}_4, \mathcal{H}_1), \quad D \in B(\mathcal{H}_2, \mathcal{H}_3).$$

According to part (a) of Theorem 2.1, $X \in A\{2\}$ if and only if there exist operators W_1 and W_2 , such that W_2DACW_1 is invertible, and X possesses the form (2.9). ■

We introduce a general representation of the Drazin inverse based on an arbitrary full-rank factorization of A^l , $l \geq k = \text{asc}(A) = \text{des}(A)$. The following theorem is a natural generalization of a Cline's result from [6], introduced for complex matrices. We shall assume that A is not a nilpotent operator, i.e. $A^D \neq 0$.

THEOREM 2.5. *Let \mathcal{X} be a Banach space. If $A \in B(\mathcal{X})$, $l \geq k = \text{asc}(A) = \text{des}(A) < \infty$ and $A^l = P_{A^l}Q_{A^l}$ is the full-rank decomposition of A^l , then*

$$A^D = P_{A^l}(Q_{A^l}AP_{A^l})^{-1}Q_{A^l}.$$

Proof. If $\text{asc}(A) = \text{des}(A) = k < \infty$, then it is well-known that $\mathcal{N}(A^l) = \mathcal{N}(A^k)$ and $\mathcal{R}(A^l) = \mathcal{R}(A^k)$ for all $l \geq k$,

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2, \tag{2.10}$$

where $\mathcal{X}_1 = \mathcal{N}(A^l)$ and $\mathcal{X}_2 = \mathcal{R}(A^l)$, $A(\mathcal{X}_i) \subset \mathcal{X}_i$ for $i = 1, 2$, $A_1 = A|_{\mathcal{X}_1}$ is nilpotent and $A_2 = A|_{\mathcal{X}_2}$ is invertible (A is not nilpotent) [3], [4]. We can write

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix}$$

with respect to the decomposition (2.10) ([3], [4]). Since $\mathcal{N}(A^l)$ and $\mathcal{R}(A^l)$ are complementary and closed subspaces of \mathcal{X} , it follows that A^l is g -invertible, so there exists the full-rank decomposition $A^l = P_{A^l}Q_{A^l}$, where $P_{A^l} \in B(\mathcal{Z}, \mathcal{X})$ is left invertible and $Q_{A^l} \in B(\mathcal{X}, \mathcal{Z})$ is right invertible, for some Banach space \mathcal{Z} . By the isomorphism theorem [3], we can take that $\mathcal{Z} = \mathcal{X}_2$. We conclude that P_{A^l} and Q_{A^l} have the following representations with respect to (2.10):

$$P_{A^l} = \begin{bmatrix} M \\ \tilde{P} \end{bmatrix} \quad \text{and} \quad Q_{A^l} = [N \quad \tilde{Q}],$$

where $\tilde{P}, \tilde{Q} \in B(\mathcal{X}_2)$, $M \in B(\mathcal{X}_2, \mathcal{X}_1)$, $N \in B(\mathcal{X}_1, \mathcal{X}_2)$. Now, P_{A^l} is left invertible and Q_{A^l} is right invertible, so P_{A^l} and Q_{A^l} are g -invertible operators, $\mathcal{N}(P_{A^l}) = \{0\}$ and $\mathcal{R}(Q_{A^l}) = \mathcal{X}_2$. It follows that $\mathcal{R}(P_{A^l}) = \mathcal{R}(A^l) = \mathcal{X}_2$ and $\mathcal{N}(Q_{A^l}) = \mathcal{N}(A^l) = \mathcal{X}_1$, so $M = 0$, $N = 0$ and

$$P_{A^l} = \begin{bmatrix} 0 \\ \tilde{P} \end{bmatrix} \quad \text{and} \quad Q_{A^l} = [0 \quad \tilde{Q}].$$

It is easy to verify that \tilde{P} is left invertible and \tilde{Q} is right invertible in $B(\mathcal{X}_2)$. But

$$\begin{bmatrix} 0 & 0 \\ 0 & A_2^l \end{bmatrix} = A^l = P_{A^l}Q_{A^l} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}\tilde{Q} \end{bmatrix},$$

so $A_2^l = \tilde{P}\tilde{Q}$. Since A_2^l is invertible, it follows that \tilde{P} and \tilde{Q} are invertible in $B(\mathcal{X}_2)$.

Now, $Q_{A^l} A P_{A^l} = \tilde{Q} A_2 \tilde{P}$ is invertible in $B(\mathcal{X}_2)$, so

$$A^D = \begin{bmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}(\tilde{Q} A_2 \tilde{P})^{-1} \tilde{Q} \end{bmatrix} = P_{A^l} (Q_{A^l} A P_{A^l})^{-1} Q. \quad \blacksquare$$

As a corollary, we get the following result.

COROLLARY 2.1. *If \mathcal{X} is a Banach space, $A \in B(\mathcal{X})$ and $\text{asc}(A) = \text{des}(A) = k < \infty$ and $A^l = P_{A^l} Q_{A^l}$ is an arbitrary full-rank decomposition of A^l , $l \geq k$, then*

- (a) $(A^D)^l = P_{A^l} (Q_{A^l} A^l P_{A^l})^{-1} Q_{A^l} = P_{A^l} (Q_{A^l} P_{A^l})^{-2} P_{A^l}$;
- (b) $AA^D = P_{A^l} (Q_{A^l} P_{A^l})^{-1} Q_{A^l}$;
- (c) *If \mathcal{X} is a Hilbert space, then $(A^D)^\dagger = (Q_{A^l})^\dagger Q_{A^l} A P_{A^l} (P_{A^l})^\dagger$.*

Proof. (a) Follows from $(A^D)^l = (A^l)^\#$ and Theorem 2.5.

(b) According to Theorem 2.5 it follows that $Q_{A^l} P_{A^l} = \tilde{Q} \tilde{P}$, so an easy computation shows that

$$P_{A^l} (Q_{A^l} P_{A^l})^{-1} Q_{A^l} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = AA^D.$$

(c) Follows from Theorem 2.1 (d) and Theorem 2.5. \blacksquare

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